Comparison of Two Interval Arithmetic

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Abstract

The article compared the classical interval arithmetic and interval arithmetic introduced by the authors. In scientific research, techniques and mass production often measurements of any quantities have to be done (length, mass, current, etc.). When measurements of the same object is repeated, performed using the same meter with the same care because of the impact of various factors, the same data is never obtained. These factors include random vibrations of parts of the instrument, physiological changes of senses of the performer, various unrecorded changes in the environment (temperature, optical, electrical and magnetic properties, etc.). Although the result of each measurement with random dispersion cannot be predicted in advance, it corresponds to the "normal distribution curve."

“Classic” interval arithmetic suggests that all interval values are equally likely. Therefore, all the results obtained with it cover all possible values and considered as "super sufficient."

Keywords: interval, interval mathematics, classical intervals, expectation, variance, interval valued function.

1. INTRODUCTION

Interval analysis is currently being actively developed in many countries. Originally interval methods have emerged as a means of automatic control of rounding errors on the computer and later turned into one of the branches of modern applied mathematics.
The first publication on interval analysis was made by R.E. Mur in 1966/1/. The fundamentals and techniques of interval analysis were systematically presented by Y.I. Shokin / 2 / in 1981.

Interval analysis is a relatively new area of computational mathematics is widely used to study the properties of mechanical systems. One of the main requirements for the quality of these systems is the requirement of stability. Using interval analysis for solving the problem of stability of mechanical systems provides a guaranteed stability criterion. But the application of interval mathematics researchers has difficulties in dealing with bulky interval equations, and these solutions are obtained "super sufficient", which is a severe limitation in practice.

In this paper, / 3 / a new interval arithmetic that takes into account the uneven distribution of values within the interval was proposed. This article focuses on the comparative analysis of interval arithmetic: classical / 1,2 / and proposed in work/ 3 /.

2. STATEMENT OF THE PROBLEM

As it is known, interval arithmetic suggests that all values within the interval are equally probable. Therefore, all the results obtained with it, and cover all possible values and are "super sufficient." Gist of the introduced arithmetic, is to assume that the values within the interval are independent normally distributed variables. Formal concept of interval $a$ was introduced in the following form:

$$a = [a - \varepsilon_a, a + \varepsilon_a] = (\delta_a, \varepsilon_a),$$

where $\bar{a}$ - middle of the interval (or expectation), $\varepsilon_a$ - width of the interval (or dispersion). For a comparative analysis of the introduced arithmetic and "classic" we will denote by $a_k = [\bar{a}_k - \delta_a, \bar{a}_k + \delta_a] = (\delta_a, \varepsilon_a)$ "classic" interval. In the future we will not specifically stipulate the form of the interval, and will distinguish them by index $k$.

3. COMPARISON OF ARITHMETIC OPERATIONS.

The operation of addition (subtraction).

$$c = a \pm b; \quad \varepsilon_c = \sqrt{\varepsilon_a^2 + \varepsilon_b^2};$$

$$c_k = a_k \pm b_k; \quad \delta_c = \delta_a + \delta_b.$$

As seen, the width of the entered interval is less than the “classical” interval. For example, for intervals $c = a + a$ и $c_k = a_k + a_k$ we get $c = 2a, c_k = 2a_k, \varepsilon_c = \sqrt{2} \cdot \varepsilon_a, \delta_c = 2 \cdot \delta_a$. Thus, centers of both intervals
Comparison of two interval arithmetic

are the same, however width of the entered interval $\sqrt{2}$ times smaller of the width of the “classic” interval.

For $n$-fold addition of intervals we get $\bar{c} = n \bar{a}$, $\bar{c}_k = n \bar{a}_k$, $\varepsilon_c = \sqrt{n} \cdot \varepsilon_a$, $\delta_c = n \cdot \delta_a$. Width of the entered interval $\sqrt{n}$ time smaller than width of the “classic” interval.

The operation of multiplication.

$\bar{c} = \bar{a} \cdot \bar{b}$; $\varepsilon_c = \sqrt{\bar{a}^2 \cdot \varepsilon_b^2 + \bar{b}^2 \cdot \varepsilon_a^2}$;

$$c_k = \left[ \min \left( \frac{(\bar{a}_k - \delta_a) \cdot (\bar{b}_k - \delta_b) \cdot (\bar{a}_k + \delta_a) \cdot (\bar{b}_k + \delta_b)}{(\bar{a}_k + \delta_a) \cdot (\bar{b}_k - \delta_b) \cdot (\bar{a}_k - \delta_a) \cdot (\bar{b}_k + \delta_b)} \right), \right.$$  
$$\left. \max \left( \frac{(\bar{a}_k - \delta_a) \cdot (\bar{b}_k - \delta_b) \cdot (\bar{a}_k + \delta_a) \cdot (\bar{b}_k + \delta_b)}{(\bar{a}_k + \delta_a) \cdot (\bar{b}_k - \delta_b) \cdot (\bar{a}_k - \delta_a) \cdot (\bar{b}_k + \delta_b)} \right) \right].$$

As seen, from the formulas above $\bar{c}_k \neq \bar{a}_k \cdot \bar{b}_k$, i.e. with “classic” multiplication occurs shift of work of centers of the intervals. For example, for intervals $c = a \cdot a$ we get $\bar{c} = \bar{a}^2$, $\varepsilon_c = \sqrt{2} \cdot \varepsilon_a \cdot |\bar{a}|$, $\bar{c}_k = \bar{a}_k^2 + \delta_a^2$, $\delta_c = 2 \cdot \delta_a \cdot |\bar{a}_k|$. Thus, width of the entered interval $\sqrt{2}$ times smaller than “classic” interval and the center of “classic” interval is shifted by the amount $\delta_a^2$.

The operation of reverse calculation interval.

$\bar{c} = \frac{1}{\bar{a}}$; $\varepsilon_c = \frac{\varepsilon_a}{\bar{a}}$

$$\bar{c}_k = \left( \frac{-\bar{a}_k}{\bar{a}_k^2 - \delta_a^2} \right); \delta_c = \left( \frac{-\bar{a}_k}{\bar{a}_k^2 - \delta_a^2} \right).$$

Thus, width of the entered interval is smaller than the width of the “classic” interval and the center of the “classic” interval is shifted by a certain value.

In the “classic” calculation of the reverse interval is assumed, that $0 \notin a_k$.

It is allowed in the input definition of the reverse interval that $0 \in a$, if only $\bar{a} \neq 0$.

The operation of division of two intervals.

$\bar{c} = \frac{\bar{a}}{\bar{b}}$; $\varepsilon_c = \sqrt{\frac{\bar{a}^2 \cdot \varepsilon_b^2}{\bar{b}^4} + \frac{\varepsilon_a^2}{\bar{b}^2}}$.
\[ c_k = (a_k - \delta_a, a_k + \delta_a)^\ast \left( \frac{1}{b_k + \delta_b}, \frac{1}{b_k - \delta_b} \right). \]

For example, for intervals \( c = \frac{a}{a} \) и \( c_k = \frac{a_k}{a_k} \) we get

\[ \bar{c} = 1, \quad \varepsilon_c = \frac{\sqrt{2} \cdot \varepsilon_a}{|a|}, \]
\[ \bar{c}_k = \frac{-a_k^2 + \delta_a^2}{a_k^2 - \delta_a^2}, \quad \delta_c = \frac{2 \cdot \delta_a \cdot |a_k|}{-a_k^2 - \delta_a^2}. \]

The center of the entered interval equals 1. The center of the “classic” interval is shifted concerning 1, even though it contains 1.

At \( \varepsilon_a = \frac{a}{2} \) и \( \delta_a = \frac{a_k}{2} \) we get

\[ c = \left( 1, \frac{1}{\sqrt{2}} \right) = \left[ 1 - \frac{1}{\sqrt{2}}, 1 + \frac{1}{\sqrt{2}} \right] = [0.293, 1.707], \]
\[ c_k = [1.666 - 1.333, 1.666 + 1.333] = [0.333, 2.999]. \]

Thus, the width of the entered interval equals 1.414 and smaller that the width of the “Classic” interval, which equals 2.666. Moreover, the center of the “classic” interval is shifted from 1 by the amount of 0.666.

3. COMPARATIVE ANALYSIS FOR THE CALCULATION OF INTERVAL FUNCTIONS

Suppose \( f(x) \) – interval-valued function of the interval argument. Then for \( c = f(a) \) we get:

For the entered interval mathematic

\[ \bar{c} = f(a), \quad \varepsilon_c = \left( \frac{df}{dx} \right)_{a} \cdot \varepsilon_a. \] (1)

for the classic interval mathematic

\[ \bar{c}_k = \left( f_{\text{max}} + f_{\text{min}} \right) \]
\[ \delta_c = \left( f_{\text{max}} - f_{\text{min}} \right) \] (2)
Comparison of two interval arithmetic

where \( f_{\min} = \min_{y \in a} f(y) \), \( f_{\max} = \max_{y \in a} f(y) \).

As seen from the formulas above for differentiable calculation functions values of the function at the entered interval mathematic (1) more constructive in the view of the limb of performed arithmetical operations. At the same time calculations by formulas (2) require solution of two optimization problems, solution of each of them generally has a necessity of conduction of interactive calculations. This raises problems with the convergence of the iterative process and the choice of starting point.

Example. Suppose given intervals are \( a = [1.750, 2.250] = (2.0, 0.250) \) and \( b = [-0.250, 1.250] = (0.5, 0.750) \). It is required to calculate the value of the interval-valued function \( \exp(a) \) and \( \exp(b) \).

Then for the entered interval mathematics formula (1) we get following results:

\[
\overline{c} = f(\overline{a}) = 7.389, \quad \overline{\varepsilon} = \left(\frac{\partial f}{\partial x}\right)_{\overline{a}} \cdot \varepsilon_{a} = 1.847.
\]

Then \( f(a) = [5.542, 9.236] = (7.389, 1.847) \).

Similarly, we calculate the value of the function on the interval \( b \):

\( f(b) = [0.412, 2.885] = (1.649, 1.238) \).

Now we calculate the value of the function on the interval \( a \) by the formulas of classical interval mathematics

\[
\overline{c}_k = f(\overline{a}) = 7.621, \quad \delta_c = 1.867.
\]

Тогда \( f(a) = [5.755, 9.488] = (7.621, 1.867) \).

Similarly we calculate the value of the function on the interval \( b \) by the formulas of classical interval mathematics

\[
\overline{c}_k = f(\overline{b}) = 2.135, \quad \delta_c = 1.365 \text{ и } f(b) = [0.779, 3.490] = (2.135, 1.365) \).

As seen from the results above, when calculating the value of the function \( \exp \) by the formulas of entered interval mathematics intervals are more “compressed” than in calculations using formulas (2).

Let’s consider an example of Reichman /2/, to show another advantage of inputted arithmetic. K. Reichman built his counterexamples to show the impossibility of applying the interval method of Gauss. In this example the dimension of the matrix is substantial, therefore the considered matrix form is:
For $(\sqrt{5} - 1)/2 \leq \alpha < 1$, using classical interval arithmetical operations, calculating the determinant by Gauss method we get then interval matrix $S(\alpha)$ is singular.

When using the input interval arithmetical operations, implemented in a form of a subroutine and function library, we get systems of linear interval algebraic equations.

$$S(\alpha)x(\alpha) = \begin{bmatrix} [1,1] \\ [1,1] \\ [1,1] \end{bmatrix}$$

For different values of $\alpha$:

$$x(0.65) = \begin{bmatrix} 0.244,0.968 \\ 0.132,1.080 \\ 0.075,1.137 \end{bmatrix} = \begin{bmatrix} 0.606,0.362 \\ 0.606,0.474 \\ 0.606,0.531 \end{bmatrix},$$

$$x(0.70) = \begin{bmatrix} 0.187,0.989 \\ 0.065,1.111 \\ -0.002,1.178 \end{bmatrix} = \begin{bmatrix} 0.588,0.401 \\ 0.588,0.523 \\ 0.588,0.590 \end{bmatrix}.$$

4. CONCLUSION

The article compares the two interval arithmetic. Displayed is the efficiency of entered interval arithmetic.

REFERENCES


Received: April 11, 2014