Topologies of Hilbert Space Effect Algebras and Scale Effect Algebras

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Abstract

In this paper, we study the topologies of scale effect algebras and Hilbert space effect algebras and obtain: (1) The interval topology and the order topology of scale effect algebras are the same and they are Hausdorff; The effect algebraic operations $\oplus$ and $\ominus$ are two variables continuous in the scale effect algebras with the interval topology. (2) We give the relationships between the interval topology, order topology, weak operator topology and strong operator topology of the Hilbert space effect algebras.

Mathematics Subject Classification: 81Q99

Keywords: Hilbert space effect algebra, Scale effect algebra, Interval topology, Order topology, Strong operator topology, Weak operator topology

1 Introduction

If a quantum-mechanical system $E$ is represented in the usual way by a Hilbert space $\mathcal{H}$, then a self-adjoint operator $A$ on $\mathcal{H}$ such that $0 \leq A \leq I$ corresponds
to an effect for $E$. In order to model unsharp quantum logics, Foulis and Bennett introduced the following quantum logic structure called an effect algebra [2].

A structure $(E, \oplus, 0, 1)$ is called an effect algebra if $0, 1$ are two distinguished elements and $\oplus$ is a partially defined binary operation on $E$ which satisfies the following conditions for any $a, b, c \in E$:

(1). If $a \oplus b$ is defined, then $b \oplus a$ is defined and $a \oplus b = b \oplus a$.

(2). If $a \oplus b$ and $(a \oplus b) \oplus c$ are defined, then $b \oplus c$ and $a \oplus (b \oplus c)$ are defined and $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.

(3). For each $a \in E$ there exists a unique $b \in E$ such that $a \oplus b$ is defined and $a \oplus b = 1$.

(4). If $a \oplus 1$ is defined, then $a = 0$.

We often denote the effect algebra $(E, \oplus, 0, 1)$ briefly by $E$. For each $a \in E$, we denote the unique $b$ in condition (3) by $a'$ and call it the orthosupplement of $a$. The sense is that if $a$ presents a proposition, then $a'$ corresponds to the negation. The operation $\oplus$ of an effect algebra $E$ can induce a new partial operation $\ominus$ and a partial order $\leq$ as follows: $a \ominus b$ is defined iff there exists $c \in E$ such that $b \oplus c$ is defined and $b \oplus c = a$, in which case we denote $c$ by $a \ominus b$; $d \leq e$ iff there exists $f \in E$ such that $d \oplus f$ is defined and $d \oplus f = e$. Thus each effect algebra is a poset. If the partial order $\leq$ of $E$ defined as above is a totally order, then $E$ is said to be a scale effect algebra. For more details on effect algebras, for example, $a \oplus b$ is defined iff $a \leq b'$, we refer to [2].

Example 1.1. [2] Let $\mathcal{H}$ be a complex Hilbert space and $E(\mathcal{H})$ be the set of self-adjoint operators on $\mathcal{H}$ that satisfy $0 \leq A \leq I$. For $A, B \in E(\mathcal{H})$, we define $\oplus$ as follows: $A \oplus B = A + B$ iff $A + B \leq I$. Then $(E(\mathcal{H}), \oplus, 0, I)$ is an effect algebra called Hilbert space effect algebra.

Hilbert space effect algebra $(E(\mathcal{H}), \oplus, 0, I)$ is important in foundational studies of quantum mechanics.

Example 1.2. [2] Let $G$ be a totally ordered abelian group. For any $u \in G$ and $0 < u$, the effect algebra $G^+[0, u] = \{g \in G|0 \leq g \leq u\}$ is a scale effect algebra.

The study of effect algebras from any aspect-algebraic, order, or state space-inevitably involves topological questions. In fact, the notions of infinite sums of elements of an effect algebra and of $\sigma$-additive states suggest an underlying topological basis. For example, consider the Hilbert space effect algebra $E(\mathcal{H})$. Let $(A_i)_{i \in \mathbb{N}}$ be a sequence of $E(\mathcal{H})$ and satisfy the following condition: For any finite set $F$ of $\mathbb{N}$, there is

$$\sum_{i \in F} A_i = \bigoplus_{i \in F} A_i \leq I.$$
Note that the algebraic structure and order structure of $E(\mathcal{H})$ can not answer the existence of $\bigoplus_{i\in\mathbb{N}} A_i$. While the topological structure of $E(\mathcal{H})$ tells us that it is just the strong operator topology limit of$(\sum_{i=1}^{n} A_i)_{n\in\mathbb{N}}$ [3]. At the same time, the Stone Representations for Boolean Algebras [10] and Priestley duality theory of distributive lattices [9] both have strongly topological implication. Therefore, study about of topologies of quantum logics has important theoretical and realistic significance.

Interval topology and order topology are the most important intrinsic topologies of quantum logics [1,4,5-8] and the weak operator topology, strong operator topology and norm topology are the nature and important topologies of Hilbert space effect algebras [3].

In this paper, we give the relationships between the interval topology, order topology, weak operator topology, strong operator topology and the norm topology of the Hilbert space effect algebras. Also, we prove that the interval topology and order topology of scale effect algebras are equivalent and they are Hausdorff and the the effect algebraic operations $\oplus$ and $\ominus$ are two variables continuous respect to the interval topology.

2 Interval topology of scale effect algebras

Let $E$ be an effect algebra and $a, b \in E$ with $a \leq b$. We use the notation $[a, b]$ for the interval $[a, b] = \{x \in E | a \leq x \leq b\}$.

By the interval topology of an effect algebra $E$, we mean a topology with closed intervals $[a, b]$ as a sub-basis of closed sets of $E$. We usually denote the interval topology by $\tau_i$.

It is clear that in a scale effect algebra $E$, $\{x \in E | x < r_1\}$ and $\{x \in E | x > r_2\}$ consist of the sub-basis of open sets of $E$ where $r_1, r_2 \in E$.

Assume that $P$ is a poset and $(a_\alpha)_{\alpha \in \Lambda}$ is a net of $P$. If $a_\alpha \leq a_\beta$ for all $\alpha, \beta \in \Lambda$ such that $\alpha \preceq \beta$, then we write $a_\alpha \uparrow$. If moreover $a = \bigvee \{a_\alpha | \alpha \in \Lambda\}$, we write $a_\alpha \uparrow a$. The meaning of $a_\alpha \downarrow$ and $a_\alpha \downarrow a$ is dual. The net $(a_\alpha)_{\alpha \in \Lambda}$ is said to be order convergent ((o)-convergent, for short) to a point $a \in P$ if there are nets $(u_\alpha)_{\alpha \in \Lambda}$ and $(v_\alpha)_{\alpha \in \Lambda}$ of $P$ such that $a \uparrow u_\alpha \leq a_\alpha \leq v_\alpha \downarrow a$, and we write $a_\alpha \xrightarrow{(o)} a$. If $\tau$ is another topology of $P$ such that each (o)-convergent net of $P$ is $\tau$- convergent, then we call $\tau$ has $C$ property. The strongest topology of $P$ which have $C$ property is called order topology and it is denoted by $\tau_o$.

It is obvious that $\tau_i \subseteq \tau_o$.

Lemma 2.1. [4] Let $(E, \oplus, 0, 1)$ be a lattice effect algebra. Then its interval topology $\tau_i$ is Hausdorff iff for any nets $(a_\alpha)_{\alpha \in \Lambda}$ and $(b_\alpha)_{\alpha \in \Lambda}$ in $E$ with $a_\alpha \xrightarrow{(\tau_i)} a$ and $b_\alpha \xrightarrow{(\tau_i)} b$, the condition $a_\alpha \leq b_\alpha$ for any $\alpha \in \Lambda$ implies $a \leq b$. 

Theorem 2.2. Let \((E, \oplus, 0, 1)\) be a scale effect algebra. Then

(1) \(\tau_i = \tau_o\) and they are Hausdorff.

(2) For any nets \((a_\alpha)_{\alpha \in \Lambda}\) and \((b_\alpha)_{\alpha \in \Lambda}\) in \(E\) with \(a_\alpha \xrightarrow{(\tau_i)} a\), \(b_\alpha \xrightarrow{(\tau_i)} b\) and \(a_\alpha \leq b'_\alpha\) for any \(\alpha \in \Lambda\), then \(a_\alpha \oplus b_\alpha \xrightarrow{(\tau_i)} a \oplus b\).

(3) For any nets \((a_\alpha)_{\alpha \in \Lambda}\) and \((b_\alpha)_{\alpha \in \Lambda}\) in \(E\) with \(a_\alpha \xrightarrow{(\tau_i)} a\), \(b_\alpha \xrightarrow{(\tau_i)} b\) and \(a_\alpha \leq b_\alpha\) for any \(\alpha \in \Lambda\), then \(a_\alpha \oplus b_\alpha \xrightarrow{(\tau_i)} a \oplus b\).

Proof. (1) \(\tau_i = \tau_o\) is obvious. Let \(a, b \in E\) with \(a \neq b\). Without any loss, we suppose \(a > b\). There are the following two cases.

(a) If there exists \(r \in E\) such that \(a > r > b\), then \(\{x \in E| x > r\}\) and \(\{x \in E| x < r\}\) are two open sets which conclude \(a\) and \(b\) respectively and \(\{x \in E| x > r\}\) \(\cap\) \(\{x \in E| x < r\}\) = \(\emptyset\). So \(\tau_i\) is Hausdorff.

(b) If there not exists \(r \in E\) such that \(a > r > b\), then \(\{x \in E| x > b\}\) and \(\{x \in E| x < a\}\) are two open sets which conclude \(a\) and \(b\) respectively and \(\{x \in E| x > b\}\) \(\cap\) \(\{x \in E| x < a\}\) = \(\emptyset\). So \(\tau_i\) is Hausdorff.

(2) From (1) and Lemma 2.1, \(a \oplus b\) is defined. If there exists \(\alpha_0 \in \Lambda\) such that \(a_\alpha = a\) or \(b_\alpha = b\) for any \(\alpha_0 \leq \alpha\), then the conclusion is correct. Otherwise, for any \(r \in E\) with \(r > 0\), there exists \(\alpha_1 \in \Lambda\) such that

\[
0 < a_{\alpha_1} \oplus a < r \quad 0 < a \oplus a_{\alpha_1} < r
\]

Let \(r_1 = a_{\alpha_1} \ominus a\) or \(a \ominus a_{\alpha_1}\), then \(0 < r_1 < r\). As \(r \ominus r_1 > 0\), there exists \(\alpha_2 \in \Lambda\) such that

\[
0 < a_{\alpha_2} \ominus a < r \ominus r_1 \quad 0 < a \ominus a_{\alpha_2} < r \ominus r_1
\]

Let \(r_2 = a_{\alpha_2} \ominus a\) or \(a \ominus a_{\alpha_2}\). Then \(0 < r_2 < r \ominus r_1\). Hence, \(0 < r_1 \ominus r_2 < r\). Denote \(\Lambda_1 = \{\alpha \in \Lambda| (a_\alpha \ominus b_\alpha) \ominus (a \ominus b) \geq 0\}\), \(\Lambda_2 = \{\alpha \in \Lambda| (a \oplus b) \ominus (a_\alpha \oplus b_\alpha) \geq 0\}\).

(a) If \(\alpha \in \Lambda_1\), then \((a_\alpha \ominus b_\alpha) \ominus (a \ominus b) \geq 0\). As \(a_\alpha \xrightarrow{(\tau_i)} a\), \(b_\alpha \xrightarrow{(\tau_i)} b\), there exists \(\alpha_0 \in \Lambda\) such that

\[
a_\alpha \ominus a < r_1 \quad a \ominus a_\alpha < r_1
\]

\[
b_\alpha \ominus b < r_2 \quad b \ominus b_\alpha < r_2
\]

for any \(\alpha_0 \leq \alpha\) and \(\alpha \in \Lambda_1\). Since \((a_\alpha \ominus b_\alpha) \ominus (a \ominus b) \geq 0\),

\[
(a_\alpha \ominus b_\alpha) \ominus (a \ominus b) = a_\alpha \ominus a \ominus b_\alpha \ominus b < r_1 \ominus r_2 < r
\]

or

\[
(a_\alpha \ominus b_\alpha) \ominus (a \ominus b) = b_\alpha \ominus b \ominus a_\alpha \ominus a < r_1 \ominus r_2 < r.
\]

That is, there is an \(\alpha_0 \in \Lambda\), such that \((a_\alpha \ominus b_\alpha) \ominus (a \ominus b) < r\) for any \(\alpha_0 \leq \alpha\) and \(\alpha \in \Lambda_1\).
(b) If $\alpha \in \Lambda_2$, then $(a \oplus b) \ominus (a_\alpha \oplus b_\alpha) \geq 0$. As $a_\alpha \xrightarrow{\tau_1} a$, \quad b_\alpha \xrightarrow{\tau_1} b$, there exists $\alpha_0 \in \Lambda$, such that

$$a_\alpha \oplus a < r_1 \text{ or } a \ominus a_\alpha < r_1$$

$$b_\alpha \oplus b < r_2 \text{ or } b \ominus b_\alpha < r_2$$

for any $\alpha_0 \leq \alpha$ and $\alpha \in \Lambda_2$. Since $(a \oplus b) \ominus (a_\alpha \oplus b_\alpha) \geq 0$,

$$(a \oplus b) \ominus (a_\alpha \oplus b_\alpha) = a \ominus a_\alpha \oplus b \ominus b_\alpha < r_1 \ominus r_2 < r$$

or

$$(a \oplus b) \ominus (a_\alpha \oplus b_\alpha) = b \ominus b_\alpha \ominus a \ominus a_\alpha < r_1 \ominus r_2 < r.$$

That is, there exists $\alpha_0 \in \Lambda$ such that $(a \oplus b) \ominus (a_\alpha \oplus b_\alpha) < r$ for any $\alpha_0 \leq \alpha$ and $\alpha \in \Lambda_2$.

In summary, for any $r \in E$ with $r > 0$, there exists $\alpha_0 \in \Lambda$ such that

$$(a_\alpha \oplus b_\alpha) \ominus (a \oplus b) < r \text{ or } (a \oplus b) \ominus (a_\alpha \oplus b_\alpha) < r$$

for any $\alpha_0 \leq \alpha$. Therefore, it is easy to prove that $a_\alpha \oplus b_\alpha \xrightarrow{\tau_1} a \oplus b$.

(3) It is similar with (2).

\[\square\]

3 Topologies of Hilbert space effect algebras

In this section, we assume that $\mathcal{H}$ is a Hilbert space and $\mathcal{B}(\mathcal{H})$ is a set of continuous linear operators on $\mathcal{H}$. Obviously, $E(\mathcal{H}) \subseteq \mathcal{B}(\mathcal{H})$. The natural partial order $\leq$ of $E(\mathcal{H})$: For $A, B \in E(\mathcal{H})$, $A \leq B \iff \forall x \in \mathcal{H}, \langle Ax, x \rangle \leq \langle Bx, x \rangle$. It is easy to see that the natural partial order is coincide with the partial order induced by the effect algebraic operations.

**Definition 3.1.** [3] Suppose that $\mathcal{V}$ is a linear space with scalar field $K$, and $\mathcal{F}$ is a family of linear functionals on $\mathcal{V}$, which separates the points of $\mathcal{V}$. For any $\rho \in \mathcal{F}$, the equation $P_\rho(x) = |\rho(x)|$ defines a semi-norm $P_\rho$ on $\mathcal{V}$. The topology generated by $\{P_\rho | \rho \in \mathcal{F}\}$ is called the weak topology induced by $\mathcal{F}$.

**Definition 3.2.** [3] The weak operator topology on $\mathcal{B}(\mathcal{H})$ is the weak topology on $\mathcal{B}(\mathcal{H})$ induced by the family $\mathcal{F}_w$ of linear functionals $\omega_{x,y} : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ defined by the equation $\omega_{x,y}(A) = \langle Ax, y \rangle$, $x, y \in \mathcal{H}$. The weak operator topology is denoted by WOT.

The family of sets of the form

$$V(T_0 : \omega_{x_1,y_1}, \ldots, \omega_{x_m,y_m}; \varepsilon) = \{T \in \mathcal{B}(\mathcal{H}) : |\langle(T-T_0)x_j, y_j \rangle| < \varepsilon (j = 1, \ldots, m)\}$$

where $\varepsilon$ is positive and $x_1, \ldots, x_m, y_1, \ldots, y_m \in \mathcal{H}$ constitutes a base of neighborhoods of $T_0$ in WOT.
Definition 3.3. [3] For any \( x \in \mathcal{H} \), the equation \( P_x(T) = \|Tx\| \) defines a semi-norm \( P_x \) on \( \mathcal{B}(\mathcal{H}) \). The family of all semi-norms \( \{P_x : x \in \mathcal{H}\} \) gives rise to a topology on \( \mathcal{B}(\mathcal{H}) \) called strong operator topology and denoted by \( \text{SOT} \).

In the strong operator topology, an element \( T_0 \in \mathcal{B}(\mathcal{H}) \) has a base of neighborhoods consisting of all sets of type

\[
V(T_0 : x_1, \cdots, x_m; \varepsilon) = \{T \in \mathcal{B}(\mathcal{H}) : \|(T - T_0)x_j\| < \varepsilon (j = 1, \cdots, m)\}
\]

where \( \varepsilon \) is positive and \( x_1, \cdots, x_m \in \mathcal{H} \).

It can be proved \( T_0 \xrightarrow{\text{WOT}} T \iff \forall x, y \in \mathcal{H}, \langle T_0x, y \rangle \rightarrow \langle Tx, y \rangle \iff \forall x \in \mathcal{H}, \|(T_0 - T)x\| \rightarrow 0. \)

Obviously, \( \text{WOT} \subseteq \text{SOT} \subseteq \|\cdot\| \) where \( \|\cdot\| \) is the norm topology of \( \mathcal{B}(\mathcal{H}) \).

If the Hilbert space \( \mathcal{H} \) is finite, then \( \text{WOT} \), \( \text{SOT} \) and \( \|\cdot\| \) of \( \mathcal{B}(\mathcal{H}) \) are coincide. If \( \mathcal{H} \) is infinite, the inclusion relationships are strict [3]. The conclusions are also correct for the relative topologies of \( E(\mathcal{H}) \).

Example 3.4. For any \( n \in \mathbb{N} \), define \( A_n : l^2 \rightarrow l^2 \) by \( A_n x = (0, \cdots, 0, x_n, 0, \cdots) \) where \( x = (x_1, x_2, \cdots, x_n, \cdots) \in l^2 \). It is easy to see that \( \{A_n\} \subseteq E(\mathcal{H}) \) and \( \|A_n\| = |x_n|^2 \rightarrow 0 \). So \( A_n \xrightarrow{\text{SOT}} 0. \) However, \( \|A_n\| = 1 \) for any \( n \in \mathbb{N} \). Hence, \( \text{SOT} \subseteq \|\cdot\| \) strictly.

Example 3.5. Define \( A : l^2 \rightarrow l^2 \) by \( Ax = (\frac{x_1}{2}, 0, \cdots) \) and for any \( n \in \mathbb{N} \), define \( A_n : l^2 \rightarrow l^2 \) by \( A_n x = (\frac{x_1 + x_n}{2}, 0, \cdots, \frac{x_1 + x_n}{2}, 0, \cdots) \) where \( x = (x_1, x_2, \cdots, x_n, \cdots) \). It is easy to see that \( \{A_n\} \subseteq E(\mathcal{H}) \) and \( A \in E(\mathcal{H}) \). \( \langle A_n x, x \rangle \rightarrow \frac{|x|^2}{2} = \langle Ax, x \rangle \) for any \( x \in l^2 \). So \( A_n \xrightarrow{\text{WOT}} A. \) However,

\[
\|A_n x - Ax\| = \|\left(\frac{x_1}{2}, 0, \cdots, 0, \frac{x_n + x_1}{2}, 0, \cdots\right)\| \rightarrow \frac{|x_1|}{2} \neq 0
\]

when \( x_1 \neq 0 \). That is, \( A_n \) is not convergent to \( A \) with respect to \( \text{SOT} \). So \( \text{WOT} \subseteq \|\cdot\| \) strictly.

Next, we study the relationships of intrinsic topologies and operator topologies of \( E(\mathcal{H}) \).

Lemma 3.6. [4] If \((a_\alpha)_{\alpha \in \Lambda}\) is a net in \( E \), then \( a_\alpha \xrightarrow{\text{(n)}} \) a iff for each subnet \((a_\gamma)_{\gamma \in \Upsilon}\) of \((a_\alpha)_{\alpha \in \Lambda}\) and for each \( r \in E \): \( a_\gamma \geq r \) for each \( \gamma \in \Upsilon \) implies \( a \geq r \) and \( a_\gamma \leq r \) for each \( \gamma \in \Upsilon \) implies \( a \leq r \).

Lemma 3.7. [3] If \((H_\alpha)_{\alpha \in \Lambda}\) is a monotone increasing set of self-adjoint operators on the Hilbert space \( \mathcal{H} \) and \( H_\alpha \leq kI \) for all \( \alpha \in \Lambda \), then \((H_\alpha)_{\alpha \in \Lambda}\) is strong-operator convergent to a self-adjoint operator \( H \), and \( H \) is the least upper bound of \((H_\alpha)_{\alpha \in \Lambda}\).
Theorem 3.8. Let WOT and SOT be the relative weak operator topology and relative strong operator topology of $E(\mathcal{H})$ and $\tau_i$ and $\tau_o$ be the interval topology and order topology of $E(\mathcal{H})$, then

$$\tau_i \subseteq \text{WOT} \subseteq \text{SOT} \subseteq \tau_o.$$ 

Proof. Let $(A_\alpha)_{\alpha \in \Lambda}$ be a net of $E(\mathcal{H})$ and $A_\alpha \xrightarrow{\text{WOT}} A$. That is, for any $x, y \in \mathcal{H}$, $\langle A_\alpha x, y \rangle \rightarrow \langle Ax, y \rangle$. Let $B \in E(\mathcal{H})$ and $(A_\gamma)_{\gamma \in \Upsilon}$ be a subnet of $(A_\alpha)_{\alpha \in \Lambda}$. Suppose $A_\gamma \geq B$ for any $\gamma \in \Upsilon$. Then $\langle A_\alpha x, x \rangle \geq \langle Bx, x \rangle$ for each $x \in \mathcal{H}$. So $\langle Ax, x \rangle \geq \langle Bx, x \rangle$. That is, $A \geq B$. Suppose $A \leq B$ for any $\gamma \in \Upsilon$. Then $\langle A_\alpha x, x \rangle \leq \langle Bx, x \rangle$ for each $x \in \mathcal{H}$. So $\langle Ax, x \rangle \leq \langle Bx, x \rangle$. That is, $A \leq B$. By Lemma 3.6, $A_\alpha \xrightarrow{\text{SOT}} A$. Therefore, $\tau_i \subseteq \text{WOT}$.

Let $A_\alpha \xrightarrow{\text{O}} A$. Then there exist two nets $(B_\alpha)_{\alpha \in \Lambda}$ and $(C_\alpha)_{\alpha \in \Lambda}$ of $E(\mathcal{H})$ such that

$$A \uparrow B_\alpha \leq A_\alpha \leq C_\alpha \downarrow A.$$ 

As $B_\alpha \uparrow A$ and $B_\alpha \leq I$, it follows from Lemma 3.7 that $B_\alpha \xrightarrow{\text{SOT}} A$. While $(I - C_\alpha) \uparrow (I - A)$, so $I - C_\alpha \xrightarrow{\text{SOT}} I - A$. That is, $C_\alpha \xrightarrow{\text{SOT}} A$. Then for any $x \in \mathcal{H}$, there exists $\alpha_0 \in \Lambda$ such that

$$\| (B_\alpha - A)x \| < \varepsilon/3, \quad \| (C_\alpha - A)x \| < \varepsilon/3$$

for any $\alpha_0 \leq \alpha$.

$$\| (A_\alpha - A)x \| = \| (A_\alpha - B_\alpha + B_\alpha - A)x \|
\leq \| (A_\alpha - B_\alpha)x \| + \| (B_\alpha - A)x \|
\leq \| (C_\alpha - B_\alpha)x \| + \| (B_\alpha - A)x \|
\leq \| (C_\alpha - A)x \| + \| (B_\alpha - A)x \| + \| (B_\alpha - A)x \|
< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$$

for any $\alpha_0 \leq \alpha$. Hence, $\| (A_\alpha - A)x \| \rightarrow 0$, and $A_\alpha \xrightarrow{\text{SOT}} A$. By the definition of order topology, $\text{SOT} \subseteq \tau_o$.

Since $\text{WOT} \subseteq \text{SOT}$, we have $\tau_i \subseteq \text{WOT} \subseteq \text{SOT} \subseteq \tau_o$. \hfill $\Box$

Corollary 3.9. If $E$ is a scale effect algebra, then $\tau_i = \text{WOT} = \text{SOT} = \tau_o$.

For example, when $\mathcal{H} = \mathcal{C}$, the topologies $\tau_i$, WOT, SOT and $\tau_o$ of $E(\mathcal{H})$ are equivalent.

ACKNOWLEDGEMENTS. This research was supported by Kumoh National Institute of Technology.
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Received: February 15, 2014