Symmetric Identities for Carlitz’s $q$-Tangent Polynomials Using $q$-Tangent Zeta Function

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Abstract
Our aim in this paper is to obtain symmetric properties for Carlitz’s $q$-tangent polynomials. We are going to find a symmetric identity for $q$-tangent zeta function. From property of the $q$-tangent zeta function, we derive some symmetric properties of Carlitz’s $q$-tangent polynomials.

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1 Introduction
The area of the Euler, Bernoulli, Genocchi, and Tangent polynomials have been studied by many authors. Those polynomials possess many interesting properties and are of great importance in pure mathematics, for example, number theory, mathematical analysis and in the calculus of finite differences. Those polynomials also have various applications in other branches of science. The tangent polynomials are defined by

$$F(t, x) = \left( \frac{2}{e^{2t} + 1} \right) e^{xt} = \sum_{n=0}^{\infty} T_n(x) \frac{t^n}{n!}, \text{ (see [4])}.$$ 

The tangent numbers, $T_n = T_n(0)$, implies that they are rational numbers. Throughout this paper, we always make use of the following notations: $N = ...$
\{1, 2, 3, \cdots\} denotes the set of natural numbers, \(\mathbb{R}\) denotes the set of real numbers, \(\mathbb{C}\) denotes the set of complex numbers. Let \(q\) be a complex number with \(|q| < 1\). Then we use the notation:

\[
[x]_q = \frac{1 - q^x}{1 - q}, \quad [x]_{-q} = \frac{1 - (-q)^x}{1 + q}.
\]

**Definition 1.1** ([3]) Let \(q\) be a complex number with \(|q| < 1\). The \(q\)-tangent polynomials are defined as

\[
\sum_{n=0}^{\infty} T_{n,q}(x) \frac{t^n}{n!} = 2 \sum_{m=0}^{\infty} (-1)^m q^m e^{[2m + x]_q t}, \quad (1.1)
\]

where we use the technical method’s notation by replacing \((T_q)^n (x)\) by \(T_{n,q}(x)\), symbolically.

In the special case \(x = 0\), \(T_{n,q}(0) = T_n\) are called the \(n\)-th \(q\)-tangent numbers. The following elementary properties of the \(q\)-tangent numbers \(T_{n,q}\) and polynomials \(T_{n,q}(x)\) are readily derived form (1.1) (see, for details, [3]). We, therefore, choose to omit details involved.

**Theorem 1.2** Let \(q \in \mathbb{C}\) with \(|q| < 1\). Then we have

\[
T_{n,q}(x) = 2 \sum_{m=0}^{\infty} (-1)^m q^m [x + 2m]_q^n.
\]

**Theorem 1.3** Let \(q \in \mathbb{C}\) with \(|q| < 1\). Then we have

\[
T_{n,q}(x) = \sum_{l=0}^{n} \left( \begin{array}{c} n \\ l \end{array} \right) [x]_q^{n-l} q^{xl} T_{l,q}
= (q^x T_q + [x]_q)^n.
\]

**Theorem 1.4** Let \(x, y \in \mathbb{C}\). Then we have

\[
T_{n,q}(x + y) = \sum_{l=0}^{n} \left( \begin{array}{c} n \\ l \end{array} \right) [x]_q^{n-l} q^{xl} T_{l,q}(y).
\]

**Theorem 1.5** (Property of complement) Let \(n\) be a positive integer. Then one has

\[
T_{n,q}^{-1}(2 - x) = (-1)^n q^{n+1} T_{n,q}(x)
\]

**Definition 1.6** ([3]) For \(s \in \mathbb{C}\) and \(\text{Re}(s) > 0\), the Hurwitz-type \(q\)-tangent zeta function are defined by

\[
\zeta_q(s, x) = 2 \sum_{n=0}^{\infty} \frac{(-1)^n q^n}{[2n + x]_q^s}.
\]
Note that $\zeta_q(s, x)$ is a meromorphic function on $\mathbb{C}$. Relation between $\zeta_q(s, x)$ and $T_{k,q}(x)$ is given by the following theorem.

**Theorem 1.7** For $k \in \mathbb{N}$, we have

$$
\zeta_q(-k, x) = T_{k,q}(x).
$$

Observe that $\zeta_q(-k, x)$ function interpolates $T_{k,q}(x)$ numbers at non-negative integers.

## 2 The alternating sums of powers of consecutive even integers

By using the similar method of [2], expect for obvious modifications, we are going to obtain the main results of Carlitz’s $q$-tangent polynomials. We also establish some interesting symmetric identities for Carlitz’s $q$-tangent polynomials by using $q$-tangent zeta function. Let $w_1$, $w_2$ be any positive odd integers. Our main result of symmetry of Carlitz’s $q$-tangent zeta function is given the following theorem, which is symmetric in $w_1$ and $w_2$.

**Theorem 2.1** Let $s \in \mathbb{C}$ with $\text{Re}(s) > 0$. Then one has

$$
[w_1]^s_q \sum_{i=0}^{w_2-1} (-1)^i q^{\frac{w_1 i}{w_2}} \zeta_{q^{w_2}} (s, w_1 x + \frac{2w_1 i}{w_2}) = [w_2]^s_q \sum_{j=0}^{w_1-1} (-1)^j q^{\frac{w_2 j}{w_1}} \zeta_{q^{w_1}} (s, w_2 x + \frac{2w_2 j}{w_1}).
$$

**Proof.** Observe that $[xy]_q = [x]_q[y]_q$ for any $x, y \in \mathbb{C}$. By substitute $w_1 x + \frac{2w_1 i}{w_2}$ for $x$ in Definition 1.5, replace $q$ by $q^{w_2}$, we derive

$$
\zeta_{q^{w_2}} (s, w_1 x + \frac{2w_1 i}{w_2}) = 2 \sum_{n=0}^{\infty} \frac{(-1)^n q^{w_2 n}}{[w_1 x + \frac{2w_1 i}{w_2}]_q^n},
$$

Since for any non-negative integer $m$ and odd positive integer $w_1$, there exist unique non-negative integer $r$ such that $m = w_1 r + j$ with $0 \leq j \leq w_1 - 1$. Hence, this can be written as

$$
\zeta_{q^{w_2}} (s, w_1 x + \frac{2w_1 i}{w_2}) = 2[w_2]^s_q \sum_{r=0}^{w_1-1} \sum_{j=0}^{w_1 r+j=0} \frac{(-1)^{w_1 r+j} q^{w_2 (w_1 r+j)}}{[w_2 (w_1 r+j) + w_1 w_2 x + 2w_1 i]_q^n}\sum_{r=0}^{w_1-1} \sum_{j=0}^{w_1 r+j=0} \frac{(-1)^{w_1 r+j} q^{w_2 (w_1 r+j)}}{[w_1 w_2 (2r + x) + 2w_1 i + 2w_2 j]_q^n}.
$$
It follows from the above equation that
\[
[w_1]_q^s \sum_{i=0}^{w_2-1} (-1)^i q^{w_1i} \zeta_{q^{w_2}} \left( s, w_1 x + \frac{2w_1i}{w_2} \right) \nonumber \\
= 2[w_1]_q^s[w_2]_q^s \sum_{i=0}^{w_1-1} \sum_{j=0}^{w_1-1} \sum_{r=0}^{\infty} \frac{(-1)^{r+i+j} q^{(w_1 w_2 r + w_1 i + w_2 j)}}{[w_1 w_2 (2r + x) + 2w_1 i + 2w_2 j]_q^s}. \tag{2.1}
\]

From the similar method, we can have that
\[
\zeta_{q^{w_1}} \left( s, w_2 x + \frac{2w_2j}{w_1} \right) = 2[w_1]_q^s \sum_{n=0}^{\infty} \frac{(-1)^n q^{w_1 n}}{[w_1 w_2 x + 2w_2 j + 2w_1 n]_q^s}.
\]

After some elementary calculations in the above, we have
\[
[w_2]_q^s \sum_{j=0}^{w_1-1} (-1)^j q^{w_2j} \zeta_{q^{w_1}} \left( s, w_2 x + \frac{2w_2j}{w_1} \right) \nonumber \\
= 2[w_1]_q^s[w_2]_q^s \sum_{i=0}^{w_1-1} \sum_{j=0}^{w_1-1} \sum_{r=0}^{\infty} \frac{(-1)^{r+i+j} q^{(w_1 w_2 r + w_1 i + w_2 j)}}{[w_1 w_2 (2r + x) + 2w_1 i + 2w_2 j]_q^s}. \tag{2.2}
\]

Thus, we complete the proof of the theorem from (2.1) and (2.2).

In Theorem 2.1, we get the following formulas for the \(q\)-tangent zeta function.

**Corollary 2.2** Let \(w_2 = 1\) in Theorem 2.1. Then we get
\[
\zeta_q(s, x) = [w_1]_q^{-s} \sum_{j=0}^{w_1-1} (-1)^j q^j \zeta_{q^{w_1}} \left( s, \frac{x + 2j}{w_1} \right).
\]

**Corollary 2.3** Let \(w_1 = 2, w_2 = 1\) in Theorem 2.1. Then we have
\[
\zeta_{q^2} \left( s, \frac{x}{2} \right) - q \zeta_{q^2} \left( s, \frac{x + 1}{2} \right) = [2]_q^s \zeta_q(s, x).
\]

**Theorem 2.4** Let \(w_1, w_2\) be odd positive integer. Then for non-negative integers \(n\), one has
\[
[w_2]_q^n \sum_{i=0}^{w_2-1} (-1)^i q^{w_1i} T_{n,q^{w_2}} \left( w_1 x + \frac{2w_1i}{w_2} \right) = [w_1]_q^n \sum_{i=0}^{w_1-1} (-1)^i q^{w_2i} T_{n,q^{w_1}} \left( w_2 x + \frac{2w_2i}{w_1} \right).
\]
Proof. By substituting $T_{n,q}(x)$ for $\zeta_q(s,x)$ in Theorem 2.1 and Theorem 1.6, we can derive that

$$\left[w_1\right]_q^{-n} \sum_{i=0}^{w_2-1} (-1)^i q^{w_1i} \zeta_q^{w_2} \left(-n, w_1 x + \frac{2w_1i}{w_2}\right) = \left[w_1\right]_q^{-n} \sum_{i=0}^{w_2-1} (-1)^i q^{w_1i} T_{n,q}^{w_2} \left(w_1 x + \frac{2w_1i}{w_2}\right),$$

$$\left[w_2\right]_q^{-n} \sum_{j=0}^{w_1-1} (-1)^i q^{w_2j} \zeta_q^{w_1} \left(-n, w_2 x + \frac{2w_2j}{w_1}\right) = \left[w_2\right]_q^{-n} \sum_{j=0}^{w_1-1} (-1)^i q^{w_2j} T_{n,q}^{w_1} \left(w_2 x + \frac{2w_2j}{w_1}\right).$$

Thus, we can complete the proof of the theorem from Theorem 2.1.

We obtain another result by applying the addition theorem for the Carlitz’s $q$-tangent polynomials.

**Theorem 2.5** Let $w_1, w_2$ be any odd positive integer. Then we have

$$\sum_{l=0}^{n} \left(\begin{array}{l} n \\ l \end{array}\right) \left[w_1\right]_q^l \left[w_2\right]_q^n T_{n-l,q}^{w_2} (w_1 x) \mathcal{T}_{n,l,q}^{w_1} (w_2)$$

$$= \sum_{l=0}^{n} \left(\begin{array}{l} n \\ l \end{array}\right) \left[w_2\right]_q^l \left[w_1\right]_q^n T_{n-l,q}^{w_1} (w_2 x) \mathcal{T}_{n,l,q}^{w_2} (w_1),$$

where $\mathcal{T}_{n,l,q}(k) = \sum_{i=0}^{k-1} (-1)^i q^{(1+2n-2l)i} [2i]_q^l$ is called as the sums of powers.

**Proof.** From the Theorem 1.3, we have

$$\sum_{i=0}^{w_2-1} (-1)^i q^{w_1i} T_{n,q}^{w_2} \left(w_1 x + \frac{2w_1i}{w_2}\right)$$

$$= \sum_{l=0}^{n} \left(\begin{array}{l} n \\ l \end{array}\right) \left[w_1\right]_q^l T_{n-l,q}^{w_2} (w_1 x) \sum_{i=0}^{w_2-1} (-1)^i q^{w_1i} q^{2(n-l)w_1i} [2i]_q^l.$$ 

Therefore, we obtain that

$$\left[w_2\right]_q^n \sum_{i=0}^{w_2-1} (-1)^i q^{w_1i} T_{n,q}^{w_2} \left(w_1 x + \frac{2w_1i}{w_2}\right)$$

$$= \sum_{l=0}^{n} \left(\begin{array}{l} n \\ l \end{array}\right) \left[w_1\right]_q^l \left[w_2\right]_q^n T_{n-l,q}^{w_2} (w_1 x) \mathcal{T}_{n,l,q}^{w_1} (w_2), \quad (2.3)$$

and

$$\left[w_1\right]_q^n \sum_{i=0}^{w_1-1} (-1)^i q^{w_2i} T_{n,q}^{w_1} \left(w_2 x + \frac{2w_2i}{w_1}\right)$$

$$= \sum_{l=0}^{n} \left(\begin{array}{l} n \\ l \end{array}\right) \left[w_2\right]_q^l \left[w_1\right]_q^n T_{n-l,q}^{w_1} (w_2 x) \mathcal{T}_{n,l,q}^{w_2} (w_1). \quad (2.4)$$

Hence, we complete the proof of the theorem by comparing (2.3) and (2.4).
3 Some symmetric identities about Carlitz’s \( q \)-tangent polynomials

In this section, we derive the symmetric results by using definition and theorem of Carlitz’s \( q \)-tangent polynomials.

**Theorem 3.1** Let \( n, m \) be any non-negative integers. Then we obtain that

\[
\sum_{k=0}^{n} \binom{n}{k} T_{m+k,q}(x + y) q^{(n-k)x} [-x]_q^{n-k} = \sum_{k=0}^{m} \binom{m}{k} T_{k+n,q}(y) q^{(n+k)x} [x]_q^{m-k}.
\]

**Proof.** Observe that

\[
[x]_q u + q^x [y + 2m]_q (u + v) = [x + y + 2m]_q (u + v) - [x]_q v. \quad (3.1)
\]

From Definition 1.1, we easily see that

\[
2 \sum_{m=0}^{\infty} (-1)^m q^m e^{[x+y+2m]_q (u+v)} = 2e^{[x]_q (u+v)} \sum_{m=0}^{\infty} (-1)^m q^m e^{q^x [y+2m]_q (u+v)}.
\]

Since \([x + y]_q = [x]_q + q^x [y]_q\), we can find out

\[
e^{-[x]_q u} 2 \sum_{m=0}^{\infty} (-1)^m q^m e^{[x+y+2m]_q (u+v)} = e^{[x]_q u} 2 \sum_{m=0}^{\infty} (-1)^m q^m e^{q^x [y+2m]_q (u+v)}. \quad (3.2)
\]

Note that

\[
2 \sum_{m=0}^{\infty} (-1)^m q^m e^{[x+y+2m]_q (u+v)} = \sum_{n=0}^{\infty} T_{n,q}(x + y) \frac{(u + v)^n}{n!} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} T_{n+m,q}(x + y) \frac{u^m v^n}{m! n!}.
\]

From the above equation, the left-hand side of (3.2) can be expressed as

\[
e^{-[x]_q u} 2 \sum_{m=0}^{\infty} (-1)^m q^m e^{[x+y+2m]_q (u+v)} = \left( \sum_{n=0}^{\infty} (-[x]_q)^n \frac{u^n}{n!} \right) \left( \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} T_{n+m,q}(x + y) \frac{u^m v^n}{m! n!} \right)
\]

\[
= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} T_{k+m,q}(x + y) (-[x]_q)^{n-k} \frac{u^m v^n}{m! n!} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} q^{(n-k)x} T_{k+m,q}(x + y) [-x]_q^{n-k} \frac{u^m v^n}{m! n!}. \quad (3.3)
\]
Observe that
\[
2 \sum_{m=0}^{\infty} (-1)^m q^m e^{q^x[y+2m]_q(u+v)} = \sum_{n=0}^{\infty} q^{nx} T_{n,q}(y) \frac{(u+v)^n}{n!} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} q^{(n+m)x} T_{n+m,q}(y) \frac{v^n u^m}{n! m!}.
\]

From the above equation, the right-hand side of (3.2) can be expressed as follows:
\[
e^{[x]_q u} \sum_{m=0}^{\infty} (-1)^m q^m e^{q^x[y+2m]_q(u+v)} = \left( \sum_{m=0}^{\infty} \frac{[x]_q^m}{m!} \right) \left( \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} q^{(n+m)x} T_{n+m,q}(y) \frac{v^n u^m}{n! m!} \right) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{m} \binom{m}{k} q^{(k+n)x} T_{n+k,q}(y) \frac{v^n u^m}{n! m!}.
\]  

By comparing the coefficients of \( \frac{v^n u^m}{n! m!} \) in (3.3) and (3.4), we assert that the theorem is right.

It is obvious that Theorem 1.3 is a special case of Theorem 3.1 by setting \( n = 0 \) and replacing \( m \) by \( n \). As another special case, in view of (2.11), we obtain that for any non-negative integers \( m, n \),
\[
(-1)^m \sum_{k=0}^{n} \binom{m}{k} q^{n+k} T_{n+k,q}(x) = (-1)^n \sum_{k=0}^{m} \binom{n}{k} q^{-m-k} T_{m+k,q-1}(1-x).
\]

**Theorem 3.2** Let \( k, n, m \) be non-negative integers. Then we have
\[
(-1)^m \sum_{k=0}^{m} \binom{m+1}{k} (k+n+1) q^{k+n} T_{k+n,q}(x) + (-1)^n \sum_{k=0}^{n+1} \binom{n+1}{k} (k+m+1) q^{-(k+m)} T_{k+m,q-1}(1-x) = 0.
\]

**References**


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