On Carlitz’s Type Twisted Tangent Numbers and Polynomials Associated with $p$-Adic Integral on $\mathbb{Z}_p$

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Abstract
In this paper we construct the Carlitz’s type twisted tangent numbers $T_{n,q,w}$ and polynomials $T_{n,q,w}(x)$. From these numbers and polynomials, we establish some interesting identities and relations.

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1 Introduction

Recently, many mathematicians have studied in the area of the $q$-analogues of the Bernoulli numbers, Euler numbers, Genocchi numbers, Tangent numbers (see [1-5]). Our aim in this paper is to define Carlitz’s type twisted tangent numbers and polynomials. We investigate some properties which are related to twisted tangent numbers and polynomials. We also derive the existence of a specific interpolation function which interpolate twisted tangent numbers and polynomials at negative integers.

Throughout this paper, we always make use of the following notations: $\mathbb{N} = \{1, 2, 3, \cdots \}$ denotes the set of natural numbers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{C}$ denotes the set of complex numbers, $\mathbb{Z}_p$ denotes the ring of
$p$-adic rational integers, $\mathbb{Q}_p$ denotes the field of $p$-adic rational numbers, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let $\nu_p$ be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of $q$-extension, $q$ is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or $p$-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q - 1|_p < p^{-1}$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. Throughout this paper we use the notation:

$$[x]_q = \frac{1 - q^x}{1 - q}.$$

First, we introduce tangent numbers $T_n$ and polynomials $T_n(x)$ (see [3]). Tangent numbers $T_n$ are defined by the generating function:

$$F(t) = \frac{2}{e^{2t} + 1} = \sum_{n=0}^{\infty} \frac{T_n}{n!} t^n. \quad (1.1)$$

We introduce tangent polynomials $T_n(x)$ as follows:

$$F(t, x) = \left(\frac{2}{e^{2t} + 1}\right) e^{xt} = \sum_{n=0}^{\infty} T_n(x) \frac{t^n}{n!}. \quad (1.2)$$

For $g \in UD(\mathbb{Z}_p) = \{g | g : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\}$, the fermionic $p$-adic invariant integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_{-1}(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x) = \lim_{N \to \infty} \sum_{0 \leq x < p^N} g(x) (-1)^x, \quad \text{(see[1])}. \quad (1.3)$$

If we take $g_1(x) = g(x + 1)$ in (1.3), then we see that

$$I_{-1}(g_1) + I_{-1}(g) = 2g(0), \quad \text{(see [1-4])}. \quad (1.4)$$

From (1.3), we obtain

$$\int_{\mathbb{Z}_p} g(x + n) d\mu_{-1}(x) = (-1)^n \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x) + 2 \sum_{l=0}^{n-1} (-1)^{n-1-l} g(l). \quad (1.5)$$

2 Carlitz’s type Twisted $q$-tangent numbers and polynomials

Our primary goal of this section is to define Carlitz’s type twisted tangent numbers $T_{n,q,w}$ and polynomials $T_{n,q,w}(x)$. We also find generating functions of twisted tangent numbers $T_{n,q,w}$ and polynomials $T_{n,q,w}(x)$ and investigate
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their properties. Let $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$ and $T_p = \cup_{N \geq 1} C_{p^N} = \lim_{N \to \infty} C_{p^N}$, where $C_{p^N} = \{w | w^{p^N} = 1\}$ is the cyclic group of order $p^N$. For $w \in T_p$, we denote by $\phi_w : \mathbb{Z}_p \to \mathbb{C}_p$ the locally constant function $x \mapsto w^x$.

For $q \in \mathbb{C}_p$ with $|1 - q|_p < 1$ and $w \in T_p$, the Carlitz’s type twisted tangent are defined by

$$T_{n,q,w} = \int_{\mathbb{Z}_p} w^x [2x]^n_q d\mu_{-1}(x). \quad (2.1)$$

By using $p$-adic integral on $\mathbb{Z}_p$, we obtain,

$$\int_{\mathbb{Z}_p} w^x [2x]^n_q d\mu_{-1}(x) = \lim_{N \to \infty} \sum_{x=0}^{p^N-1} w^x [2x]^n_q (-1)^x$$

$$= 2 \left( \frac{1}{1 - q} \right)^n \sum_{l=0}^{n} \binom{n}{l} (1)^l \frac{1}{1 + wq^{2l} \ast} \quad (2.2)$$

$$= 2 \sum_{m=0}^{\infty} (-1)^m w^m [2m]_q^n.$$

By (2.1), we have the following theorem.

**Theorem 2.1** For $w \in T_p$ and $q \in \mathbb{C}_p$ with $|q - 1|_p < 1$, we have

$$T_{n,q,w} = 2 \left( \frac{1}{1 - q} \right)^n \sum_{l=0}^{n} \binom{n}{l} (1)^l \frac{1}{1 + wq^{2l}}$$

$$= 2 \sum_{m=0}^{\infty} (-1)^m w^m [2m]_q^n.$$

We set

$$F_{q,w}(t) = \sum_{n=0}^{\infty} T_{n,q,w} \frac{t^n}{n!}.$$

By using above equation and (2.2), we have

$$\sum_{n=0}^{\infty} T_{n,q,w} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left( 2 \left( \frac{1}{1 - q} \right)^n \sum_{l=0}^{n} \binom{n}{l} (1)^l \frac{1}{1 + wq^{2l}} \right) \frac{t^n}{n!}$$

$$= 2 \sum_{m=0}^{\infty} (-1)^m w^m e^{[2m]_q t}. \quad (2.3)$$

Thus, Carlitz’s type twisted tangent numbers $T_{n,q,w}$ are defined by means of the generating function

$$F_{q,w}(t) = 2 \sum_{m=0}^{\infty} (-1)^m w^m e^{[2m]_q t}. \quad (2.4)$$
By using (2.1), we have
\[ \sum_{n=0}^{\infty} T_{n,q,w} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} w^x [2x]_q^n d\mu_{-1}(x) \frac{t^n}{n!} = \int_{\mathbb{Z}_p} w^x e^{[2x]_q t} d\mu_{-1}(x). \] (2.5)

By (2.3), (2.5), we have
\[ \int_{\mathbb{Z}_p} w^x e^{[2x]_q t} d\mu_{-1}(x) = 2 \sum_{m=0}^{\infty} (-1)^m w^m e^{[2m]_q t}. \]

Next, we introduce Carlitz’s type twisted tangent polynomials $T_{n,q,w}(x)$. The Carlitz’s type twisted tangent polynomials $T_{n,q,w}(x)$ are defined by
\[ T_{n,q,w}(x) = \int_{\mathbb{Z}_p} w^y [2y + x]_q d\mu_{-1}(y). \] (2.6)

By using $p$-adic integral, we obtain
\[ T_{n,q,w}(x) = 2 \left( \frac{1}{1 - q} \right)^n \sum_{l=0}^{n} \left( \frac{n}{l} \right) (-1)^l q^{xl} \frac{1}{1 + wq^{xl}}. \] (2.7)

We set
\[ F_{q,w}(t, x) = \sum_{n=0}^{\infty} T_{n,q,w}(x) \frac{t^n}{n!}. \] (2.8)

By using (2.7) and (2.8), we obtain
\[ F_{q,w}(t, x) = 2 \sum_{m=0}^{\infty} (-1)^m w^m e^{[2m+x]_q t}. \] (2.9)

Since $[x + 2y]_q = [x]_q + q^x[2y]_q$, we easily see that
\[ T_{n,q,w}(x) = \int_{\mathbb{Z}_p} w^y [x + 2y]_q^n d\mu_{-1}(y) \]
\[ = \sum_{l=0}^{n} \left( \frac{n}{l} \right) [x]_q^{n-l} q^{xl} T_{l,q,w} \]
\[ = (q^x T_{q,w} + [x]_q)^n \]
\[ = 2 \sum_{m=0}^{\infty} (-1)^m w^m [x + 2m]_q^n, \] (2.10)

with the usual convention of replacing $(T_{q,w})^n$ by $T_{n,q,w}$. By (1.1), (1.2), (2.3), and (2.10), we have the following remark.
Remark 2.2 Note that

1. \( T_{n,q,w}(0) = T_{n,w} \),
2. If \( w = 1 \) and \( q \to 1 \), then \( T_{n,q,w}(x) = T_n(x) \), \( T_{n,q,w} = T_{n,w} \),
3. If \( w = 1 \) and \( q \to 1 \), then \( F_{q,w}(t, x) = F_w(t, x) \), \( F_{q,w}(t) = F(t) \).

By (2.7), we obtain the following theorem.

Theorem 2.3 (Property of complement).

\[ T_{n,q-1,w}(2 - x) = (-1)^n wq^n T_{n,q,w}(x) \]

By (2.7) and after some elementary calculations, we have the following distribution relation:

Theorem 2.4 For any positive integer \( m (= \text{odd}) \), we have

\[ T_{n,q,w}(x) = [m]^n_q \sum_{a=0}^{m-1} (-1)^a w^n T_{n,q,m,w} \left( \frac{2a + x}{m} \right), \quad n \in \mathbb{Z}_+ \]

By (1.5), (2.1), and (2.6), we easily see that

\[ (-1)^{n+1} w^n T_{m,q,w}(2n) + T_{m,q,w} = 2 \sum_{l=0}^{n-1} (-1)^l w^l [2]^m_q. \]

Hence, we obtain the following theorem.

Theorem 2.5 Let \( m \in \mathbb{Z}_+ \). If \( n \equiv 0 \pmod{2} \), then

\[ w^n T_{m,q,w}(2n) - T_{m,q,w} = 2 \sum_{l=0}^{n-1} (-1)^{l+1} w^l [2]^m_q. \]

If \( n \equiv 1 \pmod{2} \), then

\[ w^n T_{m,q,w}(2n) + T_{m,q,w} = 2 \sum_{l=0}^{n-1} (-1)^l w^l [2]^m_q. \]

From (1.4), we note that

\[ 2 = w \int_{\mathbb{Z}_p} w^x e^{2x+2[2]} d\mu_{-1}(x) + \int_{\mathbb{Z}_p} w^x e^{2x} d\mu_{-1}(x) \]

\[ = \sum_{n=0}^{\infty} \left( w \int_{\mathbb{Z}_p} w^x [2x + 2]^n_q d\mu_{-1}(x) + \int_{\mathbb{Z}_p} w^x [2x]^n_q d\mu_{-1}(x) \right) \frac{t^n}{n!} \]

\[ = \sum_{n=0}^{\infty} (wT_{n,q,w}(2) + T_{n,q,w}) \frac{t^n}{n!}. \]

Therefore, we obtain the following theorem.
**Theorem 2.6** For \( n \in \mathbb{Z}_+ \), we have

\[
w T_{n,q,w}(2) + T_{n,q,w} = \begin{cases} 2, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0. \end{cases}
\]

By Theorem 2.6 and (2.10), we have the following corollary.

**Corollary 2.7** For \( n \in \mathbb{Z}_+ \), we have

\[
w \left( q^2 T_{q,w} + [2]_q \right)^n + T_{n,q,w} = \begin{cases} 2, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0, \end{cases}
\]

with the usual convention of replacing \((T_{q,w})^n\) by \(T_{n,q,w}\).

### 3 Carlitz’s type Hurwitz twisted tangent zeta functions

By using Carlitz’s type twisted tangent numbers and polynomials, Carlitz’s type twisted tangent zeta function and Hurwitz tangent zeta functions are defined. These functions interpolate the Carlitz’s type twisted tangent numbers \(T_{n,q,w}\), and polynomials \(T_{n,q,w}(x)\), respectively. Let \( q \) be a complex number with \(|q| < 1\) and \( w \) be the \( p^N \)-th root of unity. From (2.4), we note that

\[
\frac{d^k}{dt^k} F_{q,w}(t) \bigg|_{t=0} = 2 \sum_{m=0}^{\infty} (-1)^n w^n [2m]_q^k = T_{k,q,w}, \quad (k \in \mathbb{N}).
\]

By using the above equation, we are now ready to define Carlitz’s type twisted tangent zeta functions.

**Definition 3.1** Let \( s \in \mathbb{C} \) with \( \text{Re}(s) > 0 \).

\[
\zeta_w(s) = 2 \sum_{n=1}^{\infty} \frac{(-1)^n w^n}{[2n]_q^s}.
\]

(3.1)

Note that \( \zeta_w(s) \) is a meromorphic function on \( \mathbb{C} \). Note that, if \( q \to 1 \) and \( w = 1 \), then \( \zeta_w(s) = \zeta_T(s) \) which is the tangent zeta functions(see [3]). Relation between \( \zeta_w(s) \) and \( T_{k,q,w} \) is given by the following theorem.

**Theorem 3.2** For \( k \in \mathbb{N} \), we have

\[
\zeta_w(-k) = T_{k,q,w}.
\]
Observe that \( \zeta_{q,w}(s) \) function interpolates \( T_{k,q,w} \) numbers at non-negative integers. By using (2.9), we note that
\[
\left. \frac{d^k}{dt^k} F_{q,w}(t, x) \right|_{t=0} = 2 \sum_{m=0}^{\infty} (-1)^m w^m [2m + x]_q^k \tag{3.2}
\]
and
\[
\left. \left( \frac{d}{dt} \right)^k \left( \sum_{n=0}^{\infty} T_{n,q,w}(x) \frac{t^n}{n!} \right) \right|_{t=0} = T_{k,q,w}(x), \text{ for } k \in \mathbb{N}. \tag{3.3}
\]
By (3.2) and (3.3), we are now ready to define the Carlitz’s type Hurwitz twisted tangent zeta functions.

**Definition 3.3** Let \( s \in \mathbb{C} \) with \( \text{Re}(s) > 0 \).

\[
\zeta_w(s, x) = 2 \sum_{n=0}^{\infty} \frac{(-1)^n w^n}{[2n + x]_q^s}. \tag{3.4}
\]

Note that \( \zeta_w(s, x) \) is a meromorphic function on \( \mathbb{C} \). Observe that, if \( q \to 1 \) and \( w = 1 \), then \( \zeta_w(s, x) = \zeta_T(s, x) \) which is the Hurwitz tangent zeta functions (see [3]). Relation between \( \zeta_w(s, x) \) and \( T_{k,q,w}(x) \) is given by the following theorem.

**Theorem 3.4** For \( k \in \mathbb{N} \), we have
\[
\zeta_w(-k, x) = T_{k,q,w}(x).
\]

Observe that \( \zeta_w(-k, x) \) function interpolates \( T_{k,q,w}(x) \) numbers at non-negative integers.

**References**


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