Estimating the Mean of an Exponential Population 
under Progressively Type-II Censored Sample

Suchandan Kayal

Department of Mathematics
National Institute of Technology Rourkela
Rourkela-769008, India

Abstract

In this article, the problem of estimating the mean of an exponential population is studied based on progressively Type-II censored data under linex loss function. Bayes, generalized Bayes and empirical Bayes estimators are derived. Further, some admissibility and inadmissibility results in a linear class of estimators are obtained. The best scale equivariant estimator is shown to be admissible and minimax.

Mathematics Subject Classification: 62C15, 62F10

Keywords: Scale equivariant estimator, Bayes estimator, Generalized Bayes estimator, Empirical Bayes estimator, Admissibility, Minimaxity

1 Introduction

In reliability and life testing studies, it is usual practice to conduct life testing experiments to a new product before launching it to the market. During the experiment, several kind of situations (censoring schemes) may arise. For example, (a) somebody may be interested in fixing the total runtime length of the experiment (say, $T$) and look at the failures occurred during the interval $(0, T]$. Let $m$ denote the number of failures, then obviously $T$ is independent of $m$. (b) One may fix the total number of failures before starting the experiment. The experiment will not be terminated until the experimenter gets a predetermined number of failures, (say $m$). If the number of items placed for the
experiment is \( n \), then \( m \leq n \). The first scheme described above is known as Type-I censoring and the second, Type-II censoring.

In this paper, we are concerned with the progressively Type-II censored data. It is a generalization of the complete as well as Type-II censored data. In life testing experiments, the progressively Type-II censored scheme attracts many researchers from different field of science and technology because of its nice property. It permits us to remove items from the experiment at different stages, which save cost and time. The scheme is described as follows: suppose there are \( n \) items placed in a test before starting the experiment and the experiment will be stopped immediately, when we observe \( m \) number of failures. The scheme is that after \( i \)-th failure, \( w_i \) be the number of items removed from the experiment, where \( 0 \leq w_1 \leq n - 1 \), \( w_m = n - w_1 - w_2 - \ldots - w_{m-1} - m \), \( i = 2, 3, \ldots, m \). Note that, \( w_i \)'s are fixed. For detailed description on the progressively Type-II censoring, one may refer to [7]. Let \( T_{1:n}, T_{2:n}, \ldots, T_{m:n} \) denote a progressively Type-II censored random sample taken from a population having probability density function (pdf) \( f(t) \) and cumulative distribution function (cdf) \( F(t) \). If the random variable \( T_{i:n} \) takes the value \( t_{i:n} \), for \( i = 1, 2, \ldots, n \), then \( t_{1:n} < t_{2:n} < \ldots < t_{m:n} \). The joint probability density function (likelihood function) of the \( m \) progressively Type-II censored data \( T_{1:n}, T_{2:n}, \ldots, T_{m:n} \) is given by (see [7])

\[
f_{T_{1:n}, T_{2:n}, \ldots, T_{m:n}}(t_{1:n}, t_{2:n}, \ldots, t_{m:n}) = J \prod_{i=1}^{m} f(t_{i:n})[1 - F(t_{i:n})]^{w_i}, \tag{1}
\]

where \( t_{1:n} < t_{2:n} < \ldots < t_{m:n} \) and \( J = n(n - w_1 - 1) \ldots (n - w_1 - w_2 - \ldots - w_{m-1} - m + 1) \). Several authors have paid attention in estimating the unknown features of lifetime distributions under progressive Type-II censoring scheme. In this direction one may refer to [7], [12], [8], [9], [4], [13], [14], [1], [2] and [6].

We assume that the lifetime of an item follows exponential distribution. [18] considered the problem of estimation for the one parameter exponential population based on progressive Type-II censoring scheme. They studied the problem with respect to an asymmetric precautionary loss function, which is useful in the situation when underestimation is more serious than overestimation. In this article, we study the problem of estimating mean of an exponential population with respect to linex loss function due to [16]. Linex loss function is more flexible than the loss function used by [18] because it is fit for the situations where underestimation is more serious than overestimation and vice-versa. Note that the mean of the exponential distribution in the scale parameter of the distribution. Exponential distribution has significant importance in real life because of its wide applications in reliability engineering and life testing problems. For scale parameter estimation problem, always an asymmetric loss function is better choice than symmetric loss function (quadratic loss function) since symmetric loss function heavily penalizes overestimation.
The paper is organized as follows: in Section 2, model is defined. The best scale equivariant estimator (BSEE) is obtained in Section 3. Further, the Bayes, generalized Bayes and empirical Bayes estimators are obtained in Section 4. Finally, some admissibility as well as inadmissibility results are presented in Section 5.

2 Model

Assume that the random variable $T$ denote the lifetime of a unit which follows exponential distribution with pdf and cdf given by

$$f(t|\sigma) = \frac{1}{\sigma} e^{-\frac{t}{\sigma}}, \quad t > 0, \quad \sigma > 0 \quad (2)$$

and

$$F(t|\sigma) = 1 - e^{-\frac{t}{\sigma}}, \quad t > 0, \quad \sigma > 0 \quad (3)$$

respectively. From (1), the likelihood function of $T_1:n, T_2:n, \ldots, T_m:n$ can be written as

$$l(t_1:n, t_2:n, \ldots, t_m:n|\sigma) = \left(\frac{J}{\sigma^m}\right) e^{-m\sum_{i=1}^{m}(w_i + 1)t_{i:n}/\sigma}, \quad t_{1:n} < t_{2:n} < \ldots < t_{m:n}. \quad (4)$$

Therefore, the log likelihood function is

$$L = \log l(t_1:n, t_2:n, \ldots, t_m:n|\sigma) = \log J - m \log \sigma - m \sum_{i=1}^{m}(w_i + 1)t_{i:n}/\sigma, \quad (5)$$

where log denote for the logarithm with base $e$. After differentiating the log likelihood function (5) with respect to $\sigma$ and equating to zero, we get the maximum likelihood estimator (MLE) as

$$\delta_{ML} = \frac{1}{m} \sum_{i=1}^{m}(w_i + 1)t_{i:n}. \quad (6)$$

It is noted that for the derivation of the MLE and its properties, one may also refer to [3]. From the likelihood function (4), we observe that $Z = \sum_{i=1}^{m}(w_i + 1)T_{i:n}$ is a complete and sufficient statistic for $\sigma$. It is well known that $2Z/\sigma$ follows chi-square distribution with $2m$ degrees of freedom. On the other way, we can say that $Z$ follows gamma distribution with shape parameter $m$ and scale parameter $\sigma$. Denote $Z \sim G(m, \sigma)$, where the density function of $Z$ is given by

$$g(z|m, \sigma) = \frac{1}{\Gamma(m)} \frac{1}{m^m} e^{-z/\sigma} z^{m-1}, \quad z > 0. \quad (7)$$

Regarding the derivation of the distribution of $2Z/\sigma$, one may look at [17]. Note that, the MLE given in (6) is also the uniformly minimum variance unbiased estimator (UMVUE).
3 Scale Equivariant Estimator

If $\delta$ is an estimator of the parameter $\sigma$ then the linex loss function is given by

$$L(\sigma, \delta) = r \left[ e^{s \left( \frac{\delta - \sigma}{\sigma} \right)} - s \left( \frac{\delta - \sigma}{\sigma} \right) - 1 \right],$$

(8)

where $r(>0)$ and $s(\neq 0)$ are scale and shape parameters, respectively. Without loss of generality, we take $r = 1$. The linex loss function behaves like quadratic loss function when magnitude of $s$ tends to zero. We take $s$ positive when overestimation gives more penalty than underestimation and take $s$ negative for vice versa. For details, see [20].

Consider the group of scale transformations $G = \{ g_a(x) = ax, a > 0 \}$. The estimating problem under study remains invariant under the group $G$. After some simple mathematics, we get the form of the scale equivariant estimators as

$$\delta_c(Z) = cZ,$$

(9)

where $c$ is any real valued constant. The BSEE of $\sigma$ can be obtained by minimizing

$$R(\sigma, \delta_c) = E \left[ e^{s \left( \frac{cZ - \sigma}{\sigma} \right)} - s \left( \frac{cZ - \sigma}{\sigma} \right) - 1 \right].$$

(10)

Differentiating (10) with respect to $c$ and equating to zero, we get the BSEE which is given by

$$\delta_{c_0} = \frac{1}{s} \left( 1 - e^{-s/m} \right) Z.$$

(11)

Hence, we have the following theorem.

**Theorem 3.1** The BSEE of $\sigma$ with respect to the loss function (8) is $\delta_{c_0} = c_0 Z$, where $c_0 = (1/s)(1 - e^{-s/(m+1)})$.

4 Bayes, Generalized Bayes and Empirical Bayes Estimators

For the sake of completeness, we derive Bayes, Generalized Bayes and Empirical Bayes estimators, though the Bayes and the empirical Bayes estimators can be obtained from the results of [2]. To derive the Bayes estimator of $\sigma$,
we consider the conjugate prior for $\sigma$ as inverse gamma distribution with pdf given by
\[ \pi(\sigma|a, b) = \frac{b^a}{\Gamma(a)} e^{-b/\sigma} \sigma^{-(a+1)}, \quad \sigma > 0, \quad a > 0, \quad b > 0. \tag{12} \]
Therefore, the posterior density of $\sigma$ given $Z = z$ is
\[ p(\sigma|Z = z) = \frac{(b + z)(m + a)}{\Gamma(m + a)} e^{-(b + z)/\sigma} \sigma^{-(m + a + 1)}, \quad \sigma > 0. \tag{13} \]
With respect to the loss function (8), the Bayes estimator $\delta_{BE}$ can be obtained from the relation given by
\[ E[\sigma^{-1} e^{(s\delta_{BE}/\sigma)}|Z = z] = e^s E[\sigma^{-1}|Z = z]. \tag{14} \]
After solving the above equation (14), we get the Bayes estimator as
\[ \delta_{BE} = \frac{1}{s} \left(1 - e^{-s/(m + a + 1)}\right) Z + \frac{b}{s} \left(1 - e^{-s/(m + a + 1)}\right). \tag{15} \]
To derive a generalized Bayes estimator of $\sigma$, we consider the Jeffrey's prior as
\[ \pi(\sigma) = \frac{1}{\sigma}, \quad \sigma > 0. \tag{16} \]
With respect to the non-informative prior (16), the generalized Bayes estimator for $\sigma$ is given by
\[ \delta_{GB} = \frac{1}{s} \left(1 - e^{-s/(m+1)}\right) Z. \tag{17} \]
It is noted that the generalized Bayes estimator $\delta_{GB}$ given in (17) is also the BSEE. To derive the empirical Bayes estimator, we take the prior for $\sigma$ as inverse gamma distribution associated with the pdf given in (12), where shape parameter $a$ is known and scale parameter $b$ is unknown. For empirical Bayes estimator, we estimate the unknown parameter $b$ of the prior distribution, from the marginal distribution of $T = (T_{1:n}, T_{2:n}, \ldots, T_{m:n})$. The marginal density of $T$ is
\[ m(t|b) = \frac{J \Gamma(m + a)}{\Gamma(a)} \frac{1}{(b + z)^{m + a}}, \quad t_{1:n} < t_{2:n} < \ldots < t_{m:n}, \tag{18} \]
where $\ell = (t_{1:n}, t_{2:n}, \ldots, t_{m:n})$, $l(t_{1:n}, t_{2:n}, \ldots, t_{m:n}|\sigma)$ and $\pi(\sigma|a, b)$ are given in (4) and (12), respectively. The unknown parameter $b$ of the marginal distribution (18) can be estimated by any estimators. We estimate $b$ by its MLE. From (18), the MLE of $b$ can be obtained as
\[ \hat{b}_{ML} = \frac{a}{m} Z. \tag{19} \]
Substituting $\hat{b}_{ML}$ in the place of $b$ in the Bayes estimator (15), we get the empirical Bayes estimator of $\sigma$ which is given by

$$\delta_{EB} = \frac{(m + a)(1 - e^{-s/(m + a + 1)})}{ms} Z. \quad (20)$$

Interestingly, the estimators obtained above such as MLE, BSEE, UMVUE, Bayes, generalized Bayes and empirical Bayes estimators all belong to the class of estimators of the form $\delta_{A,B}(Z) = AZ + B$, where $A (\neq 0)$ and $B$ are real valued constants. This motivate us to study the properties of the class of estimators $\delta_{A,B}(Z)$.

5 Admissibility and Inadmissibility Results

In this section, we discuss about admissibility as well as inadmissibility of the class of estimators of the form $\delta_{A,B}(Z) = AZ + B$.

Theorem 5.1 With respect to the loss function (8), the estimator $\delta_{A,B}(Z)$ is admissible, for $0 < A < c_0$ and $B > 0$.

Proof: Since the loss function (8) is convex, therefore, the Bayes estimator given in (15) is unique. We observe that the coefficient of $Z$ of the Bayes estimator $\delta_{BE}$ strictly lies between $0$ and $c_0$ and the constant $(b/s)(1 - e^{-(s/(m+a+1))})$ is strictly positive for $a > 0$, $b > 0$ and $s \neq 0$.

This completes the proof.

In the following theorem, we prove an admissibility result of $\delta_{A,B}(Z) = AZ + B$ under some certain restrictions on the constants $A$ and $B$. To prove admissibility we will use the following lemma due to [5].

Lemma 5.2 ([11], p. 380): Suppose that the parameter space $\Omega$ is a non-degenerate convex subset of the real line. Let $\delta$ be an estimator with a continuous risk function and let $\{\pi_k\}$ be a sequence of (possibly improper) prior measures such that

(a) the Bayes risks $r(\pi_k, \delta)$ and $r(\pi_k, \delta_B)$ are finite for all $k$, where $\delta_B$ is the Bayes estimator with respect to the prior $\pi_k$,

(b) for any non-empty open set $\Omega_0 \subset \Omega$, there exist constants $D > 0$ and $N$ such that

$$\int_{\Omega_0} \pi_k(\theta) d\theta \geq D \quad \text{for all } k \geq N \quad \text{and}$$

(c) $r(\pi_k, \delta) - r(\pi_k, \delta_B) \to 0$ as $k \to \infty$. Then the estimator $\delta$ is admissible.

In order to apply Blyth’s lemma, we need a sequence of prior distributions $\pi_k(\sigma)$, such that the Bayes risk (with respect to the sequence of priors $\pi_k(\sigma)$) difference between the Bayes estimator and the target estimator is close to
zero for large value of \(k\). The risk and the Bayes risk (with respect to the prior \(\pi(\sigma|a,b)\) given in (12)) of \(\delta_{A,B}(Z) = AZ + B\) are given by

\[
R(\sigma, \delta_{A,B}) = \frac{e^{(sB/\sigma)} - s}{(1 - sA)^m} - msA - \frac{sB}{\sigma} + s - 1
\]

and

\[
r(\sigma, \delta_{A,B}) = \frac{e^{-s(1 - sB)} - a}{(1 - sA)^m} - msA - \frac{asB}{b} + s - 1,
\]

respectively.

**Theorem 5.3** The estimator \(\delta_{A,B}(Z) = AZ + B\) is admissible under the loss function (8), provided \(A = c_0\) and \(B \geq 0\), where \(c_0\) is given in Theorem 3.1.

**Proof:** Assume \(A = c_0\) and \(B > 0\). We consider a sequence of prior distributions for \(\sigma\) as

\[
\pi_k(\sigma) = \frac{b(1/k)}{\Gamma(1/k)} e^{-(b/\sigma)} (1 + 1/k)^m, \quad \sigma > 0.
\]

It can be shown that if \(\Omega_0\) is a non degenerate convex subset of the interval \((0, \infty)\) then, there exist constants \(D > 0\) and \(N\), such that \(\int_{\Omega_0} \pi_k(\sigma) d\sigma \geq D\) for all \(k \geq N\). Regarding the derivation of the constant \(D\), one may refer to [10]. Thus, condition (b) of the Blyth’s lemma is satisfied. The posterior distribution of \(\sigma\) given \(Z = z\) can be obtained as

\[
g(\sigma|Z = z) = \frac{(b + z)(1 + m)}{\Gamma(1 + m)} e^{-(b + z)/\sigma} (1 + 1/k + 1), \quad \sigma > 0.
\]

Thus, with respect to the loss function (8), the Bayes estimator of \(\sigma\) is given by

\[
\delta_{BE} = \frac{1}{s} \left(1 - e^{-s/(m + 1/k + 1)}\right) Z + \frac{b}{s} \left(1 - e^{-s/(m + 1/k + 1)}\right).
\]

From (22), the Bayes risk of the Bayes estimator given in (15) with respect to the sequence of prior distributions (23) is

\[
r(\sigma, \delta_{BE}^k) = \left(\frac{1}{k} + m + 1\right) \left(1 - e^{-s/(m + 1/k + 1)}\right) - \left(\frac{1}{k} + m + 1\right) + s.
\]

Moreover, consider an estimator

\[
\delta = c_0 Z + b c_0.
\]
Our aim is to prove that $\delta$ is admissible. Now using (22), the Bayes risk of the estimator (27) with respect to the prior (23) can be obtained as

$$r(\sigma, \delta) = \left( m + \frac{1}{k} + e^{s/(k(m + 1))} \right) e^{-s/(m + 1)} - \left( m + \frac{1}{k} + 1 \right) + s. \quad (28)$$

Note that, the Bayes risks of the estimators $\delta_{BE}^k$ and $\delta$ are finite. Thus, condition (a) hold. Also, from (26) and (28), we get

$$r(\sigma, \delta_{BE}^k) - r(\sigma, \delta) \to 0, \quad \text{as} \quad k \to \infty.$$  

Hence, condition (c) is verified. Therefore, the estimator $\delta$ given in (27) is admissible. Also, note that the constant $bc_0$ in (27) is always positive for $b > 0$. Therefore, we can say that the estimator $\delta_{A,B}(Z) = AZ + B$ is admissible when $A = c_0$ and $B > 0$. Admissibility of $\delta_{A,B}(Z) = AZ + B$ can be proved similarly, when $A = c_0$ and $B = 0$.

This completes the proof of the theorem.

**Remark 5.4** The BSEE $\delta_{c_0}$ is minimax, since it is admissible and having risk independent of the parameter $\sigma$.

**Theorem 5.5** The estimator $\delta_{A,B}(Z) = AZ + B$ is inadmissible when one of the following conditions hold:

(i) $A < 0$ or $B < 0$,
(ii) $A > c_0$, $B \geq 0$,
(iii) $0 \leq A < c_0$, $B = 0$.

**Proof:** (i) When $A < 0$ and $B < 0$, we notice that $P(AZ + B < 0) \neq 0$. Therefore, we can say that the estimator $\delta_{A,B}(Z)$ is dominated by $\max\{0, AZ + B\}$.

(ii) Differentiating $R(\sigma, \delta_{A,B})$ given in (21) with respect to $A$, we get

$$\frac{dR(\sigma, \delta_{A,B})}{dA} = ms \left[ \frac{e^{(sB/\sigma)} - s}{(1 - sA)^m + 1} - 1 \right], \quad (29)$$

which is always positive for $s \neq 0$. Therefore, $R(\sigma, \delta_{A,B})$ is an increasing function in $A$. Also, the risk function is continuous. Hence $\delta_{A,B}(Z) = AZ + B$ is dominated by the estimator $\delta_{c_0,B}(Z) = c_0Z + B$, when $A > c_0$ and $B \geq 0$.

(iii) Clearly, the estimator $\delta_{A,B}(Z) = AZ + B$ is inadmissible for $0 \leq A < c_0$ and $B = 0$, because when $B = 0$, risk $R(\sigma, \delta_{A,B})$ is minimum at $A = c_0$.

6 Conclusion

In this paper we have considered the problem of estimating the mean (scale parameter) of an exponential population with respect to the linex loss function.
Estimating mean of an exponential population

from a decision theoretic point of view. All the results are obtained based on
the progressively Type-II censored data. Some decision theoretic properties
of the class of estimators $\delta_{A,B}(Z)$ are studied. The BSEE is shown to be
admissible and minimax, while the MLE and empirical Bayes estimator are
inadmissible.

References

[1] A. Asgharzadeh and R. Valiollahi, Estimation based on progressively cen-
sored data from the Burr model, Inter. Math. Forum, 43 (2008), 2113 -
2121.

family under progressive Type-II censoring, J. Iranian Statist. Soc., 8
(2009), 35 - 53.


likelihood estimation for exponential distributions under general progressive
Type-II censored samples, Sankhya, 58(B) (1996), 1 - 9.


risks data from lomax distributions, Comp. Statist. Data Anal., 56 (2011),
1285 - 1303.

[7] A.C. Cohen, Progressively censored samples in life testing, Technometrics,
5 (1963), 327 - 339.

[8] A.C. Cohen and N.J. Norgaard, Progressively censored sampling in the

btribution with progressively censored samples, IEEE Trans. on Reliability,

[10] S. Kayal and S. Kumar, Estimating the Shannon entropy of several exponen-


[18] Q. Wang and L. Yang, Estimation for one-parameter exponential distribution based on progressively Type-II censored sample under a precautionary loss function, In Electronic and Mechanical Engineering and Information Technology (EMEIT), 7 (2011), 3646 - 3649.


Received: April 7, 2014