Some One-Step Embedded Hybrid Schemes of Maximal Orders

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Abstract

Hybrid schemes are known to produce lower error constants, and hence more accurate than their standard counterparts. In this research work, some explicit and implicit Linear Multi-step Methods (LMMs) of maximal orders defined on the interval $[x_n, x_{n+k}]$ are embedded into one-step hybrid schemes defined on the interval $[x_n, x_{n+1}]$. The results show that the hybrid schemes have the same orders as the standard methods for each step number. However, the hybrid schemes have much lower error constants than the corresponding standard schemes. More importantly, the hybrid schemes are consistent and zero-stable, hence convergent, unlike the standard methods which are known to be unstable.

Keywords: Explicit and implicit schemes; hybrid schemes; order; zero-stability; consistency

1 Introduction

It is a known fact that the accuracy of a linear multi-step method (LMM) increases with its order. It is not surprising therefore, that researchers try to find LMMs of the highest possible orders to achieve maximum accuracy. However, by Dhalquist’s Barrier theorem, no $k$-step explicit linear multi-step
method can have order more than $2k - 1$, while an implicit $k$-step linear multi-step method can only attain a maximum order of $2k$ [2]. The high order LMMs also have stability problems [2] as according to Dhalquist’s theorem, no convergent LMM has order more than $k + 1$ for explicit schemes and $k + 2$ for implicit methods, hence the schemes of maximal orders are unstable. Hitherto, the unstable LMMs of high orders were only used for estimating starting values for other stable numerical methods. However, attempts have been made to circumvent the Dhalquist’s barrier by developing hybrid schemes through the introduction of off-grid points into them. Off-grid points are known to improve the order and stability of a numerical methods ([3], [1]). In particular, Onumanyi, et al [4] and Sirisena et al [5] reformulated the standard Backward Differentiation Formulae (BDF) by embedding them into one-step hybrid methods and the Simpson’s method by introducing some off-grid points, resulting into increased accuracy and stability properties. In this work, we intend to explore the orders of the schemes of maximal orders to obtain hybrid schemes with better accuracy and stability properties. We hope to achieve this by embedding the standard methods of maximal orders defined on the interval $[x_n, x_{n+k}]$ into one-step hybrid schemes defined on the interval $[x_n, x_{n+1}]$. The remaining parts of this work are organised as follows: Section 2 is dedicated to explicit and implicit schemes of maximal orders, while the explicit and implicit hybrid schemes will be the subject of section 3, some conclusions will be made in section 4.

2 Explicit and Implicit Schemes of Maximal Orders

In this section, we consider the standard explicit and implicit linear multi-step methods of maximal orders. The explicit Methods are given by the the general $k$-step linear multi-step method

$$\sum_{j=0}^{k} \alpha_j y_{n+j} - \sum_{j=0}^{k-1} \beta_j f_{n+j} = 0$$

Equation (1) is now subjected to the order conditions

$$C_p = \frac{1}{p!} \sum_{j=0}^{k} j^p \alpha_j - \frac{1}{(p-1)!} \sum_{j=0}^{k-1} j^{p-1} \beta_j = 0, C_{p+1} \neq 0$$

From (2), one obtains for a $k$-step method the system
\[
\begin{bmatrix}
1 & 1 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\
0 & 1 & 2 & \cdots & k & -1 & -1 & -1 & \cdots & -1 \\
0 & \frac{1}{2} & \frac{2^2}{2} & \cdots & \frac{k^2}{2} & 0 & -1 & -2 & \cdots & -k + 1 \\
0 & \frac{1}{3!} & \frac{3^3}{3!} & \cdots & \frac{k^3}{3!} & 0 & -\frac{1}{2} & -\frac{2^2}{2} & \cdots & -\frac{(k-1)^2}{2} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \frac{1}{(p-1)!} & \frac{2^{p-1}}{(p-1)!} & \cdots & \frac{k^{p-1}}{(p-1)!} & 0 & -\frac{1}{(p-2)!} & -\frac{2^{p-2}}{(p-2)!} & \cdots & -\frac{(k-1)^{p-1}}{(p-1)!} \\
0 & \frac{1}{p!} & \frac{2^p}{p!} & \cdots & \frac{k^p}{p!} & 0 & -\frac{1}{(p-1)!} & -\frac{2^{p-1}}{(p-1)!} & \cdots & -\frac{(k-1)^p}{(p-1)!}
\end{bmatrix}
\begin{bmatrix}
\alpha_0 \\
\alpha_1 \\
\vdots \\
\alpha_k \\
\beta_0 \\
\vdots \\
\beta_{k-1}
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
\vdots \\
0
\end{bmatrix}
\tag{3}
\]

where \( \lambda \) is a free parameter. Solving the system of equations (3) for \( k = 1(1)6 \), we obtain the standard explicit discrete schemes of maximal orders.
Similarly, we obtain the implicit discrete schemes of maximal orders as

\begin{equation}
\begin{aligned}
y_{n+1} &= y_n + \frac{h}{2} [f_{n+1} + f_n] \\
y_{n+2} - y_n &= \frac{h}{3} [f_{n+2} + 4f_{n+1} + f_n] \\
y_{n+3} + \frac{27}{11} y_{n+2} - \frac{27}{11} y_{n+1} - y_n &= \frac{3h}{11} [f_{n+3} + 9f_{n+2} + 9f_{n+1} + f_n] \\
y_{n+4} + \frac{32}{5} y_{n+3} - \frac{32}{5} y_{n+1} - y_n &= \frac{6h}{25} [f_{n+4} + 16f_{n+3} + 36f_{n+2} + 16f_{n+1} + f_n] \\
y_{n+5} + \frac{1625}{137} y_{n+4} + \frac{2000}{137} y_{n+3} - \frac{2000}{137} y_{n+2} - \frac{1625}{137} y_{n+1} - y_n &= \frac{30h}{137} [f_{n+5} + 25f_{n+4} + 100f_{n+3} + 100f_{n+2} + 25f_{n+1} + f_n] \\
y_{n+6} + \frac{132}{7} y_{n+5} + \frac{375}{7} y_{n+4} - \frac{375}{7} y_{n+2} - \frac{132}{7} y_{n+1} - y_n &= \frac{10h}{49} [f_{n+6} + 36f_{n+5} + 225f_{n+4} + 400f_{n+3} + 225f_{n+2} + 36f_{n+1} + f_n]
\end{aligned}
\end{equation}

It is worth noting here that both the explicit and implicit schemes of maximal orders are not zero-stable, hence they are not very useful for the solutions of differential equations, as the solutions will not converge due to propagation of errors. However, such schemes are useful in estimating starting values for other numerical methods. In this work, we intend to explore the high orders of the schemes to develop hybrid methods which are likely to be more accurate and possess better stability properties. The next section discusses the derivation and analysis of the explicit and implicit hybrid schemes.

### 3 Explicit and Implicit Hybrid Methods

In this section, we derive some hybrid methods by embedding the standard explicit and implicit schemes of maximal orders discussed in the last section into one-step methods. The general expression for the embedded explicit hybrid schemes is given by

\begin{equation}
\sum_{j=0}^{k} \alpha_{k} y_{n+\frac{j}{k}} - \sum_{j=0}^{k-1} \beta_{k} f_{n+\frac{j}{k}} = 0
\end{equation}

The general discrete schemes in (16) are then subjected to the order conditions

\begin{equation}
C_p = \frac{1}{p!} \sum_{j=0}^{k} \left( \frac{j}{k} \right)^p \alpha_{k} - \frac{1}{(p-1)!} \sum_{j=0}^{k-1} \left( \frac{j}{k} \right)^{p-1} \beta_{k} = 0, C_{p+1} \neq 0
\end{equation}
for a scheme of order $p$, and hence we obtain the system of equations similar to (2) whose solutions for $k = 2(1)6$ yield the following hybrid schemes

$$y_{n+1} + 4y_{n+\frac{1}{2}} - 5y_n = h[2f_{n+\frac{1}{2}} + f_n]$$  \hfill (18)

$$y_{n+1} + 18y_{n+\frac{3}{4}} - 9y_{n+\frac{1}{2}} - 10y_n = h[3f_{n+\frac{3}{4}} + 6f_{n+\frac{1}{2}} + f_n]$$  \hfill (19)

$$y_{n+1} + \frac{128}{3}y_{n+\frac{3}{4}} + 36y_{n+\frac{1}{2}} + 64y_{n+\frac{1}{4}} - \frac{47}{3}y_n = h[4f_{n+\frac{3}{4}} + 18f_{n+\frac{1}{2}} + 12f_{n+\frac{1}{4}} + f_n]$$  \hfill (20)

$$y_{n+1} + \frac{475}{6}y_{n+\frac{1}{2}} + \frac{700}{3}y_{n+\frac{3}{8}} - 100y_{n+\frac{1}{8}} - \frac{575}{3}y_{n+\frac{1}{4}} - \frac{131}{6}y_n = h[5f_{n+\frac{1}{2}} + 40f_{n+\frac{3}{8}} + 60f_{n+\frac{1}{8}} + 20f_{n+\frac{1}{4}} + f_n]$$  \hfill (21)

$$y_{n+1} + \frac{642}{5}y_{n+\frac{3}{5}} + 750y_{n+\frac{5}{8}} + 400y_{n+\frac{1}{2}} - 825y_{n+\frac{3}{8}} - 426y_{n+\frac{1}{8}} - \frac{142}{5}y_n = h[6f_{n+\frac{3}{5}} + 75f_{n+\frac{5}{8}} + 200f_{n+\frac{1}{2}} + 150f_{n+\frac{3}{8}} + 30f_{n+\frac{1}{8}} + f_n]$$  \hfill (22)

The explicit hybrid schemes (19)-(22) have the same orders as their corresponding standard methods of maximal orders. However, they have much lower error constants. It is also interesting to note that the hybrid methods are consistent and zero-stable. The error constants are displayed on table 1 below.

<table>
<thead>
<tr>
<th>Step-number (k)</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Order</td>
<td>3</td>
<td>5</td>
<td>7</td>
<td>9</td>
<td>11</td>
</tr>
<tr>
<td>Error constant</td>
<td>$\frac{1}{96}$</td>
<td>$\frac{1}{156383}$</td>
<td>$\frac{1}{458762}$</td>
<td>$\frac{1}{21660837500}$</td>
<td>$\frac{1}{2011346878464}$</td>
</tr>
</tbody>
</table>

Table 1: Orders and error constants of the explicit hybrid schemes for $k = 2(1)6$.

There is a very clear difference between the error constants for the standard and the hybrid schemes as can be seen in table 2 below.

<table>
<thead>
<tr>
<th>Step-number (k)</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Standard scheme</td>
<td>$1.67 \times 10^{-1}$</td>
<td>$5.00 \times 10^{-2}$</td>
<td>$1.43 \times 10^{-2}$</td>
<td>$3.97 \times 10^{-4}$</td>
<td>$1.08 \times 10^{-3}$</td>
</tr>
<tr>
<td>Hybrid scheme</td>
<td>$1.04 \times 10^{-2}$</td>
<td>$6.86 \times 10^{-5}$</td>
<td>$2.18 \times 10^{-7}$</td>
<td>$4.06 \times 10^{-10}$</td>
<td>$4.97 \times 10^{-13}$</td>
</tr>
</tbody>
</table>

Table 2: Error constants of the standard and hybrid explicit schemes.

The implicit hybrid methods can be obtained in a similar manner, leading
to the following discrete schemes:

\[ y_{n+1} - y_n = \frac{h}{6}[f_{n+1} + 4f_{n+\frac{1}{2}} + f_n] \]  \hspace{1cm} (23)
\[ y_{n+1} + \frac{27}{11}y_{n+\frac{2}{3}} - \frac{27}{11}y_{n+\frac{1}{3}} - y_n = \frac{h}{11}[f_{n+1} + 9f_{n+\frac{2}{3}} + 9f_{n+\frac{1}{3}} + f_n] \]  \hspace{1cm} (24)
\[ y_{n+1} + \frac{32}{5}y_{n+\frac{3}{4}} - \frac{32}{5}y_{n+\frac{1}{4}} - y_n \]  \hspace{1cm} (25)

\[ = \frac{3h}{50}[f_{n+1} + 16f_{n+\frac{3}{4}} + 36f_{n+\frac{1}{4}} + 16f_{n+\frac{1}{2}} + f_n] \]
\[ y_{n+1} + \frac{1625}{137}y_{n+\frac{3}{4}} + \frac{2000}{137}y_{n+\frac{1}{2}} - \frac{2000}{137}y_{n+\frac{1}{3}} - \frac{1625}{137}y_{n+\frac{1}{4}} - y_n \]  \hspace{1cm} (26)
\[ = \frac{6h}{137}[f_{n+1} + 25f_{n+\frac{3}{4}} + 100f_{n+\frac{1}{2}} + 100f_{n+\frac{1}{3}} + 25f_{n+\frac{1}{4}} + f_n] \]
\[ y_{n+1} + \frac{132}{7}y_{n+\frac{3}{4}} + \frac{375}{7}y_{n+\frac{1}{2}} - \frac{375}{7}y_{n+\frac{1}{3}} - \frac{132}{7}y_{n+\frac{1}{4}} - y_n \]  \hspace{1cm} (27)

\[ = \frac{5h}{147}[f_{n+1} + 36f_{n+\frac{3}{4}} + 225f_{n+\frac{1}{2}} + 400f_{n+\frac{1}{3}} + 225f_{n+\frac{1}{4}} + 36f_{n+\frac{1}{2}} + f_n] \]

The implicit hybrid schemes are also consistent and zero-stable. Their error constants are as displayed in table 3.

<table>
<thead>
<tr>
<th>Step-number ((k))</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Order</td>
<td>4</td>
<td>6</td>
<td>8</td>
<td>10</td>
<td>12</td>
</tr>
<tr>
<td>Error constant</td>
<td>(-\frac{1}{280})</td>
<td>(-\frac{1}{11200})</td>
<td>(-\frac{1}{6881200})</td>
<td>(-\frac{1}{618105468750})</td>
<td>(-\frac{5}{3843683884744704})</td>
</tr>
</tbody>
</table>

Table 3: Orders and error constants of the implicit hybrid schemes for \(k = 2(1)6\).

It is clear again from table 4 that the implicit hybrid schemes are much more accurate than the standard implicit methods.

<table>
<thead>
<tr>
<th>Step-number ((k))</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Standard scheme</td>
<td>(-1.11E^{-2})</td>
<td>(-1.95E^{-3})</td>
<td>(-3.81E^{-4})</td>
<td>(-7.90E^{-5})</td>
<td>(-3.40E^{-6})</td>
</tr>
<tr>
<td>Hybrid scheme</td>
<td>(-3.47E^{-4})</td>
<td>(-8.91E^{-7})</td>
<td>(-1.45E^{-9})</td>
<td>(-1.62E^{-12})</td>
<td>(-2.60E^{-16})</td>
</tr>
</tbody>
</table>

Table 4: Error constants of the standard and hybrid implicit schemes.

## 4 Conclusions

In this research work, some explicit and implicit methods of maximal orders for \(k = 2(1)6\) have been embedded into one-step hybrid methods. While it
is a known fact that the standard schemes of maximal orders are unstable, the embedded hybrid schemes are consistent and zero-stable, hence they are convergent. Furthermore, the hybrid schemes have much lower error constants than the standard ones, which implies that they are more accurate (see tables 2 and 4).

References


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