On a KdV6 with Forcing Term:
Exact Solutions

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Abstract

The homogeneous sixth-order KdV equation (KdV6) has been studied by several authors from the point of view of its integrability and exact solutions. However, in the case that a forcing term $F(t)$ is considered, the results presented here are new. In this work, the improved generalized tanh-coth method is used to obtain exact solutions to the sixth-order bidirectional Kaup–Kupershmidt equation with forcing term (bKKf).

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1 Introduction

Many problems in the natural sciences and engineering are modeled by using nonlinear partial differential equations (NLPDE). One of the most important models in the study of propagation of the shallow water waves, is given by the Korteweg-de Vries equation (KdV) which was solved by the inverse scattering method in 1962. Since the task of finding exact solutions of NLPDE’s is one of the most active areas of investigation in applied mathematics. In the last years, a great variety of direct and computational methods have been implemented
for obtain exact solutions of these type of equations. One of the most used computational method and that have been applied successfully to solve several NLEE’s is given by the tanh-coth method [4] or the most general improved generalized tanh-coth method [1]. The general sixth-order KdV equation with forcing term is given by the following equation

\[ u_{xxxxxx} + au_{x}u_{xxxx} + bu_{xx}u_{xxx} + cu_{x}^{2}u_{xx} + du_{tt} + eu_{xxxt} + fu_{x}u_{xt} + gu_{t}u_{xx} = F(t), \] (1)

where, \(a, b, c, d, e, f, g\) are arbitrary parameters, not all zero, \(u(x, t)\) is a differentiable function in the independent variables \(x\) and \(t\), and \(F(t)\) is an external forcing function varying with the time \(t\).

In the particular case \(F(t) = 0\), several forms of (1) can be constructed changing the values of the parameters. Some of these models have been studied from the point of view of its integrability and exact solutions (see for instance [1],[3], [5] and references therein). In this work, we consider the particular sixth-order bidirectional Kaup–Kupershmidt equation with forcing term (bKKf) [3] given by the equation,

\[ u_{xxxxxx} + \frac{75}{2} u_{xx}u_{xxx} + 15u_{x}u_{xxxx} + 45(u_{x})^{2}u_{xx} - 15u_{xx}u_{t} - 15u_{x}u_{xt} - 5u_{xxxt} - 5u_{tt} = F(t), \] (2)

where \(F(t) \neq 0\), and using the improved generalized tanh-coth method [1] we obtain exact solutions for them.

## 2 Exact solutions to bKKf equation

We assume that (2) has solution of the form

\[
\begin{aligned}
\left\{ 
  u(x, t) &= v(\xi) - \frac{1}{5} \int \int F(t) dt, \\
  \xi &= x + \int h(t) dt,
\right.
\end{aligned}
\] (3)

where \(h(t)\) is a function of \(t\) to be determined later. Substituting (3) into (2) and after simplifications, we have

\[
\begin{aligned}
  \left( v''''''(\xi) + \frac{45}{4} (v''(\xi))^{2} + 15v'(\xi)v'''(\xi) + \\
  \frac{45}{3} (v'(\xi))^{3} - 15h(t)(v'(\xi))^{2} + 3q(t)v'(\xi) - 5h^{2}(t)v'(\xi) - 5h(t)v''(\xi) \right)' &= 0,
\end{aligned}
\] (4)
On a KdV6 with forcing term

where

\[ q(t) = \int F(t) dt. \]

Integrating (4) once with respect to \( \xi \) we get

\[
v''''''(\xi) + \frac{45}{4}(v''(\xi))^2 + 15v'(\xi)v'''(\xi) + \frac{45}{3}(v'(\xi))^3 - 15h(t)(v'(\xi))^2 + 3q(t)v'(\xi) - 5h^2(t)v'(\xi) - 5h(t)v'''(\xi) = c, \tag{5}\]

where \( c \) is the integration constant. As we look for the exact solutions of special form, we can set \( c = 0 \). Using the idea of the improved generalized tanh-coth method we seek the solution to (4) using the expansion

\[
v(\xi) = \sum_{i=0}^{M} a_i(t) \phi(\xi)^i + \sum_{i=M+1}^{2M} a_i(t) \phi(\xi)^{M-i}, \tag{6}\]

where \( M \) is a positive integer that will be determined by using the balance method, \( \phi = \phi(\xi) \) satisfies the Riccati equation

\[
\phi'(\xi) = \alpha(t) + \beta(t)\phi(\xi) + \gamma(t)\phi(\xi)^2, \tag{7}\]

and \( a_i(t), i = 1, 2, \ldots, 2M, \alpha(t), \beta(t), \gamma(t) \) are functions of the variable \( t \). The solution of (7) in the case \( \beta(t)^2 - 4\alpha(t)\gamma(t) \neq 0 \) is given by (see [2]):

\[
\phi(\xi) = \sqrt{\beta(t)^2 - 4\alpha(t)\gamma(t)} \tanh\left[-\frac{1}{2}\sqrt{\beta(t)^2 - 4\alpha(t)\gamma(t)}\xi + \xi_0\right] - \beta(t) \tag{8}\]

being \( \xi_0 \) an arbitrary constant.

Substituting (6) into (5), and balancing \( v'''(\xi) \) with \( v'(\xi)v'''(\xi) \) we obtain

\[ M + 5 = 2M + 4, \]

so that

\[ M = 1. \]

Therefore, (6) reduces to

\[
v(\xi) = a_0(t) + a_1(t)\phi(\xi) + a_2(t)(\phi(\xi))^{-1}. \tag{9}\]
Now, substituting (9) into (5) (with $c = 0$), taking in account (7) and equating to zero the coefficients of all powers of $\phi(\xi)$, we get a set of algebraic equations for $a_0(t), a_1(t), a_2(t), \alpha(t), \beta(t), \gamma(t), h(t)$. Solving the system with aid the Mathematica, we obtain a set of nine nontrivial solutions, however, by sake of simplicity we put here the following two solutions which are the most general of them:

$$h(t) = \pm 2\sqrt{\frac{3}{29}} \int F(t)dt, \quad a_1 = \pm \sqrt{\frac{3}{29}} \int F(t)dt \quad a_2 = 8\alpha(t),$$

$$\gamma(t) = \mp \sqrt{\frac{3}{29}} \int F(t)dt, \quad \beta(t) = 0. \quad (10)$$

$$h(t) = \pm \frac{1}{\sqrt{3}} \int F(t)dt, \quad a_1 = \pm \frac{\int F(t)dt}{3\sqrt{3}\alpha(t)}, \quad a_2 = \alpha(t),$$

$$\gamma(t) = \mp \frac{\int F(t)dt}{4\sqrt{3}\alpha(t)}, \quad \beta(t) = 0. \quad (11)$$

We consider the solution to (2) respect to (10). The other case can be obtain in a similar way. Now, using (3), (8) and (9) we have:

$$u(x, t) = a_0 \pm \sqrt{\frac{3}{29}} \int F(t)dt \quad \frac{1}{\alpha(t)} \left\{ \frac{1}{\gamma(t)} (-\sqrt{-\alpha(t)\gamma(t)} \tanh[\sqrt{-\alpha(t)\gamma(t)}(x + \int h(t)dt)]) \right\} +$$

$$8\alpha(t) \left\{ \frac{1}{\gamma(t)} (-\sqrt{-\alpha(t)\gamma(t)} \tanh[\sqrt{-\alpha(t)\gamma(t)}(x + \int h(t)dt)]) \right\}^{-1} - \frac{1}{5} \int F(t)dt, \quad (12)$$

where $h(t) = \pm 2\sqrt{\frac{3}{29}} \int F(t)dt, \quad \gamma(t) = \mp \sqrt{\frac{3}{29}} \int F(t)dt, \quad a_0$ is an arbitrary constant and $\alpha(t)$ is a function depending on $t$ which we can assumed constant and not zero.

3 Conclusions

We have used the improved generalized tanh-coth method to solve a particular case of the sixth-order bidirectional Kaup–Kupershmidt equation with forcing.
term (bKKf). The results obtained here are new in the literature with respect to this model with \( F(t) \neq 0 \). Varying the coefficients of the (1) we can obtain a variety of models of KdV6 equations with forcing term which can solved by the method used here.

References


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