Homomorphism on Bipolar-Valued Q-Fuzzy Subgroup of a Group

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Abstract

In this paper, we made an attempt to study the algebraic nature of bipolar-valued Q-fuzzy subgroups under homomorphism and anti-homomorphism and prove some results on these.

Keywords: Bipolar-valued Q-fuzzy set, bipolar-valued Q-fuzzy subgroup, bipolar-valued Q-fuzzy normal subgroup.

1. Introduction

In 1965, Zadeh [10] introduced the notion of a fuzzy subset of a set, fuzzy sets are a kind of useful mathematical structure to represent a collection of objects
whose boundary is vague. Since then it has become a vigorous area of research in different domains, there have been a number of generalizations of this fundamental concept such as intuitionistic fuzzy sets, interval-valued fuzzy sets, vague sets, soft sets etc [7]. Lee [6] introduced the notion of bipolar-valued fuzzy sets. Bipolar-valued fuzzy sets are an extension of fuzzy sets whose membership degree range is enlarged from the interval [0, 1] to [−1, 1]. In a bipolar-valued fuzzy set, the membership degree 0 means that elements are irrelevant to the corresponding property, the membership degree (0, 1] indicates that elements somewhat satisfy the property and the membership degree [−1, 0) indicates that elements somewhat satisfy the implicit counter property. Bipolar-valued fuzzy sets and intuitionistic fuzzy sets look similar each other. However, they are different each other [6, 7]. We introduce the concept of bipolar-valued Q-fuzzy subgroup under homomorphism, anti-homomorphism and established some results.

2. Preliminaries

2.1 Definition: A bipolar-valued Q-fuzzy set (BVQFS) A in X is defined as an object of the form A = { < (x, q), A^+(x, q), A^−(x, q) > / x in X and q in Q}, where A^+: X×Q→ [0, 1] and A^−: X×Q→ [−1, 0]. The positive membership degree A^+(x, q) denotes the satisfaction degree of an element (x, q) to the property corresponding to a bipolar-valued Q-fuzzy set A and the negative membership degree A^−(x, q) denotes the satisfaction degree of an element (x, q) to some implicit counter-property corresponding to a bipolar-valued Q-fuzzy set A. If A^+(x, q) ≠ 0 and A^−(x, q) = 0, it is the situation that (x, q) is regarded as having only positive satisfaction for A and if A^+(x, q) = 0 and A^−(x, q) ≠ 0, it is the situation that (x, q) does not satisfy the property of A, but somewhat satisfies the counter property of A. It is possible for an element (x, q) to be such that A^+(x, q) ≠ 0 and A^−(x, q) ≠ 0 when the membership function of the property overlaps that of its counter property over some portion of X.

2.2 Example: A = { < (a, q), 0.7, −0.4 >, < (b, q), 0.6, −0.7 >, < (c, q), 0.5, −0.8 >} is a bipolar-valued Q-fuzzy subset of X= {a, b, c }, where Q = {q }.

2.3 Definition: Let G be a group and Q be a non-empty set. A bipolar-valued Q-fuzzy subset A of G is said to be a bipolar-valued Q-fuzzy subgroup of G (BVQFSG) if the following conditions are satisfied,

(i) A^+(xy, q) ≥ min{ A^+(x, q), A^+(y, q) },
(ii) A^+(x^{-1}, q) ≥ A^−(x, q),
(iii) A^−(xy, q) ≤ max{ A^−(x, q), A^−(y, q) },
(iv) A^−(x^{-1}, q) ≤ A^−(x, q), for all x and y in G and q in Q.
2.4 Example: Let $G = \{1, -1, i, -i\}$ be a group with respect to the ordinary multiplication and $Q = \{q\}$. Then $A = \{(1, q), (0.5, 0.6), (0.4, 0.5), (i, q), (0.2, 0.4), (-i, q), (0.2, 0.4)\}$ is a bipolar-valued Q-fuzzy subgroup of $G$.

2.5 Definition: Let $(G, \cdot)$ be a group. A bipolar-valued Q-fuzzy subgroup $A$ of $G$ is said to be a bipolar-valued Q-fuzzy normal subgroup (BVQFNSG) of $G$ if $A^+(xy, q) = A^+(yx, q)$ and $A^-(xy, q) = A^-(yx, q)$, for all $x$ and $y$ in $G$ and $q$ in $Q$.

2.6 Definition: Let $G$ and $G'$ be any two groups. Then the function $f: G \rightarrow G'$ is said to be an anti-homomorphism if $f(xy) = f(y)f(x)$, for all $x$ and $y$ in $G$.

2.7 Definition: Let $X$ and $X'$ be any two sets. Let $f: X \rightarrow X'$ be any function and let $A$ be a bipolar-valued Q-fuzzy subset in $X$, $V$ be a bipolar-valued Q-fuzzy subset in $f(x) = X'$, defined by $V^+(y, q) = \sup_{x \in f^{-1}(y)} A^-(x, q)$ and $V^-(y, q) = \inf_{x \in f^{-1}(y)} A^-(x, q)$, for all $x$ in $X$, $y$ in $X'$, and $q$ in $Q$. $A$ is called a preimage of $V$ under $f$ and is denoted by $f^{-1}(V)$.

2.8 Definition: Let $A = < A^+, A^- >$ be a bipolar-valued Q-fuzzy subset of $X$. Then a bipolar-valued Q-fuzzy subset $A^0 = < A^0+, A^0->$ of $X$, is defined as $A^0+(x, q) = A^+(x, q) / A^+(e, q)$ and $A^0-(x, q) = A^-(x, q) / A^-(e, q)$, for all $x$ in $X$ and $q$ in $Q$.

3. Some Properties

3.1 Theorem: Let $(G, \cdot)$ and $(G', \cdot)$ be any two groups. The homomorphic image of a bipolar-valued Q-fuzzy subgroup of $G$ is a bipolar-valued Q-fuzzy subgroup of $G'$.

Proof: Let $(G, \cdot)$ and $(G', \cdot)$ be any two groups. Let $f : G \rightarrow G'$ be a homomorphism. Then $f(xy) = f(x)f(y)$, for all $x$ and $y$ in $G$. Let $V = f(A)$, where $A$ is a bipolar-valued Q-fuzzy subgroup of $G$. We have to prove that $V$ is a bipolar-valued Q-fuzzy subgroup of $G'$. Now, for $f(x), f(y)$ in $G'$, $V^+(f(x)f(y), q) = V^+(f(xy), q) \geq A^-(xy, q) \geq \min\{A^-(x, q), A^-(y, q)\}$ which implies that $V^+(f(x)f(y), q) \geq \min\{V^+(f(x), q), V^+(f(y), q)\}$. For $f(x)$ in $G'$, $V^+(f[x])^{-1}, q) = V^+(f(x^{-1}), q) \geq A^+(x^{-1}, q) \geq A^+(x, q)$ which implies that $V^+(f[x])^{-1}, q) \geq V^+(f(x), q)$. And $V^-(f(x)f(y), q) = V^-(f(xy), q) \leq A^-(xy, q) \leq \max\{A^-(x, q), A^-(y, q)\}$ which implies that $V^-(f(x)f(y), q) \leq \max\{V^-(f(x), q), V^-(f(y), q)\}$. Also $V^-(f(x])^{-1}, q) = V^-(f(x^{-1}), q) \leq A^-(x^{-1}, q) \leq A^-(x, q)$ which implies that $V^-(f[x])^{-1}, q) \leq V^-(f(x), q)$. Hence $V$ is a bipolar-valued Q-fuzzy subgroup of $G'$.
3.2 Theorem: Let \(( G, . )\) and \(( G', . )\) be any two groups. The homomorphic preimage of a bipolar-valued Q-fuzzy subgroup of \(G'\) is a bipolar-valued Q-fuzzy subgroup of \(G\).

Proof: Let \(( G, . )\) and \(( G', . )\) be any two groups. Let \(f : G \rightarrow G'\) be a homomorphism. Then \(f(xy) = f(x)f(y)\), for all \(x\) and \(y\) in \(G\). Let \(V = f(A)\), where \(V\) is a bipolar-valued Q-fuzzy subgroup of \(G'\). We have to prove that \(A\) is a bipolar-valued Q-fuzzy subgroup of \(G\). Let \(x\) and \(y\) in \(G\) and \(q\) in \(Q\). Now, \(A'(xy, q) = V'( f(xy), q) = V'( f(x)f(y), q) \geq \min \{ V'( f(x), q), V'( f(y), q) \} = \min \{ A'(x, q), A'(y, q) \} \) which implies that \(A'(xy, q) \geq \min \{ A'(x, q), A'(y, q) \} \) and \(A'(x^{-1}, q) = V'( f(x^{-1}), q) = V'( f(x), q) = A'(x, q)\) which implies that \(A'(x^{-1}, q) \geq A'(x, q)\). Also \(A'(x, q) = V'( f(x), q)\) \(\leq \max \{ V'( f(x), q), V'( f(y), q) \} \) which implies that \(A'(x, q) \leq \max \{ A'(x, q), A'(y, q) \} \). Hence \(A\) is a bipolar-valued Q-fuzzy subgroup of \(G\).

3.3 Theorem: Let \(( G, . )\) and \(( G', . )\) be any two groups. The anti-homomorphic image of a bipolar-valued Q-fuzzy subgroup of \(G\) is a bipolar-valued Q-fuzzy subgroup of \(G'\).

Proof: Let \(( G, . )\) and \(( G', . )\) be any two groups. Let \(f : G \rightarrow G'\) be an anti-homomorphism. Then \(f(xy) = f(y)f(x)\), for all \(x\) and \(y\) in \(G\). Let \(V = f(A)\), where \(A\) is a bipolar-valued Q-fuzzy subgroup of \(G\). We have to prove that \(V\) is a bipolar-valued Q-fuzzy subgroup of \(G'\). Now, for \(f(x), f(y)\) in \(G'\), \(V'( f(x)f(y), q) = V'( f(xy), q) \geq \min \{ A'(x, q), A'(y, q) \} \) which implies that \(V'( f(x)f(y), q) \geq \min \{ V'( f(x), q), V'( f(y), q) \} \). For \(f(x)\) in \(G'\), \(V'( f([f(x)]^{-1}), q) = V'( f(x), q) = A'(x, q)\) which implies that \(V'( f([f(x)]^{-1}), q) \geq V'( f(x), q)\). Also \(V'( f(x), q) = V'( f([f(x)]^{-1}), q) = A'(x, q)\) which implies that \(V'( f(x), q) \leq V'( f([f(x)]^{-1}), q) \). Hence \(V\) is a bipolar-valued Q-fuzzy subgroup of \(G'\).

3.4 Theorem: Let \(( G, . )\) and \(( G', . )\) be any two groups. The antihomomorphic preimage of a bipolar-valued Q-fuzzy subgroup of \(G'\) is a bipolar-valued Q-fuzzy subgroup of \(G\).

Proof: Let \(( G, . )\) and \(( G', . )\) be any two groups. Let \(f : G \rightarrow G'\) be an anti-homomorphism. Then \(f(xy) = f(y)f(x)\), for all \(x\) and \(y\) in \(G\). Let \(V = f(A)\), where \(V\) is a bipolar-valued Q-fuzzy subgroup of \(G'\). We have to prove that \(A\) is a bipolar-valued Q-fuzzy subgroup of \(G\). Now, \(A'(x, q) = V'( f(xy), q) \geq \min \{ V'( f(x), q), V'( f(y), q) \} = \min \{ A'(x, q), A'(y, q) \} \) which implies that \(A'(x, q) \geq \min \{ A'(x, q), A'(y, q) \} \).
Also $A^+(x^{-1}, q) = V^+( f(x^{-1}), q) \geq V^+( f(x), q) = A^+(x, q)$ which implies that $A^+(x^{-1}, q) \geq A^+(x, q)$. And $A^-(xy, q) = V^-( f(xy), q) = V^-( f(yf(x), q) \leq \max \{ V^-( f(x), q), V^-( f(y), q) \} = \max \{ A^-(x, q), A^-(y, q) \}$ which implies that $A^-(xy, q) \leq \max \{ A^-(x, q), A^-(y, q) \}$. Also $A^+(x^{-1}, q) = V^+( f(x^{-1}), q) = V^+( [f(x)]^{-1}, q)$, $A^-(x^{-1}, q) = V^-( f(x^{-1}), q) = V^-( [f(x)]^{-1}, q)$ which implies that $A^-(x^{-1}, q) \leq A^-(x, q)$. Hence $A$ is a bipolar-valued Q-fuzzy subgroup of $G$.

3.5 Theorem: Let $(G, \cdot)$ and $(G^I, \cdot)$ be any two groups. The homomorphic image of a bipolar-valued Q-fuzzy normal subgroup of $G$ is a bipolar-valued Q-fuzzy normal subgroup of $G^I$.

Proof: Let $(G, \cdot)$ and $(G^I, \cdot)$ be any two groups. Let $f : G \to G^I$ be a homomorphism. Then $f(xy) = f(x)f(y)$, for all $x$ and $y$ in $G$. Let $V = f(A)$, where $A$ is a bipolar-valued Q-fuzzy normal subgroup of $G$. We have to prove that $V$ is a bipolar-valued Q-fuzzy normal subgroup of $G^I$. Now, for $f(x), f(y)$ in $G^I$, $V^+( f(x)f(y), q) = V^+( f(xy), q) \geq A^+(xy, q) = A^+(yx, q) \leq V^+( f(yx), q) = V^+( f(y)f(x), q)$ which implies that $V^+( f(x)f(y), q) = V^+( f(y)f(x), q)$. And $V^-( f(x)f(y), q) = V^-( f(xy), q) \geq A^-(xy, q) = A^-(yx, q) \leq V^-( f(yx), q) = V^-( f(y)f(x), q)$ which implies that $V^-( f(x)f(y), q) = V^-( f(y)f(x), q)$. Hence $V$ is a bipolar-valued Q-fuzzy normal subgroup of $G^I$.

3.6 Theorem: Let $(G, \cdot)$ and $(G^I, \cdot)$ be any two groups. The homomorphic preimage of a bipolar-valued Q-fuzzy normal subgroup of $G^I$ is a bipolar-valued Q-fuzzy normal subgroup of $G$.

Proof: Let $(G, \cdot)$ and $(G^I, \cdot)$ be any two groups. Let $f : G \to G^I$ be a homomorphism. Then $f(xy) = f(x)f(y)$, for all $x$ and $y$ in $G$. Let $V = f(A)$, where $V$ is a bipolar-valued Q-fuzzy normal subgroup of $G^I$. We have to prove that $A$ is a bipolar-valued Q-fuzzy normal subgroup of $G$. Let $x$ and $y$ in $G$ and $q$ in $Q$. Now, $A^+(xy, q) = V^+( f(xy), q) = V^+( f(x)f(y), q) = V^+( f(y)f(x), q) = V^+( f(yx), q) = A^+(yx, q)$ which implies that $A^+(xy, q) = A^+(yx, q)$. And $A^-(xy, q) = V^-( f(xy), q) = V^-( f(x)f(y), q) = V^-( f(y)f(x), q) = V^-( f(yx), q) = A^-(yx, q)$ which implies that $A^-(xy, q) = A^-(yx, q)$. Hence $A$ is a bipolar-valued Q-fuzzy normal subgroup of $G$.

3.7 Theorem: Let $(G, \cdot)$ and $(G^I, \cdot)$ be any two groups. The anti-homomorphic preimage of a bipolar-valued Q-fuzzy normal subgroup of $G^I$ is a bipolar-valued Q-fuzzy normal subgroup of $G$.

Proof: Let $(G, \cdot)$ and $(G^I, \cdot)$ be any two groups. Let $f : G \to G^I$ be an anti-homomorphism. Then $f(xy) = f(y)f(x)$, for all $x$ and $y$ in $G$. Let $V = f(A)$, where $V$ is a bipolar-valued Q-fuzzy normal subgroup of $G^I$. We have to prove that $A$ is a bipolar-valued Q-fuzzy normal subgroup of $G$. Let $x$ and $y$ in $G$ and $q$
in Q. Now, \( A^+(xy, q) = V^+(f(xy), q) = V^+(f(y)f(x), q) = V^+(f(x)f(y), q) = V^+(f(yx), q) = A^+(yx, q) \) which implies that \( A^+(xy, q) = A^+(yx, q) \). And \( A^-(xy, q) = V^-(f(xy), q) = V^-(f(y)f(x), q) = V^-(f(x)f(y), q) = V^-(f(yx), q) = A^-(yx, q) \) which implies that \( A^-(xy, q) = A^-(yx, q) \). Hence A is a bipolar-valued Q-fuzzy normal subgroup of G.

3.8 Theorem: Let \( A = < A^+, A^- > \) be a bipolar-valued Q-fuzzy subgroup of a group G, \( A^* = < A^*, A^*^- > \) be a bipolar-valued Q-fuzzy set in G defined by \( A^+(x, q) = A^+(x, q) + 1 - A^+(e, q) \) and \( A^-(x, q) = A^-(x, q) - 1 - A^-(e, q) \), for all x in G and q in Q, where e is the identity element of G. Then \( A^* \) is a bipolar-valued Q-fuzzy subgroup of the group G.

Proof: Let x and y in G and q in Q. We have, \( A^+(xy^{-1}, q) = A^+(x, q) + 1 - A^+(e, q) \) \( \geq \) min \{ \( A^+(x, q) \), \( A^+(y, q) \) \} + 1 - \( A^+(e, q) \) = min \{ \( A^+(x, q) + 1 - A^-(e, q) \), \( A^+(y, q) + 1 - A^-(e, q) \) \} = \( A^+(x, q) \). Therefore, \( A^+(xy^{-1}, q) \) \( \geq \) min \{ \( A^+(x, q) \), \( A^+(y, q) \) \}, for all x and y in G and q in Q. Also, \( A^-(xy^{-1}, q) = A^-(x, q) - 1 - A^-(e, q) \) \( \leq \) max \{ \( A^-(x, q) \), \( A^-(y, q) \) \} - 1 - \( A^-(e, q) \) = max \{ \( A^-(x, q) \), \( A^-(y, q) \) \}. Therefore, \( A^+(xy^{-1}, q) \) \( \leq \) max \{ \( A^+(x, q) \), \( A^+(y, q) \) \}, for all x and y in G and q in Q. Hence \( A^* \) is a bipolar-valued Q-fuzzy subgroup of the group G.

3.9 Theorem: Let \( A = < A^+, A^- > \) be a bipolar-valued Q-fuzzy subgroup of a group G, \( A^* = < A^*, A^*^- > \) be a bipolar-valued Q-fuzzy set in G defined by \( A^+(x, q) = A^+(x, q) + 1 - A^+(e, q) \) and \( A^-(x, q) = A^-(x, q) - 1 - A^-(e, q) \), for all x in G and q in Q, where e is the identity element of G. Then \( (A^*)^* = A^* \).

Proof: Let x and y in G and q in Q. We have, \( (A^*)^+(x, q) = (A^*^+(x, q) + 1 - A^*(e, q) \) \( = \) \( A^+(x, q) \), \( A^-(x, q) + 1 - A^*(e, q) \) \( = \) \( A^-^-(x, q) \). Also, \( (A^*)^-(x, q) = A^-(x, q) - 1 - A^*(e, q) \) \( = \) \( A^-^-(x, q) \) \( = \) \( A^-(x, q) - 1 - A^*(e, q) \) \( = \) \( A^-(x, q) - 1 - A^*(e, q) \) \( = \) \( A^-^-(x, q) \). Therefore, \( (A^*)^* = A^* \).

3.10 Theorem: Let \( A = < A^+, A^- > \) be a bipolar-valued Q-fuzzy subgroup of a group G. Then \( A^0 = < A^0+, A^0^- > \) is a bipolar-valued Q-fuzzy subgroup of the group G.

Proof: For any x in G and q in Q, we have \( A^0^+(xy^{-1}, q) = A^-(xy^{-1}, q) / A^-(e, q) \) \( \geq \) \( [1 / A^-(e, q)] \) min \{ \( A^-(x, q) \), \( A^-(y, q) \) \} = \( [A^-(x, q) / A^-(e, q)] \), \( [A^-(y, q) / A^-(e, q)] \} = \( min \{ A^0^+(x, q), A^0^-(y, q) \} \). That is \( A^0^+(xy^{-1}, q) \) \( \geq \) min \{ \( A^0^+(x, q), A^0^-(y, q) \} \}, for all x and y in G and q in Q. Also \( A^0^-(xy^{-1}, q) = A^-(xy^{-1}, q) / A^-(e, q) \) \( \leq \) \( [1 / A^-(e, q)] \) max \{ \( A^-(x, q) \), \( A^-(y, q) \} = \( max \{ A^-(x, q) / A^-(e, q)] \), \( [A^-(y, q) / A^-(e, q)] \} = \( max \{ A^0^-(x, q), A^0^-(y, q) \} \). That is \( A^0^-(xy^{-1}, q) \) \( \leq \) \( max \{ A^0^-(x, q), A^0^-(y, q) \).
A^0(y, q) }, for all x and y in G and q in Q. Hence A^0 is a bipolar-valued Q-fuzzy subgroup of the group G.

References


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