Uniqueness of Difference Polynomials of Entire Functions

Renukadevi S. Dyavanal and Rajalaxmi V. Desai

Department of Mathematics, Karnatak University
Dharwad - 580003, India

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Abstract

In this paper, we study the uniqueness of difference polynomials of entire functions \( f \) and \( g \) and a small function \( \alpha \). The result improves the result of J. Zhang[13].

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1. Introduction and results

In this paper, the term "meromorphic" will always mean meromorphic in the complex plane \( \mathbb{C} \). We shall use the standard notation in Nevanlinna’s value distribution theory of meromorphic functions ([12]). We use notations \( \sigma(f) \) and \( \lambda(f) \) for the order and the exponent of convergence of zeros of \( f(z) \) respectively. Set \( E(a, f) = \{ z : f(z) - a = 0 \} \), where a zero point with multiplicity \( m \) is counted \( m \) times in the set. If these zero points are only counted once, then we denote the set by \( \overline{E}(a, f) \). Let \( f(z) \) and \( g(z) \) be two meromorphic functions. If \( E(a, f) = E(a, g) \), then we say that \( f(z) \) and \( g(z) \) share the value \( a \) CM; if \( \overline{E}(a, f) = \overline{E}(a, f) \), then we say that \( f(z) \) and \( g(z) \) share the value \( a \) IM.

Let \( k \) be a positive integer and \( a \in \mathbb{C} \cup \{ \infty \} \). We denote by \( N_k(r, 1/(f - a)) \) the counting function of \( a \)-points of \( f \) with multiplicity \( \leq k \), by \( N_k(r, 1/(f - a)) \)
the counting function of $a$-points of $f$ with multiplicity $\geq k$; and denote the reduced counting function by $\overline{N}_k(r,1/(f-a))$, $\overline{N}(r,1/(f-a))$, respectively. Set $N_k(r,1/(f-a))/\overline{N}(r,1/(f-a))$. Recently, value distribution of difference analogues of meromorphic functions and application of Value Distribution Theory to differential equations has become a subject of great importance ([1], [2], [7],[8],[10], [3], [6]).

In 2007, Laine and Yang [9] proved the following result.

**Theorem 1.A.** Let $f$ be a transcendental entire function of finite order and $c$ be a non-zero complex constant. Then for $n \geq 2$, $f^n(z) f(z+c)$ assumes every non-zero value $a \in \mathbb{C}$ infinitely often.

In 2010, J.Zhang [13] proved the following analogue results in difference.

**Theorem 1.B.** Let $f(z)$ be a transcendental entire function of finite order, and $\alpha(z)$ be a small function with respect to $f(z)$. Suppose that $c$ is a non-zero complex constant and $n$ is an integer. If $n \geq 2$, then $f^n(z)(f(z)−1)f(z+c)−\alpha(z)$ has infinitely many zeros.

**Theorem 1.C.** Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, and $\alpha(z)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose that $c$ is a non-zero complex constant. $n \geq 7$, if $f^n(z)(f(z)−1)f(z+c)$ and $g^n(z)(g(z)−1)g(z+c)$ share $\alpha(z)$ CM, then $f(z)\equiv g(z)$.

In the present paper, we consider the difference polynomials of entire functions of the form $f^n(z)(f(z)−1)^kf(z+c)$ and prove the following theorems.

**Theorem 1.1.** Let $f(z)$ be a transcendental entire function of finite order, and $\alpha(z)$ be a small function with respect to $f(z)$. Suppose that $c$ is a non-zero complex constant and $n$ is an integer. If $n \geq 2$, $k \geq 1$ then $f^n(z)(f(z)−1)^kf(z+c)−\alpha(z)$ has infinitely many zeros.

**Theorem 1.2.** Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, and $\alpha(z)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose that $c$ is a non-zero complex constant. $k \geq 1$, $n \geq k+6$, if $f^n(z)(f(z)−1)^kf(z+c)$ and $g^n(z)(g(z)−1)^kg(z+c)$ share $\alpha(z)$ CM, then $f(z)\equiv t g(z)$, where $t^k = 1$.

### 2. Some preliminary results

To prove our theorems we require the following Lemmas.

As mentioned in Section 1, Halburd and Korhonen [7] and Chiang and Feng [2] investigated the value distribution theory of difference expressions. A key result, which is a difference analogue of the logarithmic derivative lemma, read as follows.
Lemma 2.1. [2, 7] Let $f$ be a meromorphic function of finite order and $c$ is a nonzero complex constant. Then
\[ m \left( r, \frac{f(z) + c}{f(z)} \right) + m \left( r, \frac{f(z)}{f(z) + c} \right) = S(r, f) \]

Lemma 2.2. [2] Let $f$ be a meromorphic function of finite order $\rho$ and $c$ is a non-zero complex constant. Then, for each $\epsilon > 0$, we have
\[ T(r, f(z) + c) = T(r, f) + O(r^{\rho - 1 + \epsilon}) + O(\log r) \]
From Lemma 2.2, it is evident that $S(r, f(z) + c) = S(r, f)$.

Lemma 2.3. [2] Let $f$ be a meromorphic function with finite exponent of convergence of poles $\lambda(\frac{1}{f})$ and $c$ is a non-zero complex constant. Then, for each $\epsilon > 0$, we have
\[ N(r, f(z) + c)) = N(r, f) + O(r^{\lambda(\frac{1}{f}) - 1 + \epsilon}) + O(\log r). \]

Lemma 2.4. [11] Let $F$ and $G$ be two non-constant meromorphic functions. If $F$ and $G$ share $1$ CM, then one of the following three cases holds:
(i) $\max \{T(r, F), T(r, G)\} \leq N_2(r, 0, F) + N_2(r, 0, G) + N_2(r, F) + N_2(r, G) + S(r, F) + S(r, G)$
(ii) $F \equiv G$
(iii) $FG \equiv 1$.

Lemma 2.5. Let $f(z)$, $n$ and $c$ be as in Theorem 1.1. and $F(z) = \frac{f^n(z)(f(z) - 1)^k f(z + c)}{\alpha(z)}$.
Then $T(r, F) = (n + k + 1) T(r, f) + S(r, f)$.

Proof: Since $f$ is an entire function of finite order, we deduce from Lemma 2.1, and the standard Valiron-Mohon’ko theorem [12]) that
\[ (n+k+1)T(r, f(z)) = T(r, f^{n+1}(z)(f(z)-1)^k) + S(r, f) = m(r, f^{n+1}(z)(f(z)-1)^k) + S(r, f) \]
\[ \leq m \left( r, \frac{f^{n+1}(z)(f(z)-1)^k}{f(z)} \right) + m(r, F(z)) + S(r, f) \]
\[ = m \left( r, \frac{f(z)}{f(z) + c} \right) + m(r, F(z)) + S(r, f) \leq T(r, F(z)) + S(r, f) \] (2.1)
On the other hand, using Lemma 2.2, we have
\[ T(r, F(z)) \leq (n + k + 1) T(r, f(z)) + O(\log r) + S(r, f) \] (2.2)
Using (2.1) (2.2) we obtain $T(r, F(z)) = (n + k + 1) T(r, f(z)) + S(r, f)$ (2.3)

3. Proof of Theorems

3.1. Proof of Theorem 1.1. By Lemma 2.5, Lemma 2.2 and the Nevanlinna’s second fundamental theorem for entire functions, we have
\[ (n + k + 1) T(r, f(z)) = T(r, F(z)) + S(r, f) \leq N(r, F(z)) + N(r, 0, F(z) - \alpha(z)) + S(r, f) \]
\[ \leq N(r, 0, f^n(z)) + N(r, 0, (f(z) - 1)^k) + N(r, 0, f(z) + c) + N(r, 0, F(z) - \alpha(z)) + S(r, f) \]
\[ \leq N(r, 0, f(z)) + N(r, 0, (f(z) - 1)) + N(r, 0, f(z) + c) + N(r, 0, F(z) - \alpha(z)) + S(r, f) \] (3.1)
By 3.1, and Lemma 2.2, we have \((n+k+1)T(r, f(z)) \leq 3T(r, f(z)) + N(r, 0, F(z) - \alpha(z)) + S(r, f)\)

Thus, \(N(r, 0, F(z) - \alpha(z)) \geq (n+k-2)T(r, f(z)) + S(r, f)\).

(3.2)

Since \(n \geq 2\), \(k \geq 1\) and by 3.2, \(F(z)\) assumes \(\alpha(z)\) infinitely many times.

3.2. Proof of Theorem 1.2.

Let \(F(z) = \frac{f^n(z)(f(z)-1)^k f(z+c)}{\alpha(z)}\) and \(G(z) = \frac{g^n(z)(g(z)-1)^k g(z+c)}{\alpha(z)}\)

Then \(F(z)\) and \(G(z)\) share 1 CM except the zeros or poles of \(\alpha(z)\) and hence

\(N_2(r, F(z)) = S(r, f)\) and \(N_2(r, G(z)) = S(r, g)\) as \(f\) and \(g\) entire functions \(\ (3.3)\)

\(N_2(r, 0, f^n(z)) = N_{21}(r, 0, f^n(z)) + 2N_{22}(r, 0, f^n(z)) = 2N_{22}(r, 0, f(z)) \leq N(r, 0, f(z))\)

(3.4)

From Lemma 2.5 and 3.4, we have

\(N_2(r, 0, F(z)) = N_2(r, 0, f^n(z)) + N_2(r, 0, (f(z)-1)^k) + N_2(r, 0, f(z+c))\)

\(\leq N_2(r, 0, f(z)) + N(r, 0, (f(z)-1)^k) + N(r, f(z+c)) + S(r, f)\)

\(\leq T(r, f) + kT(r, f) + T(r, f) + S(r, f) \leq (k+2)T(r, f) + S(r, f)\)

(3.5)

Similarly,

\(N_2(r, 0, G(z)) \leq (k+2)T(r, g) + S(r, g)\)

(3.6)

By 3.3, 3.5 and 3.6, we can write

\(N_2(r, \infty, F) + N_2(r, 0, F) \leq (k+2)T(r, f) + S(r, f)\) and \(N_2(r, \infty, G) + N_2(r, 0, G) \leq (k+2)T(r, g) + S(r, g)\)

(3.7)

By Lemma 2.4, suppose (i.) holds, that is,

\(\max \{T(r, F), T(r, G)\} \leq N_2(r, 0, F) + N_2(r, 0, G) + N_2(r, F) + N_2(r, G) + S(r, F) + S(r, G)\).

Then, by 3.7, we get

\(\max \{T(r, F), T(r, G)\} \leq (k+2)T(r, f) + (k+2)T(r, g) + S(r, f) + S(r, g)\)

\(\Rightarrow (T(r, F)) + T(r, G) \leq (2k+4)(T(r, f) + T(r, g) + S(r, f) + S(r, g))\)

(3.8)

By 2.3 and 3.8 we get

\((n+k+1)(T(r, f) + T(r, g)) \leq (2k+4)(T(r, f) + T(r, g)) + S(r, f) + S(r, g)\)

\(\Rightarrow (n-k-3)(T(r, f) + T(r, g)) \leq S(r, f) + S(r, g)\). Which contradicts our hypothesis \(n \geq k+6\).

Hence by Lemma 2.4, we get either \(F(z) \equiv G(z)\) or \(F(z).G(z) \equiv 1\)

(3.9)

Suppose, \(F(z).G(z) \equiv 1\) i.e, \(f^n(z)(f(z)-1)^k f(z+c) \cdot g^n(z)(g(z)-1)^k g(z+c) = \alpha^2(z)\)

Then \(N(r, 0, f) = S(r, f)\) and \(N(r, 0, (f-1)) = S(r, f)\)

And \(\delta(0, f) + \delta(1, f) + \delta(\infty, f) = 3\). Which is contradiction to defect relation.

Hence, by 3.9 we conclude that \(F(z) \equiv G(z)\).

\(i.e., f^n(z)(f(z)-1)^k f(z+c) \equiv g^n(z)(g(z)-1)^k g(z+c)\)

(3.10)
Let $h(z) = \frac{f(z)}{g(z)}$. If $h^{n+k}(z)h(z+c) \neq 1$, then from 3.12

\[(h(z)g(z))^n(h(z)g(z) - 1)^k h(z+c)g(z+c) = g^{n}(z)(g(z) - 1)^k g(z+c)\]

(3.11)

Combining 3.11 with Binomial theorem, we derive

\[g^k(z) = \frac{(-1)^{k}C_1 g^{k-1}(z) [h^{n+k-1}(z) h(z+c) - 1]}{[h^{n+k}(z) h(z+c) - 1]} + ...\]

\[+ \frac{(-1)^{k+1} C_k g^{k}(z) [h^{n+k-k}(z) h(z+c) - 1]}{[h^{n+k}(z) h(z+c) - 1]} + ... + \frac{(-1)^{k+1} [h^{n}(z) h(z+c) - 1]}{[h^{n+k}(z) h(z+c) - 1]}\]

(3.12)

By lemma 2.2, we have $T(r, h(z+c)) = T(r, h(z)) + S(r, h)$

(3.13)

From 3.13 and using the condition $n \geq k+6$, it is easy to show that $h^{n+k}(z)h(z+c)$ is not a constant. Suppose that there exists a point $z_0$ such that $h^{n+k}(z_0)h(z_0 + c) = 1$. Since $g(z)$ is an entire function and using from 3.12 we get $h^{n}(z_0)h(z_0+c) = 1$. Hence $h^{k}(z_0) = 1$

\[: \overline{\mathcal{N}}(r, 0, h^{n+k}(z) h(z+c) - 1) \leq \overline{\mathcal{N}}(r, 0, h^{k}(z) - 1) \leq k T(r, h(z)) + O(1)\]

(3.14)

Denote $H = h^{n+k}(z)h(z+c)$, then we have

\[\overline{\mathcal{N}}(r, H) = \overline{\mathcal{N}}(r, h^{n+k}) + \overline{\mathcal{N}}(r, h(z+c)) \leq 2 T(r, h)\]

(3.15)

\[\overline{\mathcal{N}}(r, 1/H) = \overline{\mathcal{N}}(r, 1/h^{n+k}) + \overline{\mathcal{N}}(r, 1/h(z+c)) \leq 2 T(r, h)\]

(3.16)

By the Second fundamental theorem and using 3.14, 3.15, 3.16 we get

\[T(r, H) \leq \overline{\mathcal{N}}(r, H) + \overline{\mathcal{N}}(r, 0, H) + \overline{\mathcal{N}}(r, 0, H-1) + S(r, H)\]

\[\leq 2 T(r, h) + 2 T(r, h) + k T(r, h) + S(r, h) \leq (4 + k)T(r, h) + S(r, h)\]

Noting this, we have

\[(n+k)T(r, h) = T(r, h^{n+k}(z)) = T(r, \frac{H(z)}{h(z+c)}) \leq T(r, H(z)) + T(r, h(z+c)) + O(1)\]

\[\leq (4 + k)T(r, h) + T(r, h) + S(r, h) = (5 + k)T(r, h) + S(r, h)\]

Which is contradiction since $n \geq k + 6$. Hence $H = h^{n+k}(z)h(z+c) \neq 1$, then 1 is Picard’s exceptional value of $H$, then by Second fundamental theorem, we have

\[(n+k + 1)T(r, h) = T(r, H) \leq \overline{\mathcal{N}}(r, H) + \overline{\mathcal{N}}(r, 0, H) + \overline{\mathcal{N}}(r, 0, H-1) + S(r, H)\]

\[\leq 2 T(r, h) + 2 T(r, h) + S(r, h) \leq 4 T(r, h) + S(r, h)\]

(3.17)

Which is contradiction since $n \geq k + 6$. Therefore $h^{n+k}(z)h(z+c) \equiv 1$, then by 3.12, $h(z)^n h(z+c) \equiv 1$. Thus $h^k(z) \equiv 1$. Hence, we get $f(z) \equiv t g(z)$, where $t^k = 1$.

4. Open Questions

**Question 4.1.** What happens if the entire function is replaced by meromorphic function in Theorem 1.2?

**Question 4.2.** What happens if the CM sharing of small function is replaced by IM or weighted sharing of small function in Theorem 1.2?

**Question 4.3.** Is the condition on $n$ in Theorem 1.2 sharp?
**Question 4.4.** Is Theorem 1.2 true if we replace \((f(z) - 1)^k\) by more generalized differential polynomial \(P(f)\)?

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**References**


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