Regularity Criteria for the Magneto-micropolar Fluid Equations in Terms of Direction of the Velocity

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Abstract

This paper is dedicated to study of the Cauchy problem for the Magneto-micropolar fluid equations. We obtain a new regularity criterion for the system in terms of the direction of the velocity in the Morrey-Campaanto space.

Keywords: Regularity criterion, Magneto-micropolar fluid equations, Morrey-Campaanto space

1 Introduction and main results

In this paper, we are concerned with following Magneto-micropolar fluid equations in $\mathbb{R}_+ \times \mathbb{R}^3$

\[
\begin{aligned}
\partial_t u - (\mu + \chi) \Delta u + u \cdot \nabla u - b \cdot \nabla b + \nabla (P + b^2) - \chi \nabla \times \omega &= 0, \\
\partial_t \omega - \gamma \Delta \omega - \kappa \nabla \text{div} \omega + 2\chi \omega + u \cdot \nabla \omega - \chi \nabla \times u &= 0, \\
\partial_t b - \nu \Delta b + u \cdot \nabla b - b \cdot \nabla u &= 0, \\
\text{div} u &= \text{div} b = 0, \\
u(0, x) &= u_0(x), \omega(0, x) = \omega_0(x), b(0, x) = b_0(x),
\end{aligned}
\]

(1.1)

where $u(t, x) = (u_1(t, x), u_2(t, x), u_3(t, x)) \in \mathbb{R}^3$ denotes the velocity of the fluid at a point $x \in \mathbb{R}^3, t \in [0, T], \omega(t, x) \in \mathbb{R}^3, b(t, x) \in \mathbb{R}^3$ and $P(t, x) \in \mathbb{R}$ denote, respectively, the
micro-rotational velocity, the magnetic field and the hydrostatic pressure. $\mu, \chi, \kappa, \gamma, \nu$ are positive numbers associated to properties of the material: $\mu$ is the kinematic viscosity, $\chi$ is the vortex viscosity, $\kappa$ and $\gamma$ are spin viscosities, and $\frac{1}{\nu}$ is the magnetic Reynold. $u_0, \omega_0, b_0$ are initial data for the velocity, the angular velocity and the magnetic field with properties $\text{div} u_0 = 0$ and $\text{div} b_0 = 0$.

When $b = 0$, the equation (1.1) reduces to the micropolar fluid system. Micropolar fluid system was first proposed by Eringe[2] in 1966. Using linearization and an almost fixed point thereom, in 1988, Lukaszewicz[4] established the global existence of weak solutions with sufficiently regular initial data. In 1989, using the same technique, Lukaszewicz[5] proved the local and global existence and the uniqueness of the strong solutions. In 2010, Yuan [9] established regularity criteria in Lorentz(weak $L^p$) space. When both $\omega = 0$ and $\chi = 0$, then the system(1.1) reduces to be the magneto-hydrodynamic (MHD) equations, which has been studied extensively in [10-17].


In[18], A. Vasseur considered regularity criterion for 3D Navier-Stokes equations in the direction of the velocity field $\frac{u(x,t)}{|u(x,t)|}$ and showed

$$\text{div}(\frac{u}{|u|}) \in L^p(0, \infty; L^q(\mathbb{R}^3)), \text{ with } \frac{2}{p} + \frac{3}{q} \leq \frac{1}{2}, \quad q \geq 6, \text{ and } p \geq 4, \quad (1.3)$$

then $u$ is smooth on $(0, \infty) \times \mathbb{R}^3$.

Very recently, Luo[19] described the regularity criterion of weak solution to the MHD equations using the direction of the velocity field $\frac{u}{|u|}$. He showed that if the initial value $u_0, b_0 \in H^1(\mathbb{R}^3)$ with

$$\text{div}(\frac{u}{|u|}) \in L^p(0, T; L^q(\mathbb{R}^3)), \text{ with } \frac{2}{p} + \frac{3}{q} \leq \frac{1}{2}, \quad q \geq 6, \text{ and } p \geq 4, \quad (1.4)$$

and

$$b \in L^\alpha(0, \infty; L^\beta(\mathbb{R}^3)), \quad \text{ with } \frac{2}{\alpha} + \frac{3}{\beta} \leq 1, \quad \beta \geq 3,$$

then $(u, b)$ is smooth on $(0, T) \times \mathbb{R}^3$.

Motivated by the above works, we consider the Magneto-micropolar fluid equations in the Morrey-Campaanto space, which is defined in Section 2. We obtain a regularity
criterion for Magneto-micropolar fluid equations in terms of the direction of the velocity. More precisely, we will prove

**Theorem 1.1.** Let \((u, \omega, b)\) be a weak solution to Magneto-micropolar fluid equations with \(u_0, \omega_0, b_0 \in L^2(\mathbb{R}^3) \cap L^q(\mathbb{R}^3), q > 3\). If the solution \((u, \omega, b)\) of the system (1.1) satisfies the following condition

\[
\text{div}(\frac{u}{|u|}) \in L^{2/(1-r)}(0, T; \dot{\mathcal{M}}_{2,3/r}(\mathbb{R}^3)), \quad 0 < r < 1,
\]

then \((u, \omega, b)\) is smooth on \((0, T) \times \mathbb{R}^3\).

**Remark 1.1.** The definition of weak solutions to the Magneto-micropolar equations (1.1) can be found in [23], here we omit it.

## 2 Preliminaries and Lemmas

Firstly, we recall the definition and some properties of the space we are going to use. This kind of space play an important role in studying the regularity of solutions to partial differential equations (see [22] and references therein). Now we recall the definition of Morrey-Campanato spaces.

**Definition 2.1.** For \(1 < p \leq q \leq +\infty\), the Morrey-Campanato space \(\dot{\mathcal{M}}_{p,q}(\mathbb{R}^3)\) is defined as

\[
\dot{\mathcal{M}}_{p,q}(\mathbb{R}^3) = \{ f \in L^p_{\text{loc}}(\mathbb{R}^3) : \| f \|_{\dot{\mathcal{M}}_{p,q}} = \sup_{x \in \mathbb{R}^3} \sup_{R>0} R^{3/q-3/p} \| f \|_{L^p(B(x,R))} < \infty \},
\]

where \(B(x, R)\) denotes the ball of center \(x\) with radius \(R\).

It is easy to check that

\[
\| f(\lambda \cdot) \|_{\dot{\mathcal{M}}_{p,q}} = \frac{1}{\lambda^{3/q}} \| f \|_{\dot{\mathcal{M}}_{p,q}}, \quad \lambda > 0,
\]

\[
\dot{\mathcal{M}}_{p,\infty}(\mathbb{R}^3) = L^\infty(\mathbb{R}^3) \quad \text{for } 1 \leq p \leq \infty.
\]

Additionally, for \(2 \leq p \leq 3/r\) and \(0 \leq r < 3/2\) we have the following embedding relations:

\[
L^{3/r}(\mathbb{R}^3) \hookrightarrow L^{3/r,\infty}(\mathbb{R}^3) \hookrightarrow \dot{\mathcal{M}}_{p,3/r}(\mathbb{R}^3),
\]

where \(L^{p,\infty}\) denotes the weak \(L^p\)–space. The second relation

\[
L^{3/r,\infty}(\mathbb{R}^3) \hookrightarrow \dot{\mathcal{M}}_{p,\frac{3}{r'}}(\mathbb{R}^3),
\]
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is shown as follows:

\[
\|f\|_{\mathcal{M}_{p,3/r}} \leq \sup_E |E|^\frac{3}{r-3/p} \left( \int_E |f(y)|^p dy \right)^{1/p} \quad (f \in L^{3/r,\infty}(\mathbb{R}^3))
\]

\[
= \left( \sup_{R>0} |R|^{3/p} \right)^\frac{1}{p} \left( \int_{|x|>R} |f(y)|^p dy \right)^{1/p}
\]

\[
\cong \left( \sup_{R>0} \left| \{ x \in \mathbb{R}^3 : |f(y)|^p > R \} \right|^{pr/3} \right)^{1/p}
\]

\[
= \sup_{R>0} \left| \{ x \in \mathbb{R}^3 : |f(y)| > R \} \right|^{r/3}
\]

\[
\cong \|f\|_{L^{3/r,\infty}}.
\]

**Lemma 2.1.** [20] For \(0 \leq r < 3/2\), the space \(\dot{Z}_r(\mathbb{R}^3)\) is defined as the space of \(f(x) \in L^2_{\text{loc}}(\mathbb{R}^3)\) such that

\[
\|f\|_{\dot{Z}_r} = \sup_{\|g\|_{B^r_{2,1}} \leq 1} \|fg\|_{L^2} < \infty. \tag{2.5}
\]

Then \(f \in \dot{M}_{2,3/r}(\mathbb{R}^3)\) if and only if \(f \in \dot{Z}_r(\mathbb{R}^3)\) with equivalence of norms.

**Lemma 2.2.** [21] For \(0 < r < 1\), we have

\[
\|f\|_{B^r_{2,1}} \leq C\|f\|_{L^2}^{1-r/2}\|\nabla f\|_{L^2}, \tag{2.6}
\]

where \(C\) only depends on \(r\).

**Remark 2.1.** For \(2 \leq p \leq 3/r\) and \(0 \leq r < 3/2\), by the embedding (2.3) we can see that our results extend and improve the known results in [18-19].

**Remark 2.2.** Compared with the known result [19] for the MHD equations, our regularity criterion only needs velocity field \(u\). Therefore, our result improves the known results in [19].

### 3 Proofs of the main results

In this section, we prove Theorem 1.1.

**Proof of Theorem 1.1.** To prove the theorem we need the \(L^4\) a priori estimate. For this purpose, we take the inner product of the first equation of (1.1) with \(|u|^2 u\) and integrate by parts, it can be deduced that

\[
\frac{1}{4} \frac{d}{dt} \|u\|_{L^4}^4 + (\mu + \chi)\|\nabla u\|_{L^2}^2 + \frac{1}{2}(\mu + \chi)\|\nabla |u|^2\|_{L^2}^2
\]

\[
\leq \int_{\mathbb{R}^3} |P||u|^2 \nabla u |dx + 3\chi \int_{\mathbb{R}^3} |u||u|^2 \nabla u |dx - \int_{\mathbb{R}^3} |b||\nabla (|u|^2 u)||b|dx, \tag{3.1}
\]

where we used the following relations by the divergence free condition \(\text{div} u = 0\):

\[
\int_{\mathbb{R}^3} u \cdot \nabla u \cdot |u|^2 u dx = \frac{1}{2} \int_{\mathbb{R}^3} u \cdot \nabla |u|^4 dx = 0,
\]
\[
\int_{\mathbb{R}^3} \Delta u \cdot |u|^2 \, dx = - \int_{\mathbb{R}^3} |\nabla u|^2 |u|^2 \, dx - \frac{1}{2} \int_{\mathbb{R}^3} |\nabla |u|^2|^2 \, dx,
\]
\[
\int_{\mathbb{R}^3} \nabla \times \omega \cdot |u|^2 \, dx = - \int_{\mathbb{R}^3} |u|^2 \omega \cdot \nabla \times u \, dx - \int_{\mathbb{R}^3} \omega \cdot \nabla |u|^2 \times u \, dx,
\]
and
\[
|\nabla \times u| \leq |\nabla u|, \quad |\nabla |u|| \leq |\nabla u|.
\]

Similarly, we take the inner product of the second equation of (1.1) with $|\omega|^2 \omega$ and integrate by parts, it can be deduced that
\[
\frac{1}{4} \frac{d}{dt} \|\omega\|_{L^4}^4 + \gamma \|\nabla \omega\|_{L^4}^2 + \frac{\gamma}{2} \|\nabla |\omega|^2\|_{L^2}^2 + k \|\text{div} \omega\|_{L^2}^2 + 2 \chi \|\omega\|_{L^4}^4 \leq 3 \chi \int_{\mathbb{R}^3} |u| |\omega|^2 |\nabla \omega| \, dx. \tag{3.2}
\]

Using an argument similar to that used in deriving the estimate (3.1)-(3.2), it can be obtained for the third equation of (1.1) that
\[
\frac{1}{4} \frac{d}{dt} \|b\|_{L^4}^4 + \|\nabla b\|_{L^4}^2 + 2 \|\nabla |b|^2\|_{L^2}^2 \leq \int_{\mathbb{R}^3} |b| \|\nabla (|b|^2 b)\| |u| \, dx. \tag{3.3}
\]

Adding up (3.1), (3.2) and (3.3), then we obtain
\[
\frac{1}{4} \frac{d}{dt} (\|u\|_{L^4}^4 + \|\omega\|_{L^4}^4 + \|b\|_{L^4}^4) + (\mu + \chi) \|\nabla u\|_{L^2}^2 + \frac{1}{2} (\mu + \chi) \|\nabla |u|^2\|_{L^2}^2 + \gamma \|\nabla \omega\|_{L^2}^2
\]
\[
+ \frac{\gamma}{2} \|\nabla |\omega|^2\|_{L^2}^2 + k \|\text{div} \omega\|_{L^2}^2 + 2 \chi \|\omega\|_{L^4}^4 + \|\nabla b\|_{L^2}^2 + 2 \|\nabla |b|^2\|_{L^2}^2 \leq 2 \int_{\mathbb{R}^3} |P| |u|^2 |\nabla u| \, dx + 3 \chi \int_{\mathbb{R}^3} |u| |\omega|^2 |\nabla u| \, dx
\]
\[
- \int_{\mathbb{R}^3} |b| \|\nabla (|u|^2 u)\| |b| \, dx + \int_{\mathbb{R}^3} |b| \|\nabla (|b|^2 b)\| |u| \, dx
\]
\[
\triangleq I_1 + I_2 + I_3 + I_4 + I_5. \tag{3.4}
\]

Applying the Hölder inequality and the Young inequality for $I_2$, it follows that
\[
I_2 \leq \frac{\chi + \mu}{2} \|\nabla u\|_{L^2}^2 + C (\|u\|_{L^4}^4 + \|\omega\|_{L^4}^4) \tag{3.5}
\]

Arguing similarly to above it can be derived for $I_3$ that
\[
I_3 \leq \frac{\gamma}{2} \|\nabla \omega\|_{L^2}^2 + C (\|u\|_{L^4}^4 + \|\omega\|_{L^4}^4). \tag{3.6}
\]
Applying the divergence operator div to the first equation of (1.1), one formally has

\[ P = \sum_{i,j=1}^{3} R_i R_j (u_i u_j - b_i b_j), \]

where \( R_j \) denotes the \( j \)-th Riesz operator. By the boundedness of Riesz operator and applying the Hölder inequality again to obtain that

\[ \| P \|_{L^q} \leq C(\| u \|_{L^{2q}}^2 + \| b \|_{L^{2q}}^2), \quad \text{for } 1 < q < \infty. \]

Since \( (u, b) \in L^2(\mathbb{R}^3) \cap L^6(\mathbb{R}^3) \) and \( H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3) \), by the Hölder inequality and the Young inequality, we get

\[
\| Pu \|_{L^2}^2 \leq \| P \|_{L^2} \| Pu \|_{L^2} \leq C(\| u \|_{L^4}^2 + \| b \|_{L^4}^2) \| P \|_{L^2} \| u \|_{L^6}^2
\]
\[
\leq C(\| u \|_{L^4}^2 + \| b \|_{L^4}^2)(\| u \|_{L^6}^2 + \| b \|_{L^6}^2) \| \nabla |u| \|_{L^2}
\]
\[
\leq C(\| u \|_{L^4}^2 + \| b \|_{L^4}^2)^2 + \frac{\chi+\mu}{8} \| \nabla |u| \|_{L^2}^2
\]
\[
\leq C(\| u \|_{L^4}^4 + \| b \|_{L^4}^4) + \frac{\chi+\mu}{8} \| \nabla |u| \|_{L^2}^2.
\]

By (2.6) and (3.7) we find using Hölder inequality and Young inequality

\[
I_1 \leq \int_{\mathbb{R}^3} \| Pu \|_{L^2}^2 \left| \frac{u}{|u|} \cdot \nabla |u| \right| dx \leq \int_{\mathbb{R}^3} \| P \|_{L^2} \| u \|_{L^4} \| \nabla |u| \|_{L^2} \left| \frac{u}{|u|} \right| dx
\]
\[
\leq \| Pu \|_{L^2} \| u \|_{L^4} \| \nabla |u| \|_{L^2} \leq \| Pu \|_{L^2} \| u \|_{B_r^\frac{1}{2}} \| \nabla |u| \|_{M_r^\frac{1}{4}}
\]
\[
\leq \| Pu \|_{L^2} \| u \|_{L^4} \| \nabla |u| \|_{L^2} \| \nabla |u| \|_{B_r^\frac{1}{2}} \| \nabla |u| \|_{M_r^\frac{1}{4}}
\]
\[
\leq \| Pu \|_{L^2} \| u \|_{L^4} \| \nabla |u| \|_{L^2} \| \nabla |u| \|_{B_r^\frac{1}{2}} \| \nabla |u| \|_{M_r^\frac{1}{4}}
\]
\[
\leq C(\| u \|_{L^4}^4 + \| b \|_{L^4}^4) + \frac{\chi+\mu}{4} \| \nabla |u| \|_{L^2}^2 + C \| u \|_{L^4} \| \nabla |u| \|_{B_r^\frac{1}{2}} \| \nabla |u| \|_{M_r^\frac{1}{4}},
\]

here we use the fact

\[ |u| \nabla \left( \frac{u}{|u|} \right) = -\frac{u}{|u|} \cdot \nabla |u|. \]

Next we estimate the term \( I_4 \). As in [25], we have

\[ I_4 \leq \int_{\mathbb{R}^3} \| b \|_{L^2}^2 \| u \|_{L^2} \| \nabla |u| \|_{L^2} dx. \]
Since \( u \in L^2(\mathbb{R}^3) \cap L^6(\mathbb{R}^3) \), using Cauchy inequality, generalized Hölder inequality, Gagliardo-Nirenberg inequality and Sobolev imbedding theorem, we obtain

\[
I_4 \leq C |||b|^2|u|||_2 \| \nabla |u|^2 ||_2 \leq C |||b|^2|u|||_2^2 + \frac{\chi + \mu}{4} \| \nabla |u|^2 ||_2^2 \\
\leq C |||b|^2||^2_{L^6} \| u||^2_{L^3} + \frac{\chi + \mu}{4} \| \nabla |u|^2 ||^2_{L^2} \\
\leq C \| \nabla |b|^2 ||^2_{L^2} \| u||L^2 \| \nabla |u|^2 ||_{L^2} + \frac{\chi + \mu}{4} \| \nabla |u|^2 ||^2_{L^2} \\
\leq C |||b|\nabla |b|||_{L^2}^2 + \frac{\chi + \mu}{4} \| \nabla |u|^2 ||^2_{L^2}. 
\]  

(3.9)

The last term of (3.4) can be treated in the same way,

\[
I_5 \leq C \int_{\mathbb{R}^3} |b|^2|u|\nabla |b|^2|dx \leq C |||b|^2|u|||_{L^2} + \frac{1}{8} \| \nabla |b|^2 ||_{L^2} \leq C |||b|\nabla |b|||_{L^2}^2. 
\]  

(3.10)

Putting (3.5), (3.6), (3.8), (3.9) and (3.10) into (3.4), we obtain

\[
\frac{1}{4} \frac{d}{dt} (||u||^4_{L^4} + ||\omega||^4_{L^4} + ||b||^4_{L^4}) + \mu + \frac{\chi}{2} \| \nabla |u|^2 \|_{L^2}^2 + \frac{\gamma}{2} \| \nabla \omega \|_{L^2}^2 + \frac{\gamma}{2} \| \nabla |\omega|^2 \|_{L^2}^2 \\
+ k \| \text{div} |\omega|^2 \|_{L^2}^2 + 2\chi \| \omega \|_{L^4}^4 + \| \nabla |b|^2 \|_{L^2}^2 \\
\leq C(||u||^4_{L^4} + ||b||^4_{L^4} + ||\omega||^4_{L^4}) + C(||u||^4_{L^4} + ||b||^4_{L^4} + ||\omega||^4_{L^4}) \| \text{div} (\frac{u}{|u|^2}) \|_{M^2_{2,\frac{3}{r}}}^2 \\
\leq C(||u||^4_{L^4} + ||b||^4_{L^4} + ||\omega||^4_{L^4}) \left( 1 + \| \text{div} (\frac{u}{|u|^2}) \|_{M^2_{2,\frac{3}{r}}}^2 \right), 
\]  

(3.11)

which gives that by the Gronwall inequality

\[
\sup_{0<t<T} (||u||^4_{L^4} + ||\omega||^4_{L^4} + ||b||^4_{L^4}) \leq (||u_0||^4_{L^4} + ||\omega_0||^4_{L^4} + ||b_0||^4_{L^4}) \exp \left( C \int_0^T \left( 1 + \| \text{div} (\frac{u}{|u|^2}) \|_{M^2_{2,\frac{3}{r}}}^2 \right) dt \right). 
\]  

(3.12)

By standard arguments of continuation of local solutions, we conclude that the solutions \((u(t,x), \omega(t,x), b(t,x))\) can be extended beyond \( t = T \) provided that \( \text{div} (\frac{u}{|u|^2}) \in L^{2/(1-r)}(0,T; M^2_{2,\frac{3}{r}}(\mathbb{R}^3)) \), \( 0 < r < 1 \). This completes the proof of Theorem 1.1.

References


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