Some Identities on the $(h, q)$-Tangent Polynomials and Bernstein Polynomials

C. S. Ryoo

Department of Mathematics
Hannam University, Daejeon 306-791, Korea

Abstract

In this paper, we give some interesting identities on the $(h, q)$-Tangent polynomials and Bernstein polynomials.

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1 Introduction

In this paper, we investigate some properties for the $(h, q)$-tangent numbers and polynomials. By using these properties, we obtain some interesting identities on the $(h, q)$-tangent polynomials and Bernstein polynomials. Throughout this paper, let $p$ be a fixed odd prime number. The symbol, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ denote the ring of $p$-adic integers, the field of $p$-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. As well known definition, the $p$-adic absolute value is given by $|x|_p = p^{-r}$ where $x = p^r t$ with $(t, p) = (s, p) = (t, s) = 1$. When one talks of $q$-extension, $q$ is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a $p$-adic number $q \in \mathbb{C}_p$. In this paper we assume that $q \in \mathbb{C}_p$ with $|1 - q|_p < 1$. We assume that $UD(\mathbb{Z}_p)$ is the space of the uniformly
differentiable function on \( \mathbb{Z}_p \). For \( g \in UD(\mathbb{Z}_p) \), the fermionic \( p \)-adic invariant integral on \( \mathbb{Z}_p \) is defined by Kim as follows:

\[
I_{-1}(f) = \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x) = \lim_{N \to \infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x, \quad \text{see} \ [1, 2, 3, 4].
\] (1.1)

For \( n \in \mathbb{N} \), let \( g_n(x) = g(x + n) \) be translation. As well known equation, by (1.1), we have

\[
\int_{\mathbb{Z}_p} g(x + n) d\mu_{-1}(x) = (-1)^n \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x) + 2 \sum_{l=0}^{n-1} (-1)^{n-1-l} g(l). \quad (1.2)
\]

For \( h \in \mathbb{Z} \), we defined the \((h, q)\)-tangent polynomials as follows:

\[
\left( \frac{2}{q^h e^{2t} + 1} \right) e^{xt} = \sum_{n=0}^{\infty} T_{n,q}^{(h)}(x) \frac{t^n}{n!}. \quad (1.3)
\]

In the special case, \( x = 0 \), \( T_{n,q}^{(h)}(0) = T_{n,q}^{(h)} \) are called the \( n \)-th \((h, q)\)-tangent numbers (see [5]).

From (1.2) and (1.3), we have the following theorem.

**Theorem 1.1** For \( n \in \mathbb{Z}_+ \), we have

\[
\int_{\mathbb{Z}_p} q^{hy} (2y)^n d\mu_{-1}(y) = T_{n,q}^{(h)}, \quad \int_{\mathbb{Z}_p} q^{hy} (x + 2y)^n d\mu_{-1}(y) = T_{n,q}^{(h)}(x). \quad (1.4)
\]

In [1], Kim introduced \( p \)-adic extension of Bernstein polynomials as follows:

\[
B_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad \text{where} \ x \in \mathbb{Z}_p \ \text{and} \ n, k \in \mathbb{Z}_+. \quad (1.5)
\]

## 2 Some identities on the \((h, q)\)-tangent polynomials

In this section, we have some interesting identities on the Bernstein polynomials and \((h, q)\)-tangent polynomials.

From (1.3), we can derive the following recurrence formula for the \((h, q)\)-Tangent numbers:

\[
T_{0,q}^{(h)} = \frac{2}{1 + q^h}, \quad \text{and} \ q^h (T_q^{(h)} + 2)^n + T_{n,q}^{(h)} = \begin{cases} 2, & \text{if} \ n = 0, \\ 0, & \text{if} \ n > 0, \end{cases} \quad (2.1)
\]

with usual convention about replacing \((T_q^{(h)})^n\) by \( T_{n,q}^{(h)} \).
By (1.3), we easily get
\[
\sum_{n=0}^{\infty} T_{n,q-1}^{(h)}(2-x)(-1)^n \frac{t^n}{n!} = \left( \frac{2q^h}{q^he^{2t} + 1} \right) e^{xt} = q^h \sum_{n=0}^{\infty} T_{n,q}^{(h)}(x) \frac{t^n}{n!}. \tag{2.2}
\]

By (2.2), we obtain the following theorem.

**Theorem 2.1** Let \( n \in \mathbb{Z}_+ \). For \( h \in \mathbb{Z} \), we have
\[
q^h T_{n,q}^{(h)}(x) = (-1)^n T_{n,q-1}^{(h)}(2-x).
\]

By (2.1), for \( n \in \mathbb{N} \), we get
\[
q^{2h} T_{n,q}^{(h)}(4) = q^{2h} \left( \frac{2^{n+1}}{q^h + 1} \right) - q^h \left( \frac{2^{n+1}}{q^h + 1} \right) = -q^h T_{n,q}^{(h)}(2) = T_{n,q}^{(h)}. \tag{2.3}
\]

Therefore, by (2.3), we obtain the following theorem.

**Theorem 2.2** For \( n \in \mathbb{N} \), we have
\[
T_{n,q}^{(h)}(4) = \frac{2^{n+1}}{q^h} + \frac{T_{n,q}^{(h)}}{q^{2h}}. 
\]

By (2.3) and Theorem 2.2, we have the following corollary.

**Corollary 2.3** For \( n \in \mathbb{N} \), we have
\[
\frac{1}{q^h} \int_{\mathbb{Z}_p} q^{-hx}(2x+4)^n d\mu_{-1}(x) = 2^{n+1} + q^h T_{n,q-1}^{(h)}. 
\]

By (2.3), Theorem 1.1, and Corollary 2.3, we know that
\[
\int_{\mathbb{Z}_p} q^{hx}(2-2x)^n d\mu_{-1}(x) = 2^{n+1} + q^h \int_{\mathbb{Z}_p} q^{-hx}(2x)^n d\mu_{-1}(x). 
\]

Therefore, we have the following theorem.

**Theorem 2.4** For \( n \in \mathbb{N} \), we have
\[
\int_{\mathbb{Z}_p} q^{hx}(2-2x)^n d\mu_{-1}(x) = 2^{n+1} + q^h \int_{\mathbb{Z}_p} q^{-hx}(2x)^n d\mu_{-1}(x). 
\]

In (1.5), we take the fermionic \( p \)-adic invariant integral on \( \mathbb{Z}_p \) for one Bernstein polynomials as follows: For \( n, k \in \mathbb{Z}_+ \), we have
\[
\int_{\mathbb{Z}_p} q^{hx} 2^n B_{k,n}(x) d\mu_{-1}(x) = \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^{n-k-l} 2^l T_{n-l,q}^{(h)}. \tag{2.4}
\]
From the reflection symmetric properties of Bernstein polynomials, we note that
\[ B_{k,n}(x) = B_{n-k,n}(1-x), \text{ where } n, k \in \mathbb{Z}_+ \text{ and } x \in \mathbb{Z}_p. \] (2.5)
For \( n, k \in \mathbb{Z}_+ \) with \( n > k \), we have
\[
\int_{\mathbb{Z}_p} q^{hx} 2^n B_{n-k,n}(1-x) d\mu_{-1}(x)
= \left( \frac{n}{k} \right) \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} 2^l \left( 2^{n-l+1} + q^h \int_{\mathbb{Z}_p} q^{-hx} (2x)^{n-l} d\mu_{-1}(x) \right).
\]
Therefore, we have the following theorem.

**Theorem 2.5** For \( n, k \in \mathbb{Z}_+ \) with \( n > k \), we have
\[
\int_{\mathbb{Z}_p} q^{hx} 2^n B_{k,n}(x) d\mu_{-1}(x) = \left( \frac{n}{k} \right) \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} 2^l \left( 2^{n-l+1} + q^h T_{n-l,q}^{(h)} \right).
\]

By (2.4) and Theorem 2.5, we have the following theorem.

**Theorem 2.6** Let \( n, k \in \mathbb{Z}_+ \) with \( n > k \). Then we have
\[
\sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^{n-k-l} 2^l T_{n-l,q}^{(h)} = \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} 2^l \left( 2^{n-l+1} + q^h T_{n-l,q}^{(h)} \right).
\]
Let \( n_1, n_2, k \in \mathbb{Z}_+ \) with \( n_1 + n_2 > 2k \). Then we get
\[
\int_{\mathbb{Z}_p} 2^{n_1+n_2} B_{k,n_1}(x) B_{k,n_2}(x) q^{hx} d\mu_{-1}(x)
= \left( \prod_{i=1}^{2} \binom{n_i}{k} \right) \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{2k-l} 2^l \left( 2^{n_1+n_2-l+1} + q^h \int_{\mathbb{Z}_p} q^{-hx} (2x)^{n_1+n_2-l} d\mu_{-1}(x) \right).
\]
Therefore, we obtain the following theorem.

**Theorem 2.7** For \( n_1, n_2, k \in \mathbb{Z}_+ \) with \( n_1 + n_2 > 2k \), we have
\[
\int_{\mathbb{Z}_p} 2^{n_1+n_2} q^{hx} B_{k,n_1}(x) B_{k,n_2}(x) d\mu_{-1}(x)
= \left( \prod_{i=1}^{2} \binom{n_i}{k} \right) \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{2k-l} 2^l \left( 2^{n_1+n_2-l+1} + q^h T_{n_1+n_2-l,q}^{(h)} \right).
\]
For \( n_1, n_2, k \in \mathbb{Z}_+ \), by simple calculation, we easily see that
\[
\int_{\mathbb{Z}_p} 2^{n_1+n_2} B_{k,n_1}(x)B_{k,n_2}(x)q^{hx}d\mu_1(x)
= \left( \prod_{i=1}^{2} \binom{n_i}{k} \right) \sum_{l=0}^{n_1+n_2-2k} \binom{n_1+n_2-2k}{n_1+n_2-2k+l} (-1)^{n_1+n_2-2k-l} 2^l T_{n_1+n_2-l,q}^{(h)}.
\] (2.6)

Therefore, by (2.6) and Theorem 2.7, we obtain the following theorem.

**Theorem 2.8** Let \( n_1, n_2, k \in \mathbb{Z}_+ \) with \( n_1 + n_2 > 2k \). Then we have
\[
\sum_{l=0}^{2k} \binom{2k}{l} (-1)^{2k-l} 2^l \left( 2^{n_1+n_2-l+1} + q^h T_{n_1+n_2-l,q}^{(h)} \right)
= \sum_{l=0}^{n_1+n_2-2k} \binom{n_1+n_2-2k}{n_1+n_2-2k+l} (-1)^{n_1+n_2-2k-l} 2^l T_{n_1+n_2-l,q}^{(h)}.
\]

For \( n_1, n_2, n_3, k \in \mathbb{Z}_+ \) with \( N_3 = n_1 + n_2 + n_3 > 3k \), by the symmetry of Bernstein polynomials, we see that
\[
\int_{\mathbb{Z}_p} 2^{N_3} B_{k,n_1}(x)B_{k,n_2}(x)B_{k,n_3}(x)q^{hx}d\mu_1(x)
= \left( \prod_{i=1}^{3} \binom{n_i}{k} \right) \sum_{l=0}^{3k} \binom{3k}{l} (-1)^{3k-l} 2^l \left( 2^{N_3-l+1} + q^h \int_{\mathbb{Z}_p} q^{hx}(2x)^{N_3-l}d\mu_1(x) \right).
\]

Therefore, we have the following theorem.

**Theorem 2.9** For \( n_1, n_2, n_3, k \in \mathbb{Z}_+ \) with \( n_1 + n_2 + n_3 > 3k \), we have
\[
\int_{\mathbb{Z}_p} 2^{n_1+n_2+n_3} B_{k,n_1}(x)B_{k,n_2}(x)B_{k,n_3}(x)q^{hx}d\mu_1(x)
= \left( \prod_{i=1}^{3} \binom{n_i}{k} \right) \sum_{l=0}^{3k} \binom{3k}{l} (-1)^{3k-l} 2^l \left( 2^{n_1+n_2+n_3-l+1} + q^h T_{n_1+n_2+n_3-l,q}^{(h)} \right).
\]

In the same manner, multiplication of three Bernstein polynomials can be given by the following relation:
\[
\int_{\mathbb{Z}_p} 2^{N_3} B_{k,n_1}(x)B_{k,n_2}(x)B_{k,n_3}(x)q^{hx}d\mu_1(x)
= \left( \prod_{i=1}^{3} \binom{n_i}{k} \right) \sum_{l=0}^{N_3-3k} \binom{N_3-3k}{l} (-1)^{N_3-3k-l} 2^l \left( N_3 - 3k \right) T_{N_3-l,q}^{(h)};
\]

where \( n_1, n_2, n_3, k \in \mathbb{Z}_+ \) with \( N_3 = n_1 + n_2 + n_3 > 3k \).

Therefore, by Theorem 2.9, we obtain the following theorem.
Theorem 2.10 Let \( n_1, n_2, n_3, k \in \mathbb{Z}_+ \) with \( n_1 + n_2 + n_3 > 3k \). Then we have
\[
\sum_{l=0}^{3k} \binom{3k}{l} (-1)^{3k-l} 2^l \left( 2^{n_1+n_2+n_3-l+1} + q^h T_{n_1+n_2+n_3-l,q}^{(h)} \right) = \sum_{l=0}^{n_1+n_2+n_3-3k} (-1)^{n_1+n_2+n_3-3k-l} 2^l \binom{n_1+n_2+n_3-3k}{l} T_{n_1+n_2+n_3-l,q}^{(h)}.
\]

Using the above theorem and mathematical induction, we obtain the following theorem.

Theorem 2.11 Let \( m \in \mathbb{N} \). For \( n_1, n_2, \ldots, n_m, k \in \mathbb{Z}_+ \) with \( N_m = n_1 + \cdots + n_m > mk \), the multiplication of the sequence of Bernstein polynomials \( B_{k,n_1}(x), \ldots, B_{k,n_m}(x) \) with different degrees under fermionic \( p \)-adic invariant integral on \( \mathbb{Z}_p \) can be given as
\[
\int_{\mathbb{Z}_p} \left( \prod_{i=1}^{m} 2^{n_i} B_{k,n_i}(x) \right) q^h x d\mu_{-1}(x) = \left( \prod_{i=1}^{m} \binom{n_i}{k} \right) \sum_{l=0}^{mk} \binom{mk}{l} (-1)^{mk-l} 2^l \left( 2^{N_m-l+1} + q^h T_{N_m-l,q}^{(h)} \right).
\]

We also easily see that
\[
\int_{\mathbb{Z}_p} \left( \prod_{i=1}^{m} 2^{n_i} B_{k,n_i}(x) \right) q^h x d\mu_{-1}(x) = \left( \prod_{i=1}^{m} \binom{n_i}{k} \right) \sum_{l=0}^{N_m-mk} \binom{N_m-mk}{l} (-1)^{N_m-mk-l} 2^l T_{N_m-l,q}^{(h)}. \tag{2.7}
\]

By Theorem 2.11 and (2.7), we have the following corollary.

Corollary 2.12 Let \( m \in \mathbb{N} \). For \( n_1, n_2, \ldots, n_m, k \in \mathbb{Z}_+ \) with \( n_1 + \cdots + n_m > mk \), we have
\[
\sum_{l=0}^{mk} \binom{mk}{l} (-1)^{mk-l} 2^l \left( 2^{n_1+\cdots+n_m-l+1} + q^h T_{n_1+\cdots+n_m-l,q}^{(h)} \right) = \sum_{l=0}^{n_1+\cdots+n_m-mk} \binom{n_1 + \cdots + n_m - mk}{l} (-1)^{n_1 + \cdots + n_m - mk-l} 2^l T_{n_1 + \cdots + n_m-l,q}^{(h)}.
\]
(h, q)-tangent polynomials and Bernstein polynomials

References


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