On Common Fixed Point for the Property(E.A.) and OWC Maps in an IFSS Using Implicit Relation

Jong Seo Park

Department of Mathematics Education and Institute of Mathematics Education
Chinju National University of Education
Jinju 660-756, South Korea

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Abstract

In this paper, we define the IFSS and some concepts, and prove the common fixed point for four self maps using weakly compatible, property(E.A.) and OWC in an IFSS using implicit relation.

Keywords: Fixed point, property(E.A.), weakly compatible, OWC, IFSS

1. Introduction

In 2002, Aamri and El[1] defined the notion of property(E.A.), and proved some common fixed point theorems for non-compatible maps under strict contractive conditions. Wilson[11] introduced the symmetric spaces as metric-like spaces lacking the triangle inequality. Imdad et.al.[3] extended the results of [1], [4] to symmetric(semi-metric) spaces. Also, Authors([3], [12]) obtained several fixed point results in that spaces, and Hicks and Rhoades[2] established some common fixed point theorems in symmetric spaces. Recently, Samanta and Mohinta[9] established the common fixed point in fuzzy symmetric space.

In this paper, we define the IFSS and some concepts, and prove the common fixed point for four self maps using weakly compatible, property(E.A.) and OWC in an IFSS using implicit relation.
2. Preliminaries

Let us recall (see [10]) that a continuous \( t \)-norm is a operation \(* : [0, 1] \times [0, 1] \to [0, 1]\) which satisfies the following conditions: (a) \(*\) is commutative and associative, (b) \(*\) is continuous, (c) \( a * 1 = a \) for all \( a \in [0, 1] \), (d) \( a * b \leq c * d \) whenever \( a \leq c \) and \( b \leq d \) \((a, b, c, d \in [0, 1])\). Also, a continuous \( t \)-conorm is a operation \( \circ : [0, 1] \times [0, 1] \to [0, 1]\) which satisfies the following conditions: (a) \( \circ \) is commutative and associative, (b) \( \circ \) is continuous, (c) \( a \circ 0 = a \) for all \( a \in [0, 1] \), (d) \( a \circ b \geq c \circ d \) whenever \( a \leq c \) and \( b \leq d \) \((a, b, c, d \in [0, 1])\).

**Definition 2.1.** The 5-tuple \((X, U, V, *, \circ)\) is said to be an *intuitionistic fuzzy symmetric space* (shortly, IFSS) if \( X \) is an arbitrary set, \(*\) is a continuous \( t \)-norm, \( \circ \) is a continuous \( t \)-conorm and \( U, V \) are fuzzy sets on \( X^2 \times (0, \infty) \) satisfying the following conditions; for all \( x, y \in X \) and \( t > 0 \), such that

(a) \( U(x, y, t) > 0 \),
(b) \( U(x, y, t) = 1 \) if and only if \( x = y \),
(c) \( U(x, y, t) = U(y, x, t) \),
(d) \( U(x, y, \cdot) : (0, \infty) \to (0, 1] \) is continuous,
(e) \( V(x, y, t) > 0 \),
(f) \( V(x, y, t) = 0 \) if and only if \( x = y \),
(g) \( V(x, y, t) = V(y, x, t) \),
(j) \( V(x, y, \cdot) : (0, \infty) \to (0, 1] \) is continuous.

Note that \((U, V)\) is called an IFS on \( X \).

**Proposition 2.2.** Let \( X \) be IFSS. Then \( U, V \) are said to satisfy the following conditions:

(a) If for a sequence \( \{x_n\} \) in \( X \) and \( x, y \in X \),
\[
\lim_{n \to \infty} U(x_n, x, t) = 1 \quad \text{and} \quad \lim_{n \to \infty} V(x_n, x, t) = 0,
\]
\[
\lim_{n \to \infty} U(x_n, y, t) = 1 \quad \text{and} \quad \lim_{n \to \infty} V(x_n, y, t) = 0.
\]

Then \( x = y \).

(b) If for any two sequences \( \{x_n\}, \{y_n\} \) in \( X \) and \( x \in X \),
\[
\lim_{n \to \infty} U(x_n, x, t) = 1 \quad \text{and} \quad \lim_{n \to \infty} V(x_n, x, t) = 0,
\]
\[
\lim_{n \to \infty} U(x_n, y_n, t) = 1 \quad \text{and} \quad \lim_{n \to \infty} V(x_n, y_n, t) = 0.
\]

Then \( \lim_{n \to \infty} U(y_n, x, t) = 1 \) and \( \lim_{n \to \infty} V(y_n, x, t) = 0 \).

(c) If for any two sequences \( \{x_n\}, \{y_n\} \) in \( X \) and \( x \in X \),
\[
\lim_{n \to \infty} U(x_n, x, t) = 1 \quad \text{and} \quad \lim_{n \to \infty} V(x_n, x, t) = 0,
\]
\[
\lim_{n \to \infty} U(y_n, x, t) = 1 \quad \text{and} \quad \lim_{n \to \infty} V(y_n, x, t) = 0.
\]

Then \( \lim_{n \to \infty} U(x_n, y_n, t) = 1 \) and \( \lim_{n \to \infty} V(x_n, y_n, t) = 0 \).
(d) If for any two sequences \( \{x_n\}, \{y_n\} \) in \( X \) and \( x, y \in X \),
\[
\lim_{n \to \infty} U(x_n, x, t) = 1 \quad \text{and} \quad \lim_{n \to \infty} V(x_n, x, t) = 0,
\]
\[
\lim_{n \to \infty} U(y_n, y, t) = 1 \quad \text{and} \quad \lim_{n \to \infty} V(y_n, y, t) = 0,
\]
\[
\lim_{n \to \infty} U(x_n, y_n, t) = 1 \quad \text{and} \quad \lim_{n \to \infty} V(x_n, y_n, t) = 0
\]
then \( x = y \).

**Definition 2.3.** ([8]) Let \( A \) and \( B \) be two self maps of an IFSS \( X \).
(a) The order pair \((A, B)\) is strongly partially commuting if for any sequence \( \{x_n\} \subset X \),
\[
\lim_{n \to \infty} U(Ax_n, Bx_n, t) = 1 \quad \text{and} \quad \lim_{n \to \infty} V(Ax_n, Bx_n, t) = 0,
\]
then \( \lim_{n \to \infty} U(AAx_n, BAx_n, t) = 1 \) and \( \lim_{n \to \infty} V(AAx_n, BAx_n, t) = 0 \).
(b) \( A \) and \( B \) are said to be compatible if for all \( t > 0 \),
\[
\lim_{n \to \infty} U(ABx_n, BAx_n, t) = 1 \quad \text{and} \quad \lim_{n \to \infty} V(ABx_n, BAx_n, t) = 0,
\]
whenever \( \{x_n\} \) is a sequence in \( X \) such that for some \( z \in X \),
\[
\lim_{n \to \infty} U(Ax_n, z, t) = \lim_{n \to \infty} U(Bx_n, z, t) = 1, \quad \lim_{n \to \infty} V(Ax_n, z, t) = \lim_{n \to \infty} V(Bx_n, z, t) = 0.
\]
(c) A point \( z \in X \) is called coincidence point of \( A \) and \( B \) iff \( Ax = Bz \).
(d) \( A \) and \( B \) are said to be weakly compatible if \( Az = Bz \) implies that \( ABz = BAz \).
(e) \( A \) and \( B \) satisfy the property(E.A.) if there exists a sequence \( \{x_n\} \) in \( X \) such that for some \( z \in X \),
\[
\lim_{n \to \infty} U(Ax_n, z, t) = \lim_{n \to \infty} U(Bx_n, z, t) = 1, \quad \lim_{n \to \infty} V(Ax_n, z, t) = \lim_{n \to \infty} V(Bx_n, z, t) = 0.
\]

**Definition 2.4.** ([6]) The maps \( A, B, S, T : X \to X \) of an IFSS \( X \) satisfy a common property(E.A.) if there exist two sequences \( \{x_n\} \) and \( \{y_n\} \) such that for some \( z \in X \),
\[
\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} By_n = \lim_{n \to \infty} Ty_n = z.
\]

**Example 2.5.** Let \( X = [0, +\infty) \) with the IFS \( U, V \) defined by
\[
U(x, y, t) = \frac{t}{t + e^{\left|x-y\right|} - 1}, \quad V(x, y, t) = \frac{e^{\left|x-y\right|} - 1}{t + e^{\left|x-y\right|} - 1}
\]
for all \( x, y \in X \). Define \( A, B, S, T : X \to X \) as follows \( Ax = 2x + 5 \), \( Bx = 3x + 2 \), \( Sx = x + 5 \) and \( Tx = 2x + 3 \) for all \( x \in X \). Consider the sequence
$x_n = \frac{1}{n}$ and $y_n = \frac{1}{n} + 1$, $n = 1, 2, \ldots$. Then

$$\lim_{n \to \infty} U(Ax_n, 5, t) = \lim_{n \to \infty} U(Sx_n, 5, t)$$
$$= \lim_{n \to \infty} U(By_n, 5, t) = \lim_{n \to \infty} U(Ty_n, 5, t) = 1,$$

$$\lim_{n \to \infty} V(Ax_n, 5, t) = \lim_{n \to \infty} V(Sx_n, 5, t)$$
$$= \lim_{n \to \infty} V(By_n, 5, t) = \lim_{n \to \infty} V(Ty_n, 5, t) = 0.$$

Therefore $A, B, S$ and $T$ satisfy the property(E.A.) and common property(E.A.).

**Definition 2.6.** ([8]) A pair map of maps $f$ and $g$ is called occasionally weakly compatible (shortly, OWC) iff there is a point $x \in X$ which is a coincidence point of $f$ and $g$ at which $f$ and $g$ commute.

3. **Main results**

Let $\Psi = \{\phi, \psi\}$ be implicit relations set, $I = [0, 1]$, $\phi, \psi : I \to R$ be continuous functions satisfying $\phi(t) > t$ and $\psi(t) < t$ for each $t \geq 0$.

**Theorem 3.1.** Let $U, V$ be IFS for $X$ that satisfies (b) and (c) of Proposition 2.2. Suppose that $A, B, S$ and $T$ is self-mappings of IFSS $X$ such that for all $x, y \in X$ and $Ax \neq By$,

(a) IFS $U, V$ are satisfied the following conditions;

$$U(Ax, By, t) \geq \phi(\min\{U(Sx, Ty, t), U(Sx, By, t), U(Ty, By, t)\}),$$
$$V(Ax, By, t) \leq \psi(\max\{V(Sx, Ty, t), V(Sx, By, t), V(Ty, By, t)\}),$$

(b) $(A, T)$ and $(B, S)$ are weakly compatibles,

(c) $(A, S)$ and $(B, T)$ satisfy the property(E.A.),

(d) $AX \subset TX$ and $BX \subset SX$.

If the range of any one for $A, B, S$ or $T$ are a complete subspace of $X$, then $A, B, S$ and $T$ have a unique common fixed point in $X$.

**Proof.** Suppose that $(B, T)$ satisfies the property(E.A.). Then there exist a sequence $\{x_n\} \subset X$ such that for some $z \in X$,

$$\lim_{n \to \infty} U(Bx_n, z, t) = \lim_{n \to \infty} U(Tx_n, z, t) = 1$$
$$\lim_{n \to \infty} V(Bx_n, z, t) = \lim_{n \to \infty} V(Tx_n, z, t) = 0.$$

Since $BX \subset SX$, there exist a sequence $\{y_n\} \subset X$ such that $Bx_n = Sx_n$. Hence

$$\lim_{n \to \infty} U(Sy_n, z, t) = 1, \quad \lim_{n \to \infty} V(Sy_n, z, t) = 0.$$
Now, we will prove that \( \lim_{n \to \infty} U(Ay_n, z, t) = 1, \lim_{n \to \infty} V(Ay_n, z, t) = 0. \) From the condition (a),

\[
U(Ay_n, Bx_n, t) \\
\geq \phi(\min\{U(Sy_n, Tx_n, t), U(Sy_n, Bx_n, t), U(Tx_n, Bx_n, t)\}) \\
\geq \phi(\min\{U(Bx_n, Tx_n, t), 1, U(Tx_n, Bx_n, t)\}) \\
\geq \phi(U(Bx_n, Tx_n, t)) > U(Bx_n, Tx_n, t), \\
V(Ay_n, Bx_n, t) \\
\leq \psi(\max\{V(Sy_n, Tx_n, t), V(Sy_n, Bx_n, t), V(Tx_n, Bx_n, t)\}) \\
\leq \psi(\max\{V(Bx_n, Tx_n, t), 0, V(Tx_n, Bx_n, t)\}) \\
\leq \psi(V(Bx_n, Tx_n, t)) < V(Bx_n, Tx_n, t).
\]

From (c) of Proposition 2.2,

\[
\lim_{n \to \infty} U(Ay_n, Bx_n, t) = 1, \quad \lim_{n \to \infty} V(Ay_n, Bx_n, t) = 0.
\]

Therefore by (b) of Proposition 2.2,

\[
\lim_{n \to \infty} U(Ay_n, z, t) = 1, \quad \lim_{n \to \infty} V(Ay_n, z, t) = 0.
\]

Suppose that SX is a complete subspace of X. Then \( z = Su \) for some \( u \in X \). Subsequently, we have

\[
\lim_{n \to \infty} U(Ay_n, Su, t) = \lim_{n \to \infty} U(Bx_n, Su, t) \\
= \lim_{n \to \infty} U(Tx_n, Su, t) = \lim_{n \to \infty} U(Sy_n, Su, t) = 1, \\
\lim_{n \to \infty} V(Ay_n, Su, t) = \lim_{n \to \infty} V(Bx_n, Su, t) \\
= \lim_{n \to \infty} V(Tx_n, Su, t) = \lim_{n \to \infty} V(Sy_n, Su, t) = 0.
\]

From the condition (a), it follows

\[
U(Au, Bx_n, t) \\
\geq \phi(\min\{U(Su, Tx_n, t), U(Su, Bx_n, t), U(Tx_n, Bx_n, t)\}), \\
V(Au, Bx_n, t) \\
\leq \psi(\max\{V(Su, Tx_n, t), V(Su, Bx_n, t), V(Tx_n, Bx_n, t)\}).
\]

Then we have as \( n \to \infty \),

\[
\lim_{n \to \infty} U(Au, Bx_n, t) = 1, \quad \lim_{n \to \infty} V(Au, Bx_n, t) = 0.
\]

Therefore from (a) of Proposition, \( Au = Su \). Since \( ASu = SAu \), \( AAu = ASu = SAu = SSu \).
On the other hand, since $AX \subset TX$, there exist $v \in X$ such that $Au = Tv$. Hence we will prove that $Au = Bv$. If $Au \neq Bv$, then from condition (a),

\[
U(Au, Bv, t) \\
\geq \phi(\min\{U(Su, Tv, t), U(Su, Bv, t), U(Tv, Bv, t)\}) \\
= \phi(\min\{1, U(Au, Bv, t), U(Au, Bv, t)\}) \\
= \phi(U(Au, Bv, t)) > U(Au, Bv, t),
\]

\[
V(Au, Bv, t) \\
\leq \psi(\max\{V(Su, Tv, t), V(Su, Bv, t), V(Tv, Bv, t)\}) \\
= \psi(\max\{0, V(Au, Bv, t), V(Au, Bv, t)\}) \\
= \psi(V(Au, Bv, t)) < V(Au, Bv, t),
\]

which is a contradiction. Hence $Au = Su = Tv = Bv$. Also, since $(B, T)$ is a weakly compatible, $BTv = TBv$ and $TTv = T(Bv = BBv)$.

Suppose that $AAu \neq Au$. Then we have

\[
U(Au, AAu, t) = U(AAu, Bv, t) \\
\geq \phi(\min\{U(SAu, Tv, t), U(SAu, Bv, t), U(Tv, Bv, t)\}) \\
= \phi(\min\{U(AAu, Au, t), U(AAu, Au, t), 1\}) \\
\geq \phi(U(AAu, Au, t)) > U(Au, AAu, t),
\]

\[
V(Au, AAu, t) = V(AAu, Bv, t) \\
\leq \psi(\max\{V(SAu, Tv, t), V(SAu, Bv, t), V(Tv, Bv, t)\}) \\
\leq \psi(\max\{V(AAu, Au, t), V(AAu, Au, t), 0\}) \\
\leq \phi(U(AAu, Au, t)) < V(Au, AAu, t),
\]

which is a contradiction. Hence $Au = AAu = SAu$ and $Au$ is a common fixed point of $A$ and $S$. Also, we can prove that $Bv$ is a common fixed point of $B$ and $T$. Therefore $Au$ is a common fixed point of $A, B, S$ and $T$.

If $Au = Bu = Su = Tu = u$ and $Av = Bv = Sv = Tv = v$ and $u \neq v$, then form condition (a),

\[
U(u, v, t) = U(Au, Bv, t) \\
\geq \phi(\min\{U(Su, Tv, t), U(Su, Bv, t), U(Tv, Bv, t)\}) \\
\geq \phi(U(u, v, t)) > U(u, v, t),
\]

\[
V(u, v, t) = V(Au, Bv, t) \\
\leq \psi(\max\{V(Su, Tv, t), V(Su, Bv, t), V(Tv, Bv, t)\}) \\
\leq \psi(V(u, v, t)) < V(u, v, t),
\]

which is a contradiction. Hence $u = v$, that is, $A, B, S$ and $T$ have a unique common fixed point in $X$.

\[\Box\]

**Theorem 3.2.** Let $X$ be an IFSS with IFS $U, V$. Suppose that $A, B, S$ and $T$ are four self-mappings of $X$ as followings;
(a) For all \(x, \, y \in X\) and \(Ax \neq By\),
\[
U^2(Ax, By, t) \\
\geq \min\{\phi(U(Sx, Ty, t))\phi(U(Sx, Ay, t)), \phi(U(Sx, Ty, t))\phi(U(Tx, By, t)), \phi(U(Sx, Ax, t))\phi(U(Ty, By, t)), \phi(U(Sy, By, t))\phi(U(Ty, Ax, t))\},
\]
\[
V^2(Ax, By, t) \\
\leq \max\{\psi(V(Sx, Ty, t))\psi(V(Sx, Ay, t)), \psi(V(Sx, Ty, t))\psi(V(Tx, By, t)), \psi(V(Sx, Ax, t))\psi(V(Ty, By, t)), \psi(V(Sy, By, t))\psi(V(Ty, Ax, t))\},
\]

(b) The pairs \((A, S)\) and \((B, T)\) are OWC.
Then \(A, B, S\) and \(T\) have a unique common fixed point in \(X\).

Proof. From condition (b), since \((A, S)\) and \((B, T)\) are owc, there exist \(u, \, v \in X\) such that \(Au = Su\), \(ASu = SAu\), \(Bv = Tv\) and \(BTv = TBv\). If \(Au \neq Bv\), then by condition (a), we have
\[
U^2(Au, Bv, t) \\
\geq \min\{\phi(U(Su, Tv, t))\phi(U(Su, Au, t)), \phi(U(Su, Tv, t))\phi(U(Tv, Bv, t)), \phi(U(Su, Au, t))\phi(U(Tv, Bv, t)), \phi(U(Su, Bv, t))\phi(U(Tv, Au, t))\}
\]
\[
= \min\{\phi(U(Au, Bv, t))\phi(U(Au, Bv, t)), 1, \phi(U(Au, Bv, t))\phi(U(Au, Bv, t))\}
\]
\[
= \phi^2(Au, Bv, t) > U^2(Au, Bv, t),
\]
\[
V^2(Au, Bv, t) \\
\leq \max\{\psi(V(Su, Tv, t))\psi(V(Su, Au, t)), \psi(V(Su, Tv, t))\psi(V(Tv, Bv, t)), \psi(V(Su, Au, t))\psi(V(Tv, Bv, t)), \psi(V(Su, Bv, t))\psi(V(Tv, Au, t))\}
\]
\[
= \max\{\psi(V(Au, Bv, t))\psi(V(Au, Bv, t)), 0, \psi(V(Au, Bv, t))\psi(V(Au, Bv, t))\}
\]
\[
= \psi^2(V(Au, Bv, t)) < V^2(Au, Bv, t),
\]
which is a contradiction. Hence \(Au = Bv = Su = Tv\).

Now, suppose that \(AAu \neq Au\), then we have from condition (a),
\[
U^2(AAu, Bv, t) \\
\geq \min\{\phi(U(SAu, Tv, t))\phi(U(SAu, AAu, t)), \phi(U(SAu, Tv, t))\phi(U(Tv, Bv, t)), \phi(U(SAu, AAu, t))\phi(U(Tv, Bv, t)), \phi(U(SAu, Bv, t))\phi(U(Tv, AAu, t))\}
\]
\[
= \min\{\phi(U(AAu, Bv, t))\phi(U(AAu, Bv, t)), 1, \phi(U(AAu, Bv, t))\phi(U(AAu, Bv, t))\}
\]
\[
= \phi^2(AAu, Bv, t) > U^2(AAu, Bv, t),
\]
\[
V^2(AAu, Bv, t) \\
\leq \max\{\psi(V(SAu, Tv, t))\psi(V(SAu, AAu, t)), \psi(V(SAu, Tv, t))\psi(V(Tv, Bv, t)), \psi(V(SAu, AAu, t))\psi(V(Tv, Bv, t)), \psi(V(SAu, Bv, t))\psi(V(Tv, AAu, t))\}
\]
\[
= \max\{\psi(V(AAu, Bv, t))\psi(V(AAu, Bv, t)), 0, \psi(V(AAu, Bv, t))\psi(V(AAu, Bv, t))\}
\]
\[
= \psi^2(V(AAu, Bv, t)) < V^2(AAu, Bv, t),
\]
which is a contradiction. Hence $AAu = Au = SAu$. Also, similarly, $BAu = TAu = Au$. Hence $Au = Bv = Su = Tv$ is a common fixed point of $A, B, S$ and $T$.

Let $w, z (w \neq z)$ be common fixed point of $A, B, S$ and $T$. Then we have from condition (a),

$$
U^2(w, z, t) = U^2(Aw, Bz, t)
\geq \min\{\phi(U(Sw, Tz, t))\phi(U(Sw, Aw, t)), \phi(U(Sw, Tz, t))\phi(U(Tz, Bz, t)), \\
\phi(U(Sw, Aw, t))\phi(U(Tz, Bz, t)), \phi(U(Sw, Bz, t))\phi(U(Tz, Aw, t))\}
= \min\{\phi(U(w, z, t))\phi(U(w, z, t)), 1, \phi(U(w, z, t))\phi(U(w, z, t))\}
= \phi^2(U(w, z, t)) > U^2(w, z, t),
$$

$$
V^2(w, z, t) = V^2(Aw, Bz, t)
\leq \max\{\psi(V(Sw, Tz, t))\psi(V(Sw, Aw, t)), \psi(V(Sw, Tz, t))\psi(V(Tz, Bz, t)), \\
\psi(V(Sw, Aw, t))\psi(V(Tz, Bz, t)), \psi(V(Sw, Bz, t))\psi(V(Tz, Aw, t))\}
= \max\{\psi(V(w, z, t))\psi(V(w, z, t)), 0, \psi(V(w, z, t))\psi(V(w, z, t))\}
= \psi^2(V(w, z, t)) < V^2(w, z, t),
$$

which is a contradiction. Hence $w = z$, that is, $z$ is a unique common fixed point of $A, B, S$ and $T$.

References


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