On Metric Type Spaces and
Fixed Point Theorems

Olaniyi Samuel Iyiola

Department of Mathematics and Statistics
King Fahd University of Petroleum and Minerals
KFUPM, Dhahran, Saudi Arabia

Yae Ulrich Gaba

Department of Mathematics and Applied Mathematics
University of Cape Town
Rondebosch 7701, South Africa

Copyright © 2014 Olaniyi Samuel Iyiola and Yae Ulrich Gaba. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

In this article, we discuss some fixed point theorems in metric type spaces. The need to define such space came from the properties obtain on cone metric spaces and the connection between the two notions is clearly explained in [1]. In particular, we show that most of the new results are merely copies of the classic ones.

Mathematics Subject Classification: 47H10, 37C25

Keywords: Metric Type Spaces; Fixed Point; Cone Metric

1 Introduction

Cone metric spaces were introduced in [2] and many fixed point results concerning mappings in such spaces have been established. In [1], M. A. Khamsi connected this concept with a generalised form of metric space that he named
metric type space. We show that some proofs follow closely the classical proofs in the cone metric case, but generalize them. For this, we begin by showing that the class of metric type spaces is strictly larger than the class of metric spaces, hence the presented generalization are indeed, not trivial.

2 Preliminaries

In this section we recall some known definitions, notations and results concerning cones in Banach spaces.

Definition 2.1 Let $E$ be a real Banach space with norm $\| \cdot \|$ and $P$ be a subset of $E$. Then $P$ is called a cone if and only if

1. $P$ is closed, nonempty and $P \neq \{\theta\}$, where $\theta$ is the zero vector in $E$;
2. for any $a, b \geq 0$ (nonnegative real numbers), and $x, y \in P$, we have $ax + by \in P$;
3. for $x \in P$, if $-x \in P$, then $x = \theta$.

Given a cone $P$ in a Banach space $E$, we define on $E$ a partial order $\preceq$ with respect to $P$ by

$$x \preceq y \iff y - x \in P.$$ 

We also write $x \prec y$ whenever $x \preceq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{Int}(P)$ (where $\text{Int}(P)$ designates the interior of $P$).

The cone $P$ is called normal if there is a real number $C > 0$, such that for all $x, y \in E$, we have

$$\theta \preceq x \preceq y \implies \|x\| \leq C\|y\|.$$ 

The least positive number satisfying this inequality is called the normal constant of $P$. Therefore, we shall say that $P$ is a $K$-normal cone to indicate the fact that the normal constant is $K$.

The cone is called regular if every increasing sequence which is bounded from above is convergent. That is if $(x_n)$ is a sequence such that $x_n \preceq x_2 \preceq \cdots \preceq y$ for some $y \in E$ then, there exists $x^* \in E$ such that $\lim_{n \to \infty} \|x_n - x^*\| = 0$.

Lemma 2.2 (see [3])

a) Every regular cone is normal;

b) The cone $P$ is regular if every decreasing sequence which is bounded from below is convergent.
Definition 2.3 Let $X$ be a non empty set. A function $d : X \times X \rightarrow E$ is called a cone metric on $X$ if:

(d1) $\theta \preceq d(x, y)$ $\forall$ $x \in X$ and $d(x, y) = \theta$ if and only if $x = y$;

(d2) $d(x, y) = d(y, x)$ $\forall$ $x, y \in X$;

(d3) $d(x, z) \leq d(x, y) + d(y, z)$ $\forall$ $x, y, z \in X$.

The pair $(X, d)$ is called a cone metric space.

Definition 2.4 A subset $A$ of $E$ is said to be bounded from above with respect to $P$ (or upper bounded) if there exists $x_0 \in E$ such that $a \preceq x_0$ for all $a \in A$.

A subset $A$ of $E$ is said to be bounded from below with respect to $P$ (or lower bounded) if there exists $x_0 \in E$ such that $x_0 \preceq a$ for all $a \in A$.

Definition 2.5 (Compare [3]) A cone $P$ is said to be minihedral if $x \lor y := \sup\{x, y\}$ exists for all $x, y \in E$ and strongly minihedral if every subset of $E$ which is bounded from above has a supremum.

Definition 2.6 (Compare [4]) Let $(x_n)$ be a sequence in a cone metric space $(X, d)$.

(a) $(x_n)$ is convergent to $x \in X$ and we denote $\lim_{n \to \infty} x_n = x$, if for every $c \in E$ with $c \gg \theta$, there exists $n_0 \in \mathbb{N}$ such that

\[ \forall \ n, m \geq n_0 \ \ d(x_n, x) \ll c; \]

(b) $(x_n)$ is called Cauchy if for every $c \in E$ with $c \gg \theta$, there exists $n_0 \in \mathbb{N}$ such that

\[ \forall \ n, m \geq n_0 \ \ d(x_n, x_m) \ll c. \]

Definition 2.7 A cone metric space $(X, q)$ is said to be complete if every Cauchy sequence in $X$ is convergent in $X$.

Lemma 2.8 (see[2]) Let $(X, d)$ be a cone metric space over a cone $K$-normal cone $P$. The sequence $(x_n)$ converges to $x \in X$ if and only if $\lim_{n \to \infty} d(x_n, x) = \theta$.

Lemma 2.9 (see[2]) Let $(X, d)$ be a cone metric space over a cone $K$-normal cone $P$. The sequence $(x_n)$ is Cauchy if and only if $\lim_{n, m \to \infty} d(x_n, x_m) = \theta$.

More properties concerning convergence and Cauchy sequences can be found in [2]
Definition 2.10 Let \((X, q)\) be a cone metric space. A function \(f : X \to X\) is said to be **lipschitzian** if there exists some \(\kappa \in \mathbb{R}\) such that
\[
q(f(x), f(y)) \leq \kappa q(x, y) \quad \forall x, y \in X.
\]
The smallest constant which satisfies the above inequality is called the **lipschitz constant** of \(f\) and is denoted \(\text{Lip}(f)\). In particular \(f\) is said to be **contractive** if \(\text{Lip}(f) \in [0, 1)\) and **expansive** if \(\text{Lip}(f) = 1\).

As we mentioned earlier, cone metric spaces have a metric type structure. Indeed, we have the following result:

**Theorem 2.11** ([1]) Let \((X, q)\) be a cone metric space over the Banach space \(E\) with the \(K\)-normal cone \(P\). The mapping \(Q : X \times X \to [0, \infty)\) defined by
\[
Q(x, y) = \|q(x, y)\|
\]
satisfies the following properties
\begin{enumerate}[(Q1)]
    
    \item \(Q(x, x) = 0\) for any \(x \in X\);
    
    \item \(Q(x, y) = Q(y, x)\) for any \(x, y \in X\);
    
    \item \(Q(x, y) \leq K(Q(x, z_1) + Q(z_1, z_2) + \cdots + Q(z_n, y))\), for any points \(x, y, z_i \in X, i = 1, 2, \ldots, n\).
\end{enumerate}

Note that property (Q3) does not give the classical triangle inequality satisfied by a distance and there are many examples where the triangle inequality fails. We are therefore led to the following definition.

**Definition 2.12** Let \(X\) be a nonempty set, and let the function \(D : X \times X \to [0, \infty)\) satisfy the following properties:
\begin{enumerate}[(D1)]
    
    \item \(D(x, x) = 0\) for any \(x \in X\);
    
    \item \(D(x, y) = D(y, x)\) for any \(x, y \in X\);
    
    \item \(D(x, y) \leq \alpha(D(x, z_1) + D(z_1, z_2) + \cdots + D(z_n, y))\) for any points \(x, y, z_i \in X, i = 1, 2, \ldots, n\) and some constant \(\alpha > 0\).
\end{enumerate}

The triplet \((X, D, \alpha)\) is called a **metric type space**.

Let \((X, D, \alpha)\) be a metric type space. Then for each \(x \in X\) and \(\epsilon > 0\), the set
\[
B_D(x, \epsilon) = \{y \in X : D(x, y) < \epsilon\}
\]
denotes the open \(\epsilon\)-ball at \(x\) with respect to \(D\). It should be noted that the collection
\[
\{B_D(x, \epsilon) : x \in X, \epsilon > 0\}
\]
yields a base for the topology \( \tau(D) \) induced by \( D \) on \( X \). This topology will be called **metric type topology**. Hence a subset \( A \subset X \) is said to be open if for any \( a \in A \), there exists \( \varepsilon > 0 \) such that \( B_D(a, \varepsilon) \subset A \).

Also each \( x \in X \) and \( \varepsilon \geq 0 \), the set

\[
C_D(x, \varepsilon) = \{ y \in X : D(x, y) < \varepsilon \}
\]

denotes the closed \( \varepsilon \)-ball at \( x \) with respect to \( D \).

A subset \( S \) metric type space \( (X, D, \alpha) \) is said to be **bounded** if \( S \) is contained in some ball \( B_D(x, r) \) of \( X \). We also define the **diameter** \( \operatorname{diam}(A) \) of \( A \) that we denote by \( \operatorname{diam}(A) := \sup\{D(x, y) : x, y \in A\} \). Hence \( A \) is bounded if and only if \( \operatorname{diam}(A) < \infty \).

The concepts of Cauchy sequence and convergence for a metric type space are defined in the same way as defined for a metric space. For the interested reader the definitions can be obtained in [5]. Moreover, for \( \alpha = 1 \), we recover \((X, D, 1)\) as \((X, D)\), hence metric type generalizes quasi-pseudo metric. It is worth mentioning that if \((X, D, \alpha)\) is a metric type space, then for any \( \beta \geq \alpha \), \((X, D, \beta)\) is also a metric type space. Hence, in the sequel we shall denote \((X, D, \alpha)\) simply as \((X, D)\) when there is no confusion.

**Example 2.13** Let \( X = \{a, b, c\} \) and the mapping \( D : X \times X \to [0, \infty) \) defined by \( D(a, b) = 1/5 \), \( D(b, c) = 1/4 \), \( D(a, c) = 1/2 \), \( D(x, x) = 0 \) for any \( x \in X \) and \( D(x, y) = D(y, x) \) for any \( x, y \in X \). Since

\[
\frac{1}{2} = D(a, c) > D(a, b) + D(b, c) = \frac{9}{20},
\]

then we conclude that \( X \) is not a metric space. Nevertheless, with \( \alpha = 2 \), it is very easy to check that \((X, D, 2)\) is metric type space.

**Definition 2.14** A subset \( S \) metric type space \( (X, D, \alpha) \) is said to be **totally bounded** if given \( \varepsilon > 0 \) there exists a finite set of points \( \{s_1, s_2, \ldots, s_n\} \subseteq S \), called \( \varepsilon \)-net, such that given any \( s \in S \) there exists \( i \in \{1, 2, \ldots, n\} \) for which \( D(s, s_i) \leq \varepsilon \).

**Definition 2.15** Let \((X, D, \alpha)\) a metric type space. If for any sequence \((x_n)\) in \( X \), there is a subsequence \((x_{n,k})\) of \((x_n)\) such that \((x_{n,k})\) is convergent in \( X \). Then \( X \) is called a **sequentially compact** metric type space.

Some topological properties concerning metric type spaces have been established by Khamsi et al. and can be found in [5], in particular compactness and closure are characterized in metric type spaces. We recall some of these results, as we will make use of them.
Proposition 2.16 [5] Let $(X, D, \alpha)$ be a metric type space. Then for any nonempty subset $S$ of $X$ we have

1. $S$ is closed if and only if for any sequence $(x_n) \subseteq S$ which converges to $x$, we have $x \in S$;

2. if we define $\overline{S}$ to be the intersection of all closed subsets of $X$ which contains $S$, then for any $x \in \overline{S}$ and for any $\varepsilon > 0$, we have $B_D(x, \varepsilon) \cap S \neq \emptyset$.

Proposition 2.17 [5] Let $(X, D, \alpha)$ be a metric type space. A subset $S$ of $X$ is compact if and only if any sequence of $(x_n)$ of points of $S$ has a subsequence $(x_{n,k})$ which converges to a point of $S$.

Proposition 2.18 [5] Let $(X, D, \alpha)$ be a metric type space and $S$ a subset of $X$. If $S$ is compact, then $S$ is totally bounded.

3 Some Generalization Results on Metric Type Spaces

We begin by giving the following results, very familiar in the theory of metric spaces but extended here for metric type space.

Lemma 3.1 Every metric type space $(X, D, \alpha)$ is Hausdorff.

Proof. Let $x, y \in X$ such that $x \neq y$. Let’s set $\varepsilon = \frac{D(x, y)}{3\alpha}$. We claim that $A = B_D(x, \varepsilon) \cap B_D(y, \varepsilon) = \emptyset$. By the way of contradiction, assume that there is $z \in A$. Hence

\[ D(x, y) \leq \alpha(D(x, z) + D(z, y)) < \alpha \left( \frac{D(x, y)}{3\alpha} + \frac{D(x, y)}{3\alpha} \right) \leq \frac{2}{3} D(x, y), \]

which implies that $D(x, y) = 0$, i.e. $x = y$. Contradiction.

Lemma 3.2 A Cauchy sequence $(x_n)$ in a metric type space $(X, D, \alpha)$ is always bounded.

Lemma 3.3 Let $(x_n)$ be a Cauchy sequence in a metric type space $(X, D, \alpha)$. Then $(x_n)$ converges to $x$ if and only if it has a subsequence that converges to $x$.

Proof. If $(x_n)$ converges to $x$, it trivially follows that $(x_n)$ is a subsequence of itself that converges to $x$. 
Conversely, Suppose that \((x_{n,k})\) is a subsequence of \((x_n)\) that converges to \(x\). Let \(\varepsilon > 0\).

By the definition of a Cauchy sequence, there exists \(n_0 \in \mathbb{N}\) such that
\[
D(x_n, x_m) < \frac{\varepsilon}{2\alpha} \quad \text{whenever} \quad n, m \geq n_0.
\]

By the definition of convergence, there exists \(k_0 \in \mathbb{N}\) such that
\[
D(x_{n,k}, x) < \frac{\varepsilon}{2\alpha} \quad \text{whenever} \quad k \geq k_0.
\]

If we set \(L = \max\{n_0 + 1, k_0 + 1\}\), by strictly increasing sequence of natural numbers, we know that \(n_{L} := n_L \geq L > k_0\). Hence
\[
D(x_m, x) \leq \alpha[D(x_m, x_{n_L}) + D(x_{n_L}, x)] < \varepsilon \quad \text{whenever} \quad m > L.
\]

That is, \((x_n)\) converges to \(x\). Another interesting notion that we discussed here is that of precompactness. A subset \(F\) of a topological space \((E, \tau(E))\) is said to be precompact if its \(\tau(E)\)-closure \(\overline{F}\) is compact. In many metric spaces, precompactness and total boundedness are equivalent. In fact, we prove here that one implication always holds is metric type space.

**Theorem 3.4** If a subspace \(K\) of a metric type space \((X, D, \alpha)\) is precompact then it is totally bounded.

**Proof.** Suppose that \(K\) is precompact and let \(\varepsilon > 0\). Since \(\overline{K}\) is compact, hence totally bounded, there exists a finite set \(\{x_1, x_2, \ldots, x_n\} \subseteq \overline{K}\) such that if \(x \in K\) then for some \(1 \leq i \leq n\), \(D(x_i, x) \leq \varepsilon/(2\alpha)\). However since each \(x_i \in \overline{K}\), for each \(i\), there exists \(s_i \in K\) such that \(D(x_i, s_i) < \varepsilon/(2\alpha)\). It follows that for each \(x \in K\), \(D(x, s_i) < \varepsilon\) for some \(1 \leq i \leq n\), so \(\{s_1, s_2, \ldots, s_n\}\) is the desired \(\varepsilon\)-net in \(K\). Another interesting result for which we shall give an application is the Cantor’s Intersection Theorem. We state the following theorem without proving it, since the proof is exactly the same as in the metric case.

**Theorem 3.5** A metric type space \((X, D, \alpha)\) is complete if and only if given any descending sequence \((D_n)\) of nonempty bounded closed subsets of \(X\),
\[
\lim_{n \to \infty} \text{diam}(D_n) = 0 \implies \bigcap_{n=1}^{\infty} D_n \neq \emptyset.
\]

**Remark 3.6** Note that in the statement of the above theorem, the condition \(\bigcap_{n=1}^{\infty} D_n \neq \emptyset\) could have been replaced with \(\bigcap_{n=1}^{\infty} D_n\) consists of exactly one point.
Hence, another variant of Cantor’s Theorem (due to Ascoli in the metric case) is

**Theorem 3.7** A metric type space \((X, D, \alpha)\) is complete if and only if the intersection of any descending sequence \((D_n)\) of nonempty bounded closed balls in \(X\) having radii tending to 0 consists of exactly one point.

As an application of Cantor’s Theorem we have the partial converse of Theorem 3.4.

**Theorem 3.8** If a subspace \(K\) of a metric type space \((X, D, \alpha)\) is totally bounded, then it is precompact.

**Proof.** In view of Proposition 2.17, we only need to show that any sequence \((x_n)\) has a convergent subsequence. Its limit will necessarily lie in \(K\) proving that \(K\) is compact. If an infinite number of terms of \((x_n)\) are the same, we could easily select a subsequence of \((x_n)\) which is constant and there converges trivially. So by throwing some terms if necessary, we may assume that each two terms of \((x_n)\) are distinct.

Since \(K\) is is totally bounded, there exists a finite set of points \(\{s_1, s_2, \ldots, s_n\} \subseteq K\) such that given any \(s \in K\) there exists \(i \in \{1, 2, \ldots, n\}\) for which \(D(s, s_i) \leq \frac{1}{2\alpha}\).

We also know that for any \(a \in K\) there exists \(x_a \in K\) for which \(D(a, x_a) \leq \frac{1}{2\alpha}\).

For \(x_a \in K\), there exists \(i_a \in \{1, 2, \ldots, n\}\) for which \(D(x_a, s_{i_a}) \leq \frac{1}{2\alpha}\).

Hence \(D(s_{i_a}, a) \leq 1\), i.e. \(a \in B_D(s_{i_a}, 1)\).

**Conclusion** There exist points \(\{z_{1,1}, z_{2,1}, \ldots, z_{n,1}\} \subseteq K\) such that

\[
K \subseteq \bigcup_{i=1}^{n} B_D(z_{i,1}; 1).
\]

The last part of the proof follows exactly the classical theory and can be read in the literature.

## 4 First Fixed Point Results

We state here our first fixed point results.

**Theorem 4.1** Let \((X, D, \alpha)\) a complete metric type space. Suppose the mapping \(T : X \rightarrow X\) satisfies the contractive condition

\[
D(Tx, Ty) \leq kD(x, y) \quad \text{for all } x, y \in X,
\]

where \(k \in [0, 1)\) is a constant. Then \(T\) has a fixed point and for any \(x \in X\), the orbit \(\{T^n x\}\) converges to the fixed point.
Proof. Choose $x_0 \in X$. Set $x_1 = Tx_0, x_2 = Tx_1 = T^2x_0, \ldots, x_{n+1} = Tx_n = T^{n+1}x_0, \ldots$. We have

\[
D(x_n, x_{n+1}) = D(Tx_{n-1}, Tx_n) \leq kD(x_{n-1}, x_n) \\
\leq D(x_{n-1}, x_{n-2}) \leq \cdots \leq k^nD(x_0, x_1).
\]

So for $n < m$,

\[
D(x_n, x_m) \leq \alpha[D(x_n, x_{n+1}) + D(x_{n+1}, x_{n+2}) + \cdots + D(x_{m-1}, x_m)] \\
\leq \alpha(k^n + k^{n+1} + \cdots + k^{m-1})D(x_0, x_1) \leq \alpha \frac{k^n}{1-k}D(x_0, x_1).
\]

This implies $D(x_n, x_m) \to 0$ as $n, m \to \infty$. Hence $(x_n)$ is a Cauchy sequence. By completeness of $X$, there is $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. Since

\[
D(Tx^*, x^*) \leq \alpha[D(T^nx, Tx_n) + D(Tx_n, x^*)] \leq \alpha kD(x^*, x_n) + \alpha D(x_{n+1}, x^*),
\]

\[
D(Tx^*, x^*) \leq \alpha kD(x^*, x_n) + \alpha D(x_{n+1}, x^*) \to 0.
\]

Hence $D(Tx^*, x^*) = 0$, i.e. $Tx^* = x^*$.

Now if $y^*$ is a fixed point of $T$, then

\[
D(x^*, y^*) = D(Tx^*, Ty^*) \leq kD(x^*, y^*).
\]

Hence $D(x^*, y^*) = 0$ and $x^* = y^*$. Therefore the fixed point of $T$ is unique.

Corollary 4.2 Let $(X, D, \alpha)$ a complete metric type space with $0 < \alpha \leq 1$, $x_0 \in X$ an arbitrary point and $\varepsilon > 0$. Suppose the mapping $T : X \to X$ satisfies the contractive condition

\[
D(Tx, Ty) \leq kD(x, y) \quad \text{for all } x, y \in C_D(x_0, \varepsilon),
\]

where $k \in [0, 1]$ is a constant and $D(x_0, Tx_0) \leq (1-k)\varepsilon$. Then $T$ has a fixed point in $C_D(x_0, \varepsilon)$.

Proof. We only need to prove that $C_D(x_0, \varepsilon)$ is complete and $Tx \in C_D(x_0, \varepsilon)$ for all $x \in C_D(x_0, \varepsilon)$.

Suppose $\{x_n\}$ is a Cauchy sequence in $C_D(x_0, \varepsilon)$. Then $\{x_n\}$ is also a Cauchy sequence in $X$. By the completeness of $X$, there is $x \in X$ such that $\{x_n\}$ converges to $x$. We have

\[
D(x_0, x) \leq \alpha[D(x_0, x_n) + D(x_n, x)] \leq \alpha[D(x_n, x) + \varepsilon].
\]

Since $(x_n)$ converges to $x$, $D(x_n, x) \to 0$. Hence $D(x_0, x) \leq \varepsilon$, and $x \in C_D(x_0, \varepsilon)$. Therefore, $C_D(x_0, \varepsilon)$ is complete.
For every \( x \in C_D(x_0, \varepsilon) \),
\[
D(x_0, Tx) \leq \alpha\left[D(x_0, Tx_0) + D(Tx_0, Tx)\right] \\
\leq \alpha\left[(1 - k)\varepsilon + kD(x_0, x)\right] \\
\leq \alpha\left[(1 - k)\varepsilon + k\varepsilon\right] = \alpha\varepsilon.
\]
Hence \( Tx \in C_D(x_0, \varepsilon) \).

**Corollary 4.3** Let \( (X, D, \alpha) \) a complete metric type space. Suppose the mapping \( T : X \to X \) satisfies for some integer \( n \),
\[
D(T^n x, T^n y) \leq kD(x, y) \quad \text{for all } x, y \in X,
\]
where \( k \in [0, 1) \) is a constant. Then \( T \) has a fixed point.

**Proof.** From Theorem 4.1, \( T^n \) has a unique fixed point \( x^* \). But \( T^n(Tx^*) = T(T^n x^*) = Tx^* \), so \( Tx^* \) is also a fixed point of \( T^n \). Hence \( Tx^* = x^* \), \( x^* \) is a fixed point of \( T \). Since the fixed point of \( T^n \) is also fixed point of \( T \), the fixed point of \( T \) is unique.

## 5 More Results on Fixed Point Theorems

We begin with the following lemmas.

**Lemma 5.1** Let \( (y_n) \) be a sequence in a metric type space \( (X, D, \alpha) \) such that
\[
D(y_n, y_{n+1}) \leq \lambda D(y_{n-1}, y_n)
\]
for some \( \lambda > 0 \) with \( \lambda < \frac{1}{\alpha} \). Then \( (y_n) \) is Cauchy.

**Proof.** Let \( m < n \in \mathbb{N} \). From the condition \( (D2) \) in the definition of a metric type, we can write:
\[
D(y_m, y_n) \leq \alpha(D(y_m, y_{m+1}) + D(y_{m+1}, y_n)) \\
\leq \alpha D(y_m, y_{m+1}) + \alpha^2(D(y_{m+1}, y_{m+2}) + D(y_{m+2}, y_n)) \\
\vdots \\
\leq \alpha D(y_m, y_{m+1}) + \alpha^2D(y_{m+1}, y_{m+2}) + \ldots + \alpha^{n-m-1}D(y_{n-2}, y_{n-1}) + \alpha^{n-m}D(y_{n-1}, y_n).
\]
From (1) and \( \lambda < \frac{1}{\alpha} \), the above becomes
\[
D(y_m, y_n) \leq (\alpha\lambda^m + \alpha^2\lambda^{m+1} + \ldots + \alpha^{n-m-1}\lambda^{n-1})D(y_0, y_1) \\
\leq \alpha\lambda^m(1 + \alpha\lambda + \ldots + (\alpha\lambda)^{n-1})D(y_0, y_1) \\
\leq \frac{\alpha\lambda^m}{1 - \alpha\lambda}D(y_0, y_1) \to 0 \text{ as } m \to \infty.
\]
It follows that \( \{y_n\} \) is Cauchy.
Theorem 5.2 Let \((X, D, \alpha)\) be a complete metric type space, and let \(f : X \to X\) be a function such that for some \(\lambda > 0\) with \(\frac{1}{\lambda} < \frac{1}{\alpha}\),
\[
D(fx, fy) \leq \lambda(D(x, fx) + D(y, fy)) \quad \text{for all } x, y, z \in X.
\]
(2)
Then \(f\) has a unique fixed point \(z\) and for every \(x_0 \in X\), the sequence \(\{f^n(x_0)\}\) converges to \(z\).

Proof. Take an arbitrary \(x_0 \in X\) and denote \(y_n = f^n(x_0)\). Then
\[
D(y_n, y_{n+1}) = D(fy_{n-1}, fy_n) \leq \lambda(D(y_{n-1}, fy_{n-1}) + D(y_n, fy_n)) \leq \lambda(D(y_{n-1}, y_n) + D(y_n, y_{n+1})),
\]
which implies that
\[
D(y_n, y_{n+1}) \leq \frac{\lambda}{1-\lambda}D(y_{n-1}, y_n).
\]
Hence, since \(\frac{\lambda}{1-\lambda} < \frac{1}{\alpha}\), by Lemma 5.1 we have that \((y_n)\) is a Cauchy and since \((X, D, \alpha)\) be is complete, there exists \(y^*\) such that \(y_n \to y^*\). Since
\[
D(y^*, fy^*) \leq \alpha[D(fy_n, fy^*) + D(y^*, fy_n)] \leq \alpha k[D(fy_n, y_n) + D(y^*, fy^*)] + \alpha D(y_{n+1}, y^*),
\]
i.e.
\[
D(y^*, fy^*) \leq \frac{\alpha}{1-\alpha k}[kD(fy_n, y_n) + D(y_{n+1}, y^*)] \to 0.
\]
we have \(y^* = fy^*\).

For uniqueness, assume by contradiction that there exists another fixed point \(z^*\). Then
\[
D(y^*, z^*) = D(fy^*, fz^*) \leq \lambda(D(y^*, fy^*) + D(z^*, fz^*)) = 0
\]
Hence \(D(y^*, z^*) = 0\) and we conclude that \(y^* = z^*\).

Theorem 5.3 Let \((X, D, \alpha)\) be a complete metric type space, and let \(f : X \to X\) be a function such that for some \(0 \leq \lambda < \frac{1}{2}\) with \(\frac{\lambda \alpha^2}{1-\lambda \alpha} < 1\),
\[
D(fx, fy) \leq \lambda(D(fx, y) + D(x, fy)) \quad \text{for all } x, y \in X.
\]
(3)
Then \(f\) has a unique fixed point \(z\) and for every \(x_0 \in X\), the sequence \(\{f^n(x_0)\}\) \(D\)-converges to \(z\).
**Proof.** Take an arbitrary \( x_0 \in X \) and denote \( y_n = f^n(x_0) \). Then

\[
D(y_n, y_{n+1}) = D(fy_{n-1}, fy_n) \leq \lambda(D(fy_{n-1}, y_n) + D(y_{n-1}, fy_n)) \\
\leq \lambda D(y_{n-1}, y_n) + D(y_{n-1}, fy_n) \\
\leq \lambda \alpha(D(y_{n-1}, y_n) + D(y_n, y_{n+1})),
\]

which implies that

\[
D(y_n, y_{n+1}) \leq \frac{\lambda \alpha}{1 - \lambda \alpha} D(y_{n-1}, y_n).
\]

Hence, since \( \frac{\lambda \alpha}{1 - \lambda \alpha} < \frac{1}{\alpha} \), by Lemma 5.1 we have that \{\( y_n \)\} is Cauchy and since \((X, D, \alpha)\) is complete, there exists \( y^* \) such that \( y_n \to y^* \). Since

\[
D(y^*, fy^*) \leq \alpha[D(fy_n, fy^*) + D(y^*, fy_n)] \\
\leq \alpha k[D(fy^*, y_n) + D(fy_n, y^*)] + \alpha D(y_{n+1}, y^*),
\]

\[
\leq \alpha k[\alpha D(fy^*, y^*) + D(y_n, y^*)] + \alpha D(y_{n+1}, y^*) + \alpha D(y_{n+1}, y^*),
\]

i.e.

\[
D(y^*, fy^*) \leq \frac{\alpha}{1 - \alpha^2 k}[\alpha^2 k D(y_n, y^*) + \alpha k D(y_{n+1}, y^*) + \alpha D(y_{n+1}, y^*)] \to 0.
\]

we have \( y^* = fy^* \).

For uniqueness, assume by contradiction that there exists another fixed point \( z^* \). Then

\[
D(y^*, z^*) = D(fy^*, fz^*) \leq \lambda(D(fy^*, z^*) + D(y^*, fz^*)) = 2\lambda D(y^*, z^*)
\]

Hence \( D(y^*, z^*) = 0 \) and we conclude that \( y^* = z^* \).

We can then without any doubt write down the following two results.

**Theorem 5.4** Let \((X, D, \alpha)\) be a metric type space, and let \( f : X \to X \) be a function such that for some \( \lambda > 0 \) with \( \lambda < \frac{1}{\alpha} \) and any \( \gamma > 0 \),

\[
D(fx, fy) \leq \lambda D(x, y) + \gamma D(fx, y) \text{ for all } x, y \in X.
\]

Then \( f \) has a unique fixed point \( z \) and for every \( x_0 \in X \), the sequence \{\( f^n(x_0) \)\} converges to \( z \).

**Corollary 5.5** Let \((X, D, \alpha)\) be a complete metric type space, and let \( f : X \to X \) be a function such that for some \( \lambda_1, \lambda_3, \lambda_4, \lambda_5 > 0 \) with \( \frac{\lambda_1 + \lambda_3 + \alpha \lambda_5}{1 - \lambda_2 - \alpha \lambda_5} < \frac{1}{\alpha} \) and any \( \lambda_2 > 0 \)

\[
D(fx, fy) \leq \lambda_1 D(x, y) + \lambda_2 D(fx, y) + \lambda_3 D(x, fx) + \lambda_4 D(y, fy) + \lambda_5 D(x, fy),
\]

for all \( x, y \in X \). Then \( f \) has a unique fixed point \( z \) and for every \( x_0 \in X \), the sequence \{\( f^n(x_0) \)\} converges to \( z \).
6 Extensions of Banach’s Principle

The following theorems are just extensions of Banach’s fixed point theorem in the context of a metric type space. We comment briefly on the fact that continuity and sequential continuity in metric type spaces are expressed exactly in the same way as in the metric case, using the $\varepsilon, \delta$ argument.

**Theorem 6.1** Let $(X, D, \alpha)$ be a compact metric type space, and let $T : X \to X$ be a function such that

$$D(Tx, Ty) < D(x, y) \text{ for all } x, y \in X, x \neq y. \quad (5)$$

Then $T$ has a unique fixed point $z$ and for every $x_0 \in X$, the sequence $\{T^n(x_0)\}$ converges to $z$.

**Proof.** To prove the existence of a fixed point, let’s introduce the mapping $\psi : X \to [0, \infty)$ by setting

$$\psi(x) = D(x, Tx), \quad x \in X.$$  

Then $\psi$ is continuous and bounded from below. So $\psi$ assumes its minimum value at some point $x_0 \in X$.

Since $x_0 \neq Tx_0$ implies

$$\psi(Tx_0) = D(Tx_0, T^2x_0) < d(x_0, Tx_0) = \psi(x_0),$$

it must be the case that $x_0 = Tx_0$.

Now, let $x \in X$ and consider the sequence $\{D(T^n x, x_0)\}$. If $T^n x \neq x_0$,

$$D(T^{n+1}x, x_0) = D(T^{n+1}x, Tx_0) < D(T^n x, x_0),$$

then $\{D(T^n x, x_0)\}$ is strictly decreasing. Consequently the limit

$$r = \lim_{n \to \infty} D(T^n x, x_0)$$

exists and $r \geq 0$. Also since $X$ is compact, the sequence $\{T^n x\}$ has a convergent subsequence $(T^{n_k} x)$, say $\lim_{k \to \infty} T^{n_k} x = z \in X$. Hence we have

$$r = D(z, x_0) = \lim_{k \to \infty} D(T^{n_k} x, x_0) = \lim_{k \to \infty} D(T^{n_k+1} x, x_0) = D(Tz, x_0)$$

But if $z \neq x_0$ then $D(Tz, x_0) = D(Tz, Tx_0) < D(z, x_0)$. This proves that $r = 0$ and that $\lim_{n \to \infty} T^n x = x_0$.

Let $\mathcal{S}$ denote the class of those functions $\gamma : [0, \infty) \to [0, 1)$ which satisfies the condition $\gamma(t_n) \to \eta \leq 1 \implies t_n \to 0$. 


Theorem 6.2 Let \((X, D, \alpha)\) be a complete metric type space with \(\alpha \geq 1\), let \(T : X \to X\), and suppose there exists \(\gamma \in S\) such that for each \(x, y \in X\),

\[
D(Tx, Ty) \leq \gamma(D(x, y))D(x, y).
\]

Then \(T\) has a unique fixed point \(z\) and for every \(x_0 \in X\), the sequence \(\{T^n(x_0)\}\) converges to \(z\).

Proof. Fix \(x \in X\) and let \(x_n = T^n(x), n = 1, 2, \ldots\). We break the proof into two steps.

Step 1 We show that \(\lim_{n \to \infty} D(x_n, x_{n+1}) = 0\).

From (6), we can infer that the sequence \(\{D(x_n, x_{n+1})\}\) is monotone decreasing and bounded from below, so \(\lim_{n \to \infty} D(x_n, x_{n+1}) = r \geq 0\). Assume \(r \neq 0\).

Then (6) implies

\[
\frac{D(x_{n+1}, x_{n+2})}{D(x_n, x_{n+1})} \leq \gamma(D(x_n, x_{n+1})), \quad n = 1, 2, \ldots.
\]

Letting \(n \to \infty\) we see that \(1 \leq \lim_{n \to \infty} \gamma(D(x_n, x_{n+1}))\), and since \(\gamma \in S\) this implies \(r = 0\). This contradiction establishes Step 1.

Step 2 We show that \(\{x_n\}\) is a Cauchy sequence.

Assume \(\limsup_{m, n \to \infty} D(x_n, x_m) > 0\). By property \((D3)\)

\[
D(x_n, x_m) \leq \alpha[D(x_n, x_{n+1}) + D(x_{n+1}, x_{m+1}) + D(x_{m+1}, x_m)],
\]

so by (6)

\[
D(x_n, x_m) \leq (1 - \alpha \gamma(D(x_n, x_m)))^{-1} \alpha[D(x_n, x_{n+1}) + D(x_{m+1}, x_m)]
\]

Under the assumption \(\limsup_{m, n \to \infty} D(x_n, x_m) > 0\) Step 1 now implies

\[
\limsup_{m, n \to \infty} (1 - \alpha \gamma(D(x_n, x_m)))^{-1} = \infty,
\]

from which

\[
\limsup_{m, n \to \infty} \gamma(D(x_n, x_m)) = \frac{1}{\alpha} \leq 1.
\]

Since \(\gamma \in S\) this implies \(\limsup_{m, n \to \infty} D(x_n, x_m) = 0\), which is again a contradiction.

Conclusion: Let \(x \in X\). Since \(X\) is complete and the sequence \(\{T^n x\}\) is a Cauchy sequence, \(\lim_{n \to \infty} T^n x = z \in X\). From (6), we obtain that \(T(z) = z\).

Indeed

\[
D(Tz, Tx_n) = D(Tz, x_{n+1}) \leq \gamma(D(z, x_n))D(z, x_n) \leq D(z, x_n).
\]

Again, from (6) \(z\) is the unique fixed point.
References


Received: May 15, 2014