Multiple $q$-Tangent Zeta Functions 
and $q$-Tangent Polynomials

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Abstract
In this paper, we introduce the $q$-Tangent polynomials $T_{n,q}^{(k)}(x)$ of higher order $k$. We also construct multiple $q$-Tangent zeta function which interpolates the $q$-Tangent numbers $T_{n,q}^{(k)}(x)$ of higher order $k$ at negative integers. Some interesting results and relationships are obtained.

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1 Introduction
Throughout this paper, we always make use of the following notations: $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, $\mathbb{C}$ denotes the set of complex numbers, $\mathbb{Z}_p$ denotes the ring of $p$-adic rational integers, $\mathbb{Q}_p$ denotes the field of $p$-adic rational numbers, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let $\nu_p$ be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of $q$-extension, $q$ is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or $p$-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q - 1|_p < p^{-\nu_p(p)}$, so that $= \exp(x \log q)$ for $|x|_p \leq 1$. For

\[ g \in UD(\mathbb{Z}_p) = \{g \mid g : \mathbb{Z}_p \rightarrow \mathbb{C}_p \text{ is uniformly differentiable function}\}, \]
the fermionic \( p \)-adic invariant integral on \( \mathbb{Z}_p \) is defined by Kim as follows:

\[
I_{-1}(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x) = \lim_{N \to \infty} \sum_{0 \leq x < p^N} g(x)(-1)^x, \quad (\text{see}[4]). (1.1)
\]

If we take \( g_1(x) = g(x + 1) \) in (1.1), then we see that

\[
I_{-1}(g_1) + I_{-1}(g) = 2g(0), \quad (\text{see}[4-5]). (1.2)
\]

From (1.1), we obtain

\[
\int_{\mathbb{Z}_p} g(x + n) d\mu_{-1}(x) = (-1)^n \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x) + 2 \sum_{l=0}^{n-1} (-1)^{n-1-l} g(l). (1.3)
\]

Let us define the \( q \)-tangent numbers \( T_{n,q} \) and polynomials \( T_{n,q}(x) \) as follows(see [8]):

\[
I_{-1}(q^y e^{2yt}) = \int_{\mathbb{Z}_p} q^y e^{2yt} d\mu_{-1}(y) = \sum_{n=0}^{\infty} T_{n,q} \frac{t^n}{n!} , \quad (1.4)
\]

\[
I_{-1}(q^y e^{(2y + x)t}) = \int_{\mathbb{Z}_p} q^y e^{(x + 2y)t} d\mu_{-1}(y) = \sum_{n=0}^{\infty} T_{n,q}(x) \frac{t^n}{n!} . \quad (1.5)
\]

By (1.4) and (1.5), we obtain the following Witt’s formula.

**Theorem 1.1** ([8]) For \( n \in \mathbb{Z}_+ \), we have

\[
\int_{\mathbb{Z}_p} q^x (2x)^n d\mu_{-1}(x) = T_{n,q},
\]

\[
\int_{\mathbb{Z}_p} q^y (x + 2y)^n d\mu_{-1}(y) = T_{n,q}(x).
\]

Numerous properties of tangent number are known. Many mathematicians have studied in the area of the analogues of the Bernoulli numbers, Euler numbers, and Genocchi numbers(see [1-10]). Our aim in this paper is to define the \( q \)-Tangent polynomials \( T_{n,q}^{(k)}(x) \) of higher order \( k \). We also derive the existence of a specific interpolation function which interpolate \( q \)-Tangent polynomials \( T_{n,q}^{(k)}(x) \) of higher order \( k \) at negative integers.

## 2 \( q \)-Tangent polynomials of higher order

In this section, we assume that \( q \in \mathbb{C}_p \). We use the notation

\[
\sum_{k_1=0}^m \ldots \sum_{k_n=0}^m = \sum_{k_1,\ldots,k_n=0}^m .
\]
Now, using multiple of \(p\)-adic integral, we introduce the \(q\)-Tangent polynomials of higher order \(T_{n,q}^{(k)}(x)\): For \(k \in \mathbb{N}\), we define
\[
\sum_{n=0}^{\infty} T_{n,q}^{(k)}(x) \frac{t^n}{n!} = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{x_1 + \ldots + x_k + x_1 \cdots x_k t} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k).
\]

By using Taylor series of \(e^{(x+2x_1 + \ldots + 2x_k)t}\) in the above equation, we obtain
\[
\sum_{n=0}^{\infty} \left( \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{x_1 + \ldots + x_k} (x + 2x_1 + \ldots + 2x_k)^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k) \right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} T_{n,q}^{(k)}(x) \frac{t^n}{n!}
\]

By comparing coefficients of \(\frac{t^n}{n!}\) in the above equation, we arrive at the following theorem.

**Theorem 2.1** For positive integers \(n, k\), we have
\[
T_{n,q}^{(k)}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{x_1 + \ldots + x_k} (x + 2x_1 + \ldots + 2x_k)^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k).
\]

By (1.4), the \(q\)-Tangent polynomials of higher order, \(T_{n,q}^{(k)}(x)\) are defined by means of the following generating function
\[
F_{q}^{(k)}(x, t) = \left( \frac{2}{qe^{2t} + 1} \right)^k e^{xt} = \sum_{n=0}^{\infty} T_{n,q}^{(k)}(x) \frac{t^n}{n!}, \quad (2.1)
\]

It follows from (2.1) that
\[
\lim_{q \to 1} F_{q}^{(k)}(x, t) = \left( \frac{2e^t}{e^{2t} + 1} \right)^k e^{xt}.
\]

This gives a generating function of the Tangent polynomials of higher order. Thus we have the following limit relationship:
\[
\lim_{q \to 1} T_{n,q}^{(k)}(x) = T_{n}^{(k)}(x),
\]

which yields the Tangent polynomials of higher order as a limit as \(q\) approaches 1 (see [6]).

By using (2.1), the \(q\)-Tangent numbers of higher order, \(T_{n,q}^{(k)}\) are defined by the following generating function
\[
\left( \frac{2}{qe^{2t} + 1} \right)^k = \sum_{n=0}^{\infty} T_{n,q}^{(k)} \frac{t^n}{n!}, \quad |2t + \log q| < \pi. \quad (2.2)
\]
When $k = 1$, above (2.1) and (2.2) will become the corresponding definitions of the $q$-Tangent polynomials $T_{n,q}(x)$ and the $q$-Tangent numbers $T_{n,q}$ (see [8]). Observe that for $x = 0$, the equation (2.1) reduces to (2.2).

**Corollary 2.2** For positive integers $n, k$, we have

$$T^{(k)}_{n,q} = \int_{\mathbb{Z}_p} \ldots \int_{\mathbb{Z}_p} q^{x_1 + \ldots + x_k}(2x_1 + \ldots + 2x_k)^n d\mu_{-1}(x_1) \ldots d\mu_{-1}(x_k).$$

By using binomial expansion in Theorem 2.1, we obtain

$$T^{(k)}_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} x^{n-l} \int_{\mathbb{Z}_p} \ldots \int_{\mathbb{Z}_p} q^{x_1 + \ldots + x_k}(2x_1 + \ldots + 2x_k)^l d\mu_{-1}(x_1) \ldots d\mu_{-1}(x_k).$$

By Corollary 2.2, we arrive at the following theorem.

**Theorem 2.3** For positive integers $n, k$, we have

$$T^{(k)}_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} x^{n-l} T^{(k)}_{l,q}.$$ 

We define distribution relation of the $q$-Tangent polynomials of higher order as follows: For $m \in \mathbb{N}$ with $m \equiv 1 \pmod{2}$, we obtain

$$\sum_{n=0}^{\infty} T^{(k)}_{n,q}(x) \frac{t^n}{n!} = \left( \frac{2}{qe^{2t} + 1} \right) \left( \frac{2}{qe^{2t} + 1} \right) \ldots \left( \frac{2}{qe^{2t} + 1} \right) e^{xt}$$

$$= \left( \frac{2}{qe^{2mt} + 1} \right)^k \sum_{a_1, \ldots, a_k=0}^{m-1} (-q)^{a_1 + \ldots + a_k} e^{\frac{2a_1 + \ldots + 2a_k + x}{m}} (mt)^n.$$

From the above, we obtain

$$\sum_{n=0}^{\infty} T^{(k)}_{n,q}(x) \frac{t^n}{n!} = \sum_{a_1, \ldots, a_k=0}^{m-1} (-q)^{a_1 + \ldots + a_k} \sum_{n=0}^{\infty} T^{(k)}_{n,q} \left( \frac{2a_1 + \ldots + 2a_k + x}{m} \right) (mt)^n \frac{t^n}{n!}.$$

By comparing coefficients of $\frac{t^n}{n!}$ in the above equation, we arrive at the following theorem.

**Theorem 2.4** (Distribution relation of the $q$-Tangent polynomials of higher order). For $m \in \mathbb{N}$ with $m \equiv 1 \pmod{2}$, we have

$$T^{(k)}_{n,q}(x) = m^n \sum_{a_1, \ldots, a_k=0}^{m-1} (-q)^{a_1 + \ldots + a_k} T^{(k)}_{n,q} \left( \frac{2a_1 + \ldots + 2a_k + x}{m} \right).$$
By (2.1), we have
\[
\sum_{n=0}^{\infty} T_{n,q}^{(k)}(x) \frac{t^n}{n!} = 2^k \sum_{a_1, \ldots, a_k = 0}^{\infty} (-q)^{a_1 + \ldots + a_k} e^{(2a_1 + \ldots + 2a_k + x)t} \sum_{m=0}^{\infty} \left( \frac{m + k - 1}{m} \right) (-1)^m q^m e^{(2m + x)t}.
\] (2.3)

From the above, we obtain
\[
\sum_{n=0}^{\infty} T_{n,q}^{(k)}(x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left( \sum_{a_1, \ldots, a_k = 0}^{\infty} (-q)^{a_1 + \ldots + a_k} (x + 2a_1 + \ldots + 2a_k)^n \right) \frac{t^n}{n!}
\]
\[
= \sum_{n=0}^{\infty} \left( \sum_{m=0}^{\infty} \left( \frac{m + k - 1}{m} \right) (-1)^m q^m (2m + x)^n \right) \frac{t^n}{n!}
\]

By comparing coefficients of \( \frac{t^n}{n!} \) in the above equation, we arrive at the following theorem.

**Theorem 2.5** For positive integers \( n, k \), we have
\[
T_{n,q}^{(k)}(x) = 2^k \sum_{a_1, \ldots, a_k = 0}^{\infty} (-q)^{a_1 + \ldots + a_k} (2a_1 + \ldots + 2a_k + x)^n \sum_{m=0}^{\infty} \left( \frac{m + k - 1}{m} \right) (-1)^m q^m (2m + x)^n.
\] (2.4)

By definition of the \( q \)-Tangent polynomials of higher order, we have the following addition theorem.

**Theorem 2.6** (Addition theorem of the \( q \)-Tangent polynomials of higher order). For \( k \in \mathbb{N} \), we have
\[
T_{n,q}^{(k)}(x + y) = \sum_{l=0}^{n} \binom{n}{l} T_{l,q}^{(k)}(x) y^{n-l}.
\]

3 Multiple \( q \)-Tangent zeta function

In this section, we assume that \( q \in \mathbb{C} \) with \( |q| < 1 \). We define multiple \( q \)-Tangent zeta function. This function interpolates the \( q \)-Tangent numbers of higher order at negative integers. By using (2.1), we have
\[
F_q^{(k)}(x, t) = 2^k \sum_{a_1, \ldots, a_k = 0}^{\infty} (-q)^{a_1 + \ldots + a_k} e^{(2a_1 + \ldots + 2a_k + x)t} = \sum_{n=0}^{\infty} T_{n,q}^{(k)}(x) \frac{t^n}{n!}.
\]
For \( s, x \in \mathbb{C} \) with \( \text{Re}(x) > 0 \), we can derive the following Eq. (3.1) form the Mellin transformation of \( F_q^{(k)}(x, t) \).

\[
\frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} F_q^{(k)}(x, -t) dt = 2^k \sum_{a_1, \ldots, a_k = 0}^{\infty} \frac{(-1)^{a_1 + \cdots + a_k} q^{a_1 + \cdots + a_k}}{(2a_1 + \cdots + 2a_k + x)^s} \tag{3.1}
\]

For \( s, x \in \mathbb{C} \) with \( \text{Re}(x) > 0 \), we define Hurwitz’s type multiple \( q \)-Tangent zeta function as follows:

**Definition 3.1** For \( s, x \in \mathbb{C} \) with \( \text{Re}(x) > 0 \), we define

\[
\zeta_q^{(k)}(s, x) = 2^k \sum_{a_1, \ldots, a_k = 0}^{\infty} \frac{(-1)^{a_1 + \cdots + a_k} q^{a_1 + \cdots + a_k}}{(2a_1 + \cdots + 2a_k + x)^s}, \tag{3.2}
\]

For \( s = -l \) in (3.2) and using (2.4), we arrive at the following theorem.

**Theorem 3.2** For positive integer \( l \), we have

\[
\zeta_q^{(k)}(-l, x) = T_{l,q}^{(k)}(x). \tag{3.3}
\]

By (3.3), we define multiple \( q \)-Tangent zeta function as follows:

**Definition 3.3** For \( s \in \mathbb{C} \), we define

\[
\zeta_q^{(k)}(s) = 2^k \sum_{m=1}^{\infty} \binom{m+k-1}{m} (-1)^m q^m (2m)^s. \tag{3.4}
\]

The function \( \zeta_q^{(k)}(s) \) interpolates the number \( T_{n,q}^{(k)} \) at negative integers. Substituting \( s = -n \) with \( n \in \mathbb{Z}_+ \) into (3.4), and using (3.3), we obtain the following theorem:

**Theorem 3.4** Let \( n \in \mathbb{Z}_+ \), We have

\[
\zeta_q^{(k)}(-n) = E_{n,q}^{(k)}. \tag{3.5}
\]

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References


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