Dynamic Pricing in the Presence of Competition
with Reference Price Effect

S. Kachani¹, Y. Oumanar¹,² and N. Raissi²

¹Department of Industrial Engineering and Operations Research
Columbia University, New York, United States

²Department of Applied Mathematics
Mohammed V University, Rabat, Morocco

Abstract

This paper presents a reference price model for competition in a
duopoly market for a single product. We derive Markov-Perfect equi-
librium pricing policies. We provide a closed-form heuristic and demon-
strate that it is very close to the equilibrium. We also derive closed-form
solutions for pricing policies of a retailer who is an optimizer when its
competitor follows one of three different suboptimal policies, and prove
monotone convergence of these solutions. Finally, we use our results to
analyze the impact of competition by comparing the revenue from the
equilibrium policy with the one from a non-competition policy.

Mathematics Subject Classification: 90C39, 90C46, 91B24, 90B99

Keywords: Dynamic Pricing, Reference Price, Management Decision mak-
ing, Dynamic Optimization

1 Introduction

Traditional literature in pricing assumes that there is no interaction between
the firms and their customers. While this assumption is often crucial to the
analysis of the models, it can be very unrealistic. As a result, more sophisticated pricing models that incorporate customer behavior have emerged in recent years. A branch of the dynamic pricing literature has taken a particular interest in the concept of a reference price. Elmaghraby and Keskinocak (2003) noted that this is an important element that, in the past, was largely missing from the academic literature and price optimization softwares.

For businesses that have repeated interaction with customers, price expectation is formed through past experiences. Subsequently, the utility of consumption is affected when there is a mismatch between the current and the expected price, and when the reference price is below the observed price, customers perceive gains and purchasing becomes more attractive (Hardie et al. (1993)). The most common framework for the formation of the reference price is an exponentially weighted average of the past observed prices, where consumers are given memory parameters associated with past prices (see Winer (1985, 1986), Kalwani et al. (1990), Briesch et al. (1997), Putler (1992) and Greenleaf (1995) for more details). For a review of the theoretical foundations and modeling of the reference price concept we refer the reader to Winer (1988).

Recent papers focus on developing dynamic pricing models with reference price effect to find optimal pricing strategies that would maximize retailers’ profit. This paper presents a dynamic pricing model in the presence of competition incorporating the concept of customer’s reference price. The concept of reference price provides a good framework for modeling competition. The economics literature models competition in the market by directly assuming that the demand of a product is a function of it’s competitor’s price, in addition to its own price. We propose an alternative model where the retailers compete through their influences on the reference price.

2 Dynamic Pricing Model

We consider the case of two retailers each selling a single product (For a Multi-Product case, refer to (Kachani et al. (2014a)). The demand for the product from the store $i$ is a function of the price $p$ and the reference price $r$ and is denoted by $D_i(p, r)$. It is given by

$$D_i(p_i, r) = a_i - b_i p_i + c_i (r - p_i), \quad \text{for } i = 1, 2$$

where $a_i$, $b_i$, and $c_i$ are positive.

Competition is modeled through the dynamics of the reference price. The reference price has a memory coefficient $\alpha$ as in the monopoly case (Kachani et al. (2014b)). We let $\mu_i$ be the influence factor of store $i$ on the reference price. We can view $\mu_i$ as a measure of the store $i$’s presence in the market.
The dynamics of the reference price in this case are:

\[ g(p_1, p_2, r) = \alpha r + (1 - \alpha)(\mu_1 p_1 + \mu_2 p_2) \]

where \(0 \leq \alpha < 1\).

The retailer chooses a pricing policy that maximizes its total revenue. Hence, the Bellman equation we need to solve for the optimal pricing policy is given by:

\[ V^*_1(r) = \sup_{p_1} D_1(p_1, r)p_1 + \beta V^*_1(g(p_1, p_2, r)) \quad (1) \]

3 Optimal pricing policy with a suboptimal competitor

In this section, we derive the optimal pricing policy for a retailer when:

- its competitor’s price is constant
- its competitor’s price is a linear function of the retailer’s own price
- its competitor follows a myopic pricing policy

We also prove the monotone convergence in the three cases.

3.1 Competitor’s price is constant

**Theorem 3.1** The optimal pricing policy for a retailer whose competitor’s price is constant at \(p_2\) is given by

\[ p_1^*(r) = \frac{(c_1 + 2(1 - \alpha)\alpha \beta \gamma_1 \mu_1) r + a_1 + (1 - \alpha)\beta \delta_1 \mu_1 + 2p_2(1 - \alpha)^2 \beta \gamma_1 \mu_1 \mu_2}{2(b_1 + c_1) - (1 - \alpha)^2 \beta \gamma_1 \mu_1^2} \quad (2) \]

where

\[
\gamma_1 = \frac{1}{2(1 - \alpha)^2 \beta \mu_1^2} \left( b_1(1 - \alpha^2 \beta) + c_1(1 - \alpha^2 \beta - \alpha \beta \mu_1(1 - \alpha)) - \sqrt{(b_1(1 - \alpha^2 \beta) + c_1(1 - \alpha^2 \beta - \alpha \beta \mu_1(1 - \alpha))^2 - c_1^2(1 - \alpha)^2 \beta \mu_1^2)} \right) 
\]

\[
\delta_1 = \frac{a_1 c_1 + 2a_1(1 - \alpha)\alpha \beta \gamma_1 \mu_1 + 2p_2(1 - \alpha)\beta \gamma_1(2(b_1 + c_1)\alpha + (1 - \alpha)\mu_1)\mu_2}{2(b_1 + c_1)(1 - \alpha \beta) - c_1(1 - \alpha)\beta \mu_1 - 2(1 - \alpha)^2 \beta \gamma_1 \mu_1^2} 
\]

\[
\epsilon_1 = \frac{(a_1 + (1 - \alpha)\beta \delta_1 \mu_1)^2 + 4p_2(1 - \alpha)\beta \mu_2((b_1 + c_1)(\delta_1 + p_2(1 - \alpha)\gamma_1 \mu_2) + a_1(1 - \alpha)\gamma_1 \mu_1)}{4(1 - \beta)(b_1 + c_1 - (1 - \alpha)^2 \beta \gamma_1 \mu_1^2)} 
\]
To prove this result, we postulate that the value function of retailer 1 is given by

\[ V^*_1(r) = \gamma_1 r^2 + \delta_1 r + \epsilon_1 \]

The Bellman equation (1) becomes

\[
V^*_1(r) = \sup_{p_1} \left[ -b_1 - c_1 + (1 - \alpha)^2 \beta \gamma_1 \right] p_1^2 \\
\quad + [a_1 + c_1 r + 2r(1 - \alpha)\alpha \beta \mu_1 \gamma_1 + (1 - \alpha)\beta \mu_1 \delta_1 + 2p_2(1 - \alpha)^2 \beta \mu_1 \mu_2 \gamma_1] p_1 \\
\quad + \beta((\alpha r + (1 - \alpha)\mu_2 p_2)^2 \gamma_1 + (\alpha r + (1 - \alpha)\mu_2 p_2) \delta_1 + \epsilon_1) \tag{3}
\]

Since the competitor’s price \( p_2 \) is constant, the optimal price can be obtained by simply optimizing the Bellman equation (3) over \( p_1 \). Solving \( \frac{\partial V^*_1(r)}{\partial p_1} = 0 \) yields the optimal pricing policy (2).

Then, by substituting (2) into (3), and collecting the coefficients of the polynomial, we can solve for \( \gamma_1 \), \( \delta_1 \) and \( \epsilon_1 \) and get the desired results.

Next, we need to establish that the postulate is valid by showing that the coefficients of the value function are nonnegative real numbers. The expression for \( \gamma_1 \) is the smaller root of the quadratic equation

\[
(1 - \alpha)^2 \beta \mu_1^2 \gamma_1^2 - (b_1 (1 - \alpha^2 \beta) + c_1 (1 - \alpha^2 \beta - \alpha \beta \mu_1 (1 - \alpha))) \gamma_1 + c_1^2 = 0 \tag{4}
\]

Suppose that (4) has real roots, then it is easy to see that \( \gamma_1 \) is positive because the coefficients of the quadratic and the constant term are positive, whereas the coefficient of the linear term is negative. To show that the roots are real, we rewrite the term inside the square root as

\[
(b_1 (1 - \alpha^2 \beta) + c_1 (1 - \alpha^2 \beta - \alpha \beta \mu_1 (1 - \alpha)))^2 - c_1^2 (1 - \alpha)^2 \beta \mu_1 \\
= (1 - \alpha^2 \beta)(b_1^2 (1 - \alpha^2 \beta) + 2b_1 c_1 (1 - \alpha \beta (\alpha (1 - \mu_1) + \mu_1)) \\
+ c_1^2 (1 - \beta (\alpha + \mu_1 (1 - \alpha))^2))
\]

which is nonnegative.

To show that \( \delta_1 \) is well defined and positive, we need to show that

\[
\gamma_1 < \frac{2b_1 (1 - \alpha \beta) + c_1 (2 - \alpha \beta (2 - \mu_1) - \beta \mu_1)}{2(1 - \alpha)^2 \beta \mu_1^2}
\]

This is true if and only if

\[
b_1 (1 - \alpha^2 \beta) + c_1 (1 - \alpha^2 \beta - \alpha \beta \mu_1 (1 - \alpha)) \\
- \sqrt{(b_1 (1 - \alpha^2 \beta) + c_1 (1 - \alpha^2 \beta - \alpha \beta \mu_1 (1 - \alpha)))^2 - c_1^2 (1 - \alpha)^2 \beta \mu_1} \\
< 2b_1 (1 - \alpha \beta) + c_1 (2 - \alpha \beta (2 - \mu_1) - \beta \mu_1) \\
\Leftrightarrow -b_1 (1 - 2 \alpha \beta + \alpha^2 \beta) - c_1 (1 - \alpha \beta (1 - \mu)(2 - \alpha) - \beta \mu_1) \\
< \sqrt{(b_1 (1 - \alpha^2 \beta) + c_1 (1 - \alpha^2 \beta - \alpha \beta \mu_1 (1 - \alpha)))^2 - c_1^2 (1 - \alpha)^2 \beta \mu_1)}
\]
Which holds because the left hand side is negative. Finally, \( \epsilon_1 \) is well defined and positive if and only if

\[
\gamma_1 < \frac{b_1 + c_1}{(1 - \alpha)^2 \beta \mu_1^2}
\]

(5)

Which is again true because \( b_1 \) and \( c_1 \) are positive and

\[
b_1(1 - \alpha^2 \beta) + c_1(1 - \alpha^2 \beta - \alpha \beta \mu_1(1 - \alpha)) < 2(b_1 + c_1)
\]

In addition, we can also derive the expression for the transition of the reference price under the optimal pricing policy. It is given by

\[
g^*(r) = \frac{(2(b_1 + c_1)\alpha + c_1(1 - \alpha)\mu_1)r + (1 - \alpha)(\mu_1 a_1 + (1 - \alpha)\beta \delta_1 \mu_1) + 2(b_1 + c_1)p_2 \mu_2}{2(b_1 + c_1) - (1 - \alpha)^2 \beta \gamma_1 \mu_1^2}
\]

(6)

Finally, the steady state reference price is given by:

\[
r^{*\ast} = \frac{a_1 \mu_1 + (1 - \alpha) \beta \delta_1 \mu_1^2 + 2(b_1 + c_1)p_2 \mu_2}{2b_1 + c_1(2 - \mu_1) - 2(1 - \alpha) \beta \gamma_1 \mu_1^2} = \frac{a_1(1 - \alpha \beta) \mu_1 + p_2(2b_1(1 - \alpha \beta) + c_1(2 - \alpha \beta(2 - \mu_1) - \beta \mu_1)) \mu_2}{2b_1(1 - \alpha \beta) + c_1(2 - 2(\alpha \beta(1 - \mu_1) - \mu_1 - \beta \mu_1)}
\]

(7)

The last equality is from substituting the expressions for \( \delta_1 \), and using (4) to complete the square in the denominator.

### 3.2 Competitor’s Price is a Linear Function of the Retailer’s Price

**Proposition 3.2** The optimal pricing policy for a retailer whose competitor’s price is a linear function of its own price, namely, \( p_2 = \theta p_1 + \rho \), with \( \theta > 0 \) and \( \rho > 0 \), is given by

\[
p_1^*(r) = \frac{(c_1 + 2(1 - \alpha)\alpha \beta \gamma_1(\mu_1 + \theta \mu_2))r + a_1 + (1 - \alpha)\beta \delta_1(\mu_1 + \theta \mu_2) + 2(1 - \alpha)^2 \beta \gamma_1 \mu_2(\mu_1 + \theta \mu_2)\rho}{2(b_1 + c_1) - (1 - \alpha)^2 \beta \gamma_1(\mu_1 + \mu_2 \theta)^2}
\]

(8)

where

\[
\gamma_1 = \frac{1}{2(1 - \alpha)^2 \beta(\mu_1 + \mu_2 \theta)^2} \left( (b_1 + c_1)(1 - \alpha^2 \beta) - c_1(1 - \alpha)\alpha \beta(\mu_1 + \mu_2 \theta) \right)
\]

\[
- \sqrt{((b_1 + c_1)(1 - \alpha^2 \beta) - c_1(1 - \alpha)\alpha \beta(\mu_1 + \mu_2 \theta))^2 - c_1^2(1 - \alpha)^2 \beta(\mu_1 + \mu_2 \theta)^2}
\]

\[
\delta_1 = \frac{a_1(c_1 + 2(1 - \alpha)\alpha \beta \gamma_1(\mu_1 + \mu_2 \theta)) + 2(1 - \alpha)\beta \gamma_1 \mu_2(2b_1 a_1 + c_1(\mu_1 + \mu_2 \theta + \alpha(2 - \mu_1 - \mu_2 \theta))\rho}{2b_1(1 - \alpha \beta) + c_1(2 - \beta(\mu_1 + \mu_2 \theta - \alpha(2 - \mu_1 - \mu_2 \theta)) - 2(1 - \alpha)^2 \beta \gamma_1(\mu_1 + \mu_2 \theta)^2}
\]

\[
\epsilon_1 = \frac{1}{4(1 - \beta)(b_1 + c_1) - (1 - \alpha)^2 \beta \gamma_1(\mu_1 + \mu_2 \theta)^2} \left( (a_1 + (1 - \alpha)\beta \delta_1(\mu_1 + \mu_2 \theta))^2 + 4(1 - \alpha) \beta \mu_2((b_1 + c_1) \delta_1 + a_1(1 - \alpha) \gamma_1(\mu_1 + \mu_2 \theta)) \rho + 4(b_1 + c_1)(1 - \alpha)^2 \beta \gamma_1 \mu_2^2 \rho^2 \right)
\]
The transition of the reference price under the optimal pricing policy is given by:

\[ \gamma^* = \sup_{p_1} \left[ b_1 - c_1 + (1 - \alpha)^2 \beta \gamma_1 (\mu_1 + \mu_2 \theta) \right] p_1^2 + a_1 c_1 r + (1 - \alpha) \beta (\mu_1 + \mu_2 \theta) (2 r \alpha \gamma_1 + \delta_1 + 2 (1 - \alpha) \gamma_1 \mu_2 \rho) p_1 + 1 \beta ((\alpha r + (1 - \alpha) \mu_2 \rho) (\alpha r \gamma_1 + \delta_1 + (1 - \alpha) \gamma_1 \mu_2 \rho) + \epsilon_1) \]

Similarly, solving \( \partial V_1 / \partial p_1 = 0 \) yields the optimal pricing policy (8). Then, by substituting (8) into (9), and collecting the coefficients of the polynomial, we solve for \( \gamma_1, \delta_1, \) and \( \epsilon_1 \) and get the desired result.

Following the same approach in the proof of theorem 3.1, we can easily verify that \( \gamma_1, \delta_1, \) and \( \epsilon_1 \) are well defined, positive and real.

The transition of the reference price under the optimal pricing policy is given by

\[ g^*(r) = \frac{1}{2(b_1 + c_1 - (1 - \alpha)^2 \beta \gamma_1 (\mu_1 + \mu_2 \theta)^2) (1 - \alpha)((\mu_1 + \mu_2 \theta) (a_1 + (1 - \alpha) \beta \delta_1 (\mu_1 + \mu_2 \theta)) + c_1 (r \mu_1 + \mu_2 \theta + \alpha (2 - \mu_1 - \mu_2 \theta)) + 2 (1 - \alpha) \mu_2 \rho) + 2 b_1 (r \alpha + (1 - \alpha) \mu_2 \rho) \}

And the steady state reference price is given by:

\[ r^{**} = \frac{a_1 (\mu_1 + \mu_2 \theta) + (1 - \alpha) \beta \delta_1 ((\mu_1 + \mu_2 \theta)^2 + 2 (b_1 + c_1) \mu_2 \rho)}{2(b_1 + c_1 - (1 - \alpha)^2 \beta \gamma_1 (\mu_1 + \mu_2 \theta)^2)} \]

### 3.3 Competitor is a Myopic Optimizer

A retailer who is a myopic optimizer prices the product to maximize the current-period revenue, i.e.,

\[ p^*_2(r) = \arg \max \left[ a_2 + c_2 r \right] \]

**Proposition 3.3** The optimal pricing policy for a retailer whose competitor is a myopic optimizer is given by

\[ p^*_1(r) = \frac{a_1 c_1 + (1 - \alpha) \beta \mu_1 (2 r \alpha \gamma_1 + \delta_1 + 2 (1 - \alpha) \gamma_1 \mu_2 (\phi + r \psi))}{2 (b_1 + c_1 - (1 - \alpha)^2 \beta \gamma_1 \mu_1^2)} \]
where \( \phi = \frac{a_{1}}{2(b_{2}+c_{2})} \), \( \psi = \frac{c_{2}}{2(b_{2}+c_{2})} \), and

\[
\gamma_{1} = \frac{1}{2(1-\alpha)^{2}\beta\mu_{1}^{2}} \left( b_{1}(1-\beta(\alpha + (1-\alpha)\mu_{2}\psi)^{2}) + c_{1}(1-\beta(\alpha + (1-\alpha)\mu_{2}\psi)(\alpha + (1-\alpha)(\mu_{1} + \mu_{2}\psi))) - \left( (b_{1}(1-\beta(\alpha + (1-\alpha)\mu_{2}\psi)^{2}) + c_{1}(1-\beta(\alpha + (1-\alpha)\mu_{2}\psi)(\alpha + (1-\alpha)(\mu_{1} + \mu_{2}\psi)))^{2}\right)
\]

\[
- c_{1}^{2}(1-\alpha)^{2}\beta\mu_{1}^{2})^{1/2}\right)
\]

\[
\delta_{1} = \left( 2(1-\alpha)\beta\gamma_{1}\mu_{2}\phi(2b_{1}(\alpha + (1-\alpha)\mu_{2}\psi) + c_{1}(2\alpha + (1-\alpha)(\mu_{1} + \mu_{2}\psi))) + a_{1}(c_{1} + 2(1-\alpha)\beta\gamma_{1}\mu_{1}(\alpha + (1-\alpha)\mu_{2}\psi)) \right) / \left( 2b_{1}(1-\alpha\beta - (1-\alpha)\beta\mu_{2}\psi) + c_{1}(2-\alpha\beta - (1-\alpha)\beta\mu_{2}\psi) - \beta(\mu_{1} - \mu_{2}\psi) - (2(1-\alpha)^{2}\beta\gamma_{1}\mu_{1}^{2}) \right)
\]

\[
\epsilon_{1} = \frac{1}{4(1-\beta)(b_{1} + c_{1} - (1-\alpha)^{2}\beta\gamma_{1}\mu_{1}^{2})} \left( (a_{1} + (1-\alpha)\beta\mu_{1}\mu_{2}\phi)^{2} + 4(b_{1} + c_{1})(1-\alpha)^{2}\beta\gamma_{1}\mu_{2}\phi^{2} + 4(1-\alpha)\beta((b_{1} + c_{1})\delta_{1} + a_{1}(1-\alpha)\gamma_{1}\mu_{1}\mu_{2}\phi) \right)
\]

Let \( \phi = \frac{a_{1}}{2(b_{2}+c_{2})} \) and \( \psi = \frac{c_{2}}{2(b_{2}+c_{2})} \). Then, the myopic pricing policy of the competitor can be written as \( p_{2}(r) = \psi r + \phi \). Since \( p_{2}(r) \) is not a function of \( p_{1} \), we get the optimal pricing policy (13) by substituting (12) into (2), which is the optimal pricing policy of theorem 3.1, when the competitor’s price is constant.

Similarly, by substituting (13) into the value function (3) in the proof of theorem 3.1, and collecting the coefficients of the polynomial, we solve for \( \gamma_{1}, \delta_{1}, \) and \( \epsilon_{1} \) and get the desired results.

Following the same approach in the proof of theorem 3.1, we can easily verify that \( \gamma_{1}, \delta_{1}, \) and \( \epsilon_{1} \) are well defined, positive and real.

The transition of the reference price under the optimal pricing policy is given by

\[
g^{*}(r) = \frac{1}{2(b_{1} + c_{1} - (1-\alpha)^{2}\beta\gamma_{1}\mu_{1}^{2})} \left( (2b_{1} + c_{1})(\alpha + (1-\alpha)\mu_{2}\psi) + c_{1}\mu_{1}(1-\alpha))r + (1-\alpha)\mu_{1}(a_{1} + (1-\alpha)\beta\delta_{1}\mu_{1} + 2(b_{1} + c_{1})\mu_{2}\phi) \right)
\]

(14)

And the steady state reference price is given by:

\[
r^{**} = \frac{a_{1}\mu_{1} + (1-\alpha)\beta\delta_{1}\mu_{1}^{2} + 2(b_{1} + c_{1})\mu_{2}\phi}{2b_{1}(1-\mu_{2}\psi) + c_{1}(2 - \mu_{1} - 2\mu_{2}\psi) - 2(1-\alpha)\beta\gamma_{1}\mu_{1}^{2}}
\]

\[
= \left( a_{1}\mu_{1}(1-\alpha\beta - (1-\alpha)\beta\mu_{2}\psi) + (2(b_{1} + c_{1})(1-\alpha\beta - (1-\alpha)\beta\mu_{2}\psi) - c_{1}(1-\alpha)\mu_{1}\beta)\mu_{2}\phi \right) / \left( 2b_{1}(1-\mu_{2}\psi)(1-\alpha\beta - (1-\alpha)\beta\mu_{2}\psi) + c_{1}(2(1-\alpha\beta) - (1-2\alpha\beta + 2\beta)(\mu_{1} + 2\mu_{2}\psi) + 2(1-\alpha)\beta\mu_{2}\psi(\mu_{1} + \mu_{2}\psi)) \right)
\]

(15)
3.4 Monotonocity and Convergence

**Proposition 3.4** When the competitor prices according to the assumptions of theorem 3.1, proposition 3.2, or proposition 3.3, the sequence of optimal prices \( p_1^* \) and reference prices \( r^* \) converge monotonically to their respective steady states.

1. Competitor’s price is constant:
   Condition (5) implies that \( p_1^*(r) \) in (2) and \( g^*(r) \) in (6) are positive and strictly increasing for all \( r \geq 0 \). Since \( g^*(r) \) is linear in \( r \), it suffices to show that \( r^{**} \) is positive and finite. It is easy to see from the expression of \( r^{**} \) in (7) that this is true.

2. Competitor’s price is a linear function of the retailer’s price:
   Similarly, we need to show that \( p_1^*(r) \) in (8) and \( g^*(r) \) in (10) are positive and strictly increasing for all \( r \geq 0 \), and the steady state price \( r^{**} \) in (11) is positive and finite. All of these are true if \( \gamma_1 \) in proposition 3.2 satisfies the following inequality:

\[
\gamma_1 < \frac{b_1 + c_1}{(1 - \alpha)^2 \beta (\mu_1 + \mu_2 \theta)^2} \quad (16)
\]

(16) holds because \( b_1 \) and \( c_1 \) are positive, and \( (b_1 + c_1)(1 - \alpha^2 \beta) - c_1(1 - \alpha)\alpha\beta(\mu_1 + \mu_2 \theta) < 2(b_1 + c_1) \).

3. Competitor is a myopic optimizer:
   First, we need to show that \( p_1^*(r) \) in (13) and \( g^*(r) \) in (14) are positive and strictly increasing for all \( r \geq 0 \). Since \( \psi \geq 0 \) and \( \phi \geq 0 \), this is true if \( \gamma_1 \) in proposition 3.3 satisfies the following inequality:

\[
\gamma_1 < \frac{b_1 + c_1}{(1 - \alpha)^2 \beta \mu_1^2} \quad (17)
\]

(17) holds because \( b_1 \) and \( c_1 \) are positive, \( \psi < 1 \), and

\[
b_1(1 - \beta (\alpha + (1 - \alpha)\mu_2 \psi)^2) + c_1(1 - \beta)
\]

\[
(\alpha + (1 - \alpha)\mu_2 \psi)(\alpha + (1 - \alpha)(\mu_1 + \mu_2 \psi)) < 2(b_1 + c_1)
\]

Finally, the steady state price \( r^{**} \) in (15) is well defined and positive because

\[
2(1 - \alpha \beta) - (1 - 2\alpha \beta + 2\beta)(\mu_1 + 2\mu_2 \psi) + 2(1 - \alpha)\beta \mu_2 \psi(\mu_1 + \mu_2 \psi)
\]

\[
eq 2\alpha \beta(1 - \mu_2 \psi)(1 - \mu_1 - \mu_2 \psi) + \mu_1(1 - \beta + 2\beta \mu_2 \psi) + 2(1 - \mu_2 \psi)(1 - \beta \mu_2 \psi) > 0
\]
4 Markov Perfect Equilibrium when both retailers are optimizers

In this section, we consider the case where both retailers are optimizers. In this setting, the retailers are playing an infinitely-repeated game. Since they are both maximizing their total discounted revenue, the Markov Perfect (MP) equilibrium prices can be determined by solving the following Bellman equations simultaneously:

\[ V_1^*(r) = \sup_{p_1} D_1(p_1, r)p_1 + \beta V_1^*(r)(g(p_1, p_2^*(r), r)) \]
\[ V_2^*(r) = \sup_{p_2} D_2(p_2, r)p_2 + \beta V_2^*(r)(g(p_1^*(r), p_2, r)) \]

To solve this system of equations, we postulate that the value function for each retailer is given by

\[ V_i^*(r) = \gamma_i r^2 + \delta_i r + \epsilon_i, \quad \text{for} \ i = 1, 2 \]

and the optimal pricing policy is given by

\[ p_i^*(r) = \zeta_i r + \nu_i, \quad \text{for} \ i = 1, 2 \] (18)

First, let us look at the Bellman equation for retailer 1

\[ V_1^*(r) = \sup_{p_1} D_1(p_1, r)p_1 + \beta V_1^*(r)(g(p_1, p_2^*(r), r)) \]
\[ = \sup_{p_1}(a_1 - b_1 p_1 + c_1 (r - p_1))p_1 \]
\[ + \beta \left( \gamma_1(\alpha r + (1 - \alpha)\mu_1 p_1 + (1 - \alpha)\mu_2(\zeta_2 r + \nu_2))^2 \right) \]
\[ + \beta \left( \delta_1(\alpha r + (1 - \alpha)\mu_1 p_1 + (1 - \alpha)\mu_2(\zeta_2 r + \nu_2)) + \epsilon_1 \right) \]
\[ = \sup_{p_1}(a_1 - b_1 - c_1 + (1 - \alpha_1)^2 \beta \gamma_1 \mu_1^2 r_1^2) \]
\[ + [a_1 + c_1 r + 2r(1 - \alpha)\alpha \beta \gamma_1 \mu_1 + (1 - \alpha)\beta \delta_1 \mu_1 + 2(1 - \alpha)^2 \beta \gamma_1 \mu_1 \mu_2(\zeta_2 r + \nu_2)]p_1 \]
\[ + \beta(r^2 \alpha^2 \gamma_1 + r\alpha \delta_1 + \epsilon_1 + (1 - \alpha)\mu_2(\zeta_2 r + \nu_2))(2r\alpha \gamma_1 + \delta_1 + (1 - \alpha)\gamma_1 \mu_2(\zeta_2 r + \nu_2)) \]

The optimal price can be determined by taking the partial derivative of (19) with respect to \( p_1 \), and setting it equal to zero. Solving \( \frac{\partial V(r)}{\partial p_1} = 0 \), we get

\[ p_1^*(r) = \frac{(c_1 + 2 \beta \gamma_1 \mu_1(\alpha - \alpha^2 + (1 - \alpha)^2 \zeta_2 \mu_2))r + a_1 + (1 - \alpha)\beta \mu_1(\delta_1 + 2(1 - \alpha)\gamma_1 \mu_2 \nu_2)}{2(b_1 + c_1 - (1 - \alpha)^2 \beta \gamma_1 \mu_1^2)} \]

By collecting the coefficients of \( r \), and equating them to the postulate in (18), we get:

\[ \zeta_1 = \frac{c_1 + 2 \beta \gamma_1 \mu_1(\alpha - \alpha^2 + (1 - \alpha)^2 \zeta_2 \mu_2)}{2(b_1 + c_1 - (1 - \alpha)^2 \beta \gamma_1 \mu_1^2)} \] (20)
\[ \nu_1 = \frac{a_1 + (1 - \alpha)\beta \mu_1(\delta_1 + 2(1 - \alpha)\gamma_1 \mu_2 \nu_2)}{2(b_1 + c_1 - (1 - \alpha)^2 \beta \gamma_1 \mu_1^2)} \] (21)
Substituting (18) into (19), and collecting the coefficients of the polynomial, we can solve for \( \gamma_1, \delta_1 \) and \( \epsilon_1 \) and obtain the following solutions:

\[
\begin{align*}
\gamma_1 &= \frac{\zeta_1(c_1(1 - \zeta_1) - b_1\zeta_1)}{1 - \beta(a + (1 - a)(\zeta_1\mu_1 + \zeta_2\mu_2))^2} \\[4pt]
\delta_1 &= \frac{a_1\zeta_1 + c_1\nu_1(1 - 2\zeta_1) - 2(b_2\zeta_1\nu_1 - (1 - a)\beta\gamma_1(\alpha + (1 - a)(\zeta_1\mu_1 + \zeta_2\mu_2))(\mu_1\nu_1 + \mu_2\nu_2))}{1 - \beta(a + (1 - a)(\zeta_1\mu_1 + \zeta_2\mu_2))} \\[4pt]
\epsilon_1 &= \frac{a_1\nu_1 - (b_1 + c_1)\nu_1^2 + (1 - a)\beta(\mu_1\nu_1 + \mu_2\nu_2)(\delta_1 + (1 - a)\gamma_1(\mu_1\nu_1 + \mu_2\nu_2))}{1 - \beta}
\end{align*}
\]

(22) (23) (24)

Solving (20) for \( \gamma_1 \), we get:

\[
\gamma_1 = \frac{2b_1\zeta_1 - c_1(1 - 2\zeta_1)}{2(1 - a)\beta\mu_1(\alpha + (1 - \alpha)(\zeta_1\mu_1 + \zeta_2\mu_2)}
\]

(25)

Finally, we equate (22) to (25) to get

\[
(1 - a)\beta\mu_1(2(b_1 + c_1)(\alpha + (1 - a)\zeta_2\mu_2) + (1 - a)c_1\mu_1)\zeta_1^2 \\
- 2(b_1 + c_1)(1 - \beta(\alpha + (1 - a)\zeta_2\mu_2)^2)c_1(1 - \beta(\alpha + (1 - a)\zeta_2\mu_2)^2) = 0
\]

(26)

In addition, we have the following expression for retailer 2:

\[
(1 - a)\beta\mu_2(2(b_2 + c_2)(\alpha + (1 - a)\zeta_1\mu_1) + (1 - a)c_2\mu_2)\zeta_2^2 \\
- 2(b_2 + c_2)(1 - \beta(\alpha + (1 - a)\zeta_1\mu_1)^2)\zeta_2 + c_2(1 - \beta(\alpha + (1 - a)\zeta_1\mu_1)^2) = 0
\]

(27)

By following a similar procedure, we can solve (21) for \( \delta_1 \), and equate it to (23) to get

\[
(2(1 - a)^2 \beta\gamma_1\mu_1^2 - b_1(2 - 2\beta(\alpha + (1 - a)\zeta_2\mu_2))) - c_1(2 - \beta(\mu_1 + 2\zeta_2\mu_2) \\
+ \alpha(2 - \mu_1 - 2\zeta_2\mu_2))\nu_1 + 2(1 - a)^2 \beta\gamma_1\mu_1\mu_2\nu_2 + a_1(1 - a)\beta - (1 - a)\beta\zeta_2\mu_2) = 0
\]

(28)

Similarly, for retailer 2 we have

\[
(2(1 - a)^2 \beta\gamma_2\mu_2^2 - b_2(2 - 2\beta(\alpha + (1 - a)\zeta_1\mu_1))) - c_2(2 - \beta(\mu_2 + 2\zeta_1\mu_1) \\
+ \alpha(2 - \mu_2 - 2\zeta_1\mu_1))\nu_2 + 2(1 - a)^2 \beta\gamma_2\mu_2\mu_1\nu_1 + a_2(1 - a)\beta - (1 - a)\beta\zeta_1\mu_1) = 0
\]

(29)

Now, we can solve for the equilibrium pricing policies \( p_1^*(r) \) and \( p_2^*(r) \) by simultaneously solving (26) and (27) for \( \zeta_1 \) and \( \zeta_2 \), and (28) and (29) for \( \nu_1 \) and \( \nu_2 \).

Even though closed-form expressions for \( \zeta_1 \) and \( \zeta_2 \) do not exist, we can determine their values by easily solving equations (26) and (27) numerically. Once
\( \zeta_1 \) and \( \zeta_2 \) are determined, \( \gamma_1 \) and \( \gamma_2 \) are given by (22). Equations (28) and (29) can be solved analytically for the closed form solutions of \( \zeta_1 \) and \( \zeta_2 \). However, their expressions are cumbersome and are omitted here. Finally, \( \delta_1 \) and \( \delta_2 \) are given by (23), and \( \epsilon_1 \) and \( \epsilon_2 \) are given by (24).

We propose here a heuristic that approximates the MP equilibrium prices for which we provide a closed form expression. We will illustrate numerically that the solutions of this heuristic are very close to the equilibrium prices obtained by numerically solving the implicit equations given above.

The heuristic assumes that each retailer maximizes its revenue assuming that its competitor’s pricing policy is a function of its price and the reference price will remain constant. Under this assumption, each retailer solves for the best response pricing function \( p_1(p_2, r) \) and \( p_2(p_1, r) \). This best response function is the same as the optimal pricing policy given by (2). Since \( \delta_1 \) is linear and \( \epsilon_1 \) is quadratic in \( p_2 \), we rewrite them as

\[
\begin{align*}
\delta_1 &= \delta_{1,1} p_2 + \delta_{1,0} \\
\epsilon_1 &= \epsilon_{1,2} p_2^2 + \epsilon_{1,1} p_2 + \epsilon_{1,0}
\end{align*}
\]

Then, the best response pricing function in (2) can be rewritten as

\[
p_1(p_2, r) = A_1 p_2 + B_1 r + C_1
\]

where

\[
A_1 = \frac{(1 - \alpha)\beta\delta_{1,1}\mu_1 + 2(1 - \alpha)^2\beta\gamma_1\mu_1\mu_2}{2(b_1 + c_1 - (1 - \alpha)^2\beta\gamma_1\mu_1^2)}
\]
\[
B_1 = \frac{c_1 + 2(1 - \alpha)\alpha\beta\gamma_1\mu_1}{2(b_1 + c_1 - (1 - \alpha)^2\beta\gamma_1\mu_1^2)}
\]
\[
C_1 = \frac{a_1 + (1 - \alpha)\beta\delta_{1,0}\mu_1}{2(b_1 + c_1 - (1 - \alpha)^2\beta\gamma_1\mu_1^2)}
\]

The equilibrium pricing policy for this heuristic is the solution of

\[
p_1^* = A_1 p_2^* + B_1 r + C_1
\]
\[
p_2^* = A_2 p_1^* + B_2 r + C_2
\]

which is given by

\[
p_1^*(r) = E_1 r + F_1
\]
\[
p_2^*(r) = E_2 r + F_2
\]

Where

\[
E_1 = \frac{A_1 B_2 + B_1}{1 - A_1 A_2}, \quad F_1 = \frac{A_1 C_2 + C_1}{1 - A_1 A_2}
\]
The transition of the reference price in this case is given by
\[
g^*(r) = g(p_1^*(r), p_2^*(r), r)
= [\alpha + (1 - \alpha)(\mu_1 E_1 + \mu_2 E_2)]r + (1 - \alpha)(\mu_1 F_1 + \mu_2 F_2)
\]

And the steady-state reference price is given by:
\[
r^{**} = \frac{\mu_1 F_1 + \mu_2 F_2}{1 - \mu_1 E_1 - \mu_2 E_2}
\]

To determine the accuracy of the heuristic, we randomly select 10,000 sets of demand parameters and compare the equilibrium prices \(p^*(r)\) and the total discounted revenue \(V^*(r)\). For each set of parameters, we compute:

- **i)** The percentage difference in revenue:
  \[
  \max_{i=1,2} \left\{ \frac{|V_i^*(r_0) - V_i^*(r_0)|}{V_i^*(r_0)} \right\}
  \]

- **ii)** The average percentage difference in price during the path from \(r_0\) to \(r^{**}\):
  \[
  \max_{i=1,2} \left\{ \frac{1}{T} \sum_{t=0}^{T} \left| \frac{p_i^*(r_t) - p_i^*(r_t)}{p_i^*(r_t)} \right| \right\}
  \]

Where \(T\) is the first time the reference price reaches the steady state \(r^{**}\) within a certain tolerance error \(\varepsilon\) (for table 1, we use \(\varepsilon = 10^{-6}\)).

Table 1 summarizes the data and results of this analysis. We find that the median error of the total revenues and the prices are 0.003% and 1.4% respectively. The heuristic gives a very accurate estimate of the total discounted revenue, with a mean error of 1%, and 95\(^{th}\) percentile of only 4.31%. The error on the average difference in price is higher, with a mean of 4% and a 95\(^{th}\) percentile of 22%.

5 **Comparison between Equilibrium Policy and Policy that Ignores Competition**

In this section, we study the effects of competition in our model. We compare the prices and the revenues under the equilibrium policy to the policy where both retailers maximize their revenue without considering the existence of the other. In other words, they follow the optimal pricing policy of the monopoly case (Kachani et al. (2014)). Figure 1 shows the transition of the equilibrium
Table 1: Discrepancy between Prices and Revenues of Heuristic and Equilibrium Policy

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>Unif(0,100)</td>
</tr>
<tr>
<td>b</td>
<td>Unif(0,2)</td>
</tr>
<tr>
<td>c</td>
<td>Unif(0,100)</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>Unif(0,1)</td>
</tr>
<tr>
<td>$\beta$</td>
<td>Unif(0,1)</td>
</tr>
<tr>
<td>$\mu_1$</td>
<td>Unif(0,1)</td>
</tr>
<tr>
<td>$r_0$</td>
<td>Unif(0,100)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>St. Dev.</th>
<th>50th PCT</th>
<th>90th PCT</th>
<th>95th PCT</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total Revenue</td>
<td>1.01%</td>
<td>0.0756</td>
<td>0.02%</td>
<td>1.71%</td>
<td>4.31%</td>
</tr>
<tr>
<td>Average Price</td>
<td>4.17%</td>
<td>0.1256</td>
<td>0.63%</td>
<td>13.51%</td>
<td>21.61%</td>
</tr>
</tbody>
</table>

prices and the reference price, when the demand for the two products are identical, but their prices have different weights $\mu_i$ on the reference price. As expected, we find that prices are significantly lower when retailers are considering competition.

Figure 1: Transition of Prices and Reference Prices when Retailers Ignore Competition

Next, we study the effects of different demand parameters. We compute the loss of revenue when both retailers ignore the competition. The loss of revenue may be negative, meaning that the retailers may receive higher revenue when they both ignore competition. Figure 2 shows the effect of the magnitude of demand $a$ on the revenue loss. When the retailers have equal influence on the reference price, both their revenues increase when competition is ignored, and
the gain is increasing in $a$. When $\mu = (0.75, 0.25)$, both retailers still gain from ignoring competition, and the more influential retailer achieves more gain. However, when all of the influence lies with one retailer, ignoring the competition has no impact on that retailer’s revenue, but significantly reduces the revenue of the other. The loss of revenue of the retailer with no influence is increasing in $a$.

Figure 2: Effects of Magnitude of Demand on Revenue Loss

Figure 3 shows the effect of the price sensitivity of demand $b$ on the loss of revenue. Similarly, when the retailers have equal influence on the reference price, they receive more revenue when competition is ignored. However, the gain itself is decreasing in the price sensitivity $b$. When $\mu = (0.75, 0.25)$, both retailers gain from ignoring the competition, with the more influential retailer gaining more. Finally, the retailer that does not have any influence on the reference price experiences a loss of revenue when the competition is ignored. The loss in this case is decreasing in $b$.

Figure 3: Effects of Price Sensitivity of Demand on Revenue Loss

Figure 4 shows the impact of the magnitude of the reference price effect factor $c$ on the loss of revenue when both retailers ignore competition. We observe
that the loss/gain of the revenue is a non-monotonic function of the magnitude of the reference price effect factor $c$. For instance, when the two retailers have equal influence on the reference price, their revenue gains are initially increasing with $c$, but become decreasing once $c$ exceeds a certain threshold. The loss of revenue of the retailer that has no influence on the reference price is increasing in $c$.

![Figure 4: Effects of Magnitude of Reference Price Effects on revenue Loss](image)

Overall, we find the retailers are better off when they both ignore competition. However, this is only when both retailers have some influence on the reference price. The right panels of figures 1, 2, and 3 show that the loss revenue of retailer 1 is never positive when $\mu_1 \geq 0.5$. In fact, it is only positive when $\mu_1$ is close to zero (in these examples, when $\mu_1 \leq 0.18$).

## 6 Conclusion

In this paper, we considered a new competitive model where retailers interact through their influences on the customers’ reference price. In the case of two retailers selling a single product, we derived closed-form solutions for pricing policies for a retailer who is an optimizer when its competitor is following three different suboptimal policies and proved monotone convergence of these policies. When both retailers are optimizers, we provided a system of cubic equations, which can be solved numerically for the MP equilibrium prices. We provided a closed-form heuristic solution that is very close to the MP equilibrium prices. We found that competition reduces revenue in that revenue from equilibrium prices is lower than revenue from the case where both retailers ignore competition and optimize their prices. We intend to extend our model to consider pricing of multiple products in a duopoly market.
References


Received: April 15, 2014