Nonlinear Problem on Hyperelastic Deformation
of the Shell of Average Thickness FEM

M. K. Sagdatullin
Kazan national research university
420015 Kazan, Russia

D. V. Berezhnoi
Kazan Federal University
420008 Kazan, Russia

Abstract

In work the numerical research of finite deformations of isotropic hyperelastic bodies is resulted. In the first section it is resulted resolving linearise the equation in a current configuration and the parities defining speed of change stress tensor Cauchy-Euler as linear function from tensor of a spatial gradient of speed are deduced. In the second section within the limits of finite-element technique numerical realization of research algorithm of finite deformations isotropic hyperelastic bodies is considered. In the third section numerical decisions of some deformation problems of hyperelastic bodies are resulted.

Keywords: a method of finite elements, finite strains, metric tensor

Introduction

Research of deformation of difficult constructions from the hyperelastic materials supposing considerable deformations, is possible only with application of modern computing methods. Some statements of problem and possible algorithms of the solving are presented in works [1-7]. They are based on fundamental decisions of nonlinear mechanics of the continuous environments
stated in works [7-9]. It is used the Lagrange description of kinematics of the continuous environment. In case of use of the Cartesian coordinates as Lagrange structural parities take the elementary form. It is represented expedient to construct and realize algorithm of the solving generally curvilinear and not orthogonal Lagrange coordinates.

In the present work within the limits of the finite-element analysis the algorithm of the decision of a problem of numerical modeling of finite deformations of hyperelastic environments is resulted. In particular in the first section it is resulted linearise the equation on a step load, and expressions for speed stress tensor as linearly depending from a spatial gradient speed tensor.

1. The resolving equation on a step load

Let’s accept as a base parity the variation equation of a principle of virtual speeds in a current configuration which for statics problems can be written down in a kind

\[
\left( \int_{V} (\sigma') \cdot (\delta d) dV = \int_{V} \textbf{f}' \cdot \delta dV + \int_{S} \textbf{t}' \cdot \delta dS, \right. \tag{1.1}
\]

Where \( \textbf{f}' \) - a vector of the set external volume forces, \( \textbf{t}' \) - a vector of the set pressure on a surface part on \( S \) which power boundary conditions are defined. The technology of calculations represents method consecutive loads with definition of the current metrics as the basis for calculations. Deformation process we will present as sequence of equilibrium conditions which are realized at preset values of external forces \( \textbf{f}' \) and \( \textbf{t}' \).

We will define as the basic unknown size a vector of speed which \( \textbf{\dot{v}} \) can be treated as a vector of an increment of a current configuration at transition to a condition, \( \textbf{v}_{0} \),

\[
\textbf{\dot{v}} = \Delta \textbf{u} = \textbf{v}_{0} - \textbf{v}, \tag{1.2}
\]

where

\[
\textbf{v} = \textbf{v}'(\xi, \xi, \xi). \tag{1.3}
\]

The resolving equation on a current step is under construction by linearise equations (1.1) in the assumption \( \partial^{i} \textbf{v}' / \partial x' \ll 1 \). It is as a result had

\[
\int_{V} \left[ \left( \sigma' \right) \cdot \left( \delta 
abla d \right) - \frac{1}{2} \left( \sigma' \right) \cdot \left[ \left( \delta 
abla h \right) \cdot \left( \nabla h \right)^{T} + \left( \nabla h \right) \cdot \left( \delta 
abla h \right)^{T} \right] \right] dV + \int_{S} \left[ \left( \textbf{t}' \right) \cdot \left( \delta 
abla h \right) \right] dS = \int_{V} \textbf{f}' \cdot \delta dV + \int_{S} \textbf{t}' \cdot \delta dS - \textbf{f}' \cdot \delta \textbf{v} - \left( \textbf{t}' \right) \cdot \delta \textbf{S}.
\]

Completely to define this equation it is necessary to construct expression of speed of stress for \( \left( \sigma' \right) \) a known configuration through \( \textbf{v} \) an unknown vector of
speed (1.2) in the form of linear function. We will consider the general case of an isotropic material. Fairly
\[
\sigma = \frac{1}{2} \left[ \psi_1 + \psi_2 I_1 + \psi_3 I_2 \right] (g) + \frac{1}{2} \left[ \psi_1 + \psi_2 I_1 \right] (g) -
\phi_2 (A) - \psi_3 I_2 \left( A^{-1} \right) + \psi_3 I_1 \left( A^{-1} \right).
\]
Further we will paint each of the composed
\[
\psi_i = \rho \frac{\partial \psi_i}{\partial I_1} \frac{\partial I_1}{\partial A} \cdot (\dot{A}) + \rho \frac{\partial \psi_i}{\partial I_2} \frac{\partial I_2}{\partial A} \cdot (\dot{A}) + \rho \frac{\partial \psi_i}{\partial I_3} \frac{\partial I_3}{\partial A} \cdot (\dot{A}) =
\]
\[
= \left\{ \rho \frac{\partial \psi_1}{\partial I_1} \frac{\partial I_1}{\partial A} (g) + \rho \frac{\partial \psi_3}{\partial I_3} \frac{\partial I_3}{\partial A} \left[ I_1 (g) - (A) \right] + \rho \frac{\partial \psi_3}{\partial I_3} I_2 \left( A^{-1} \right) \right\} \cdot (\dot{A}).
\]
Here following representations were used
\[
i_1 = \left( \frac{\partial I_1}{\partial A} \right) \cdot (\dot{A}) = (g) \cdot (\dot{A}), i_2 = \left( \frac{\partial I_2}{\partial A} \right) \cdot (\dot{A}) = [I_1 (g) - (A)] \cdot (\dot{A}), i_3 = \left( \frac{\partial I_3}{\partial A} \right) \cdot (\dot{A}) = I_3 \left( A^{-1} \right) \cdot (\dot{A}).
\]
It is easy to receive
\[
\left( \dot{A}^{-1} \right) = \left( A^{-1} \right) \cdot \left[ (g) - (\dot{A}) \cdot (A^{-1}) \right].
\]
Thus it was possible to construct expressions of all composed in the ratio (1.3) through two tensor speeds (A) and (g). And all tensors which are curtailed with them are unequivocally defined by a current configuration and \( i_p \) can be calculated under corresponding formulas. It is necessary to express tensors (A) through (g) spatial tensor a gradient of deformations in the form of linear dependence.
We have
\[
\left( \dot{A} \right) = \left( i_d \right) - \left( i_h \right) ^T \cdot \left( \dot{A} \right) + \left( \dot{A} \right) \cdot \left( i_h \right).
\]
That is \( i_p \) linear function from \( i_h \).
Then
\[
\left( g \right) = 2 \left( \dot{A} \right) + \left( \dot{B}^{-1} \right).
\]
The parity is fair
\[
\left( \dot{B}^{-1} \right) = \left( \dot{F}^{-1} \right) ^T \cdot (G) \cdot \left( F^{-1} \right) + \left( \dot{F}^{-1} \right) ^T \cdot (G) \cdot \left( \dot{F}^{-1} \right).
\]
It is obvious, that
\[
\left( \dot{F}^{-1} \right) = -\left( F^{-1} \right) \cdot \left( \dot{F} \right) \cdot \left( F^{-1} \right) = -\left( F^{-1} \right) \cdot \left( h \right).
\]
Then
\[
\left( \dot{B}^{-1} \right) = \left( \dot{h} \right) ^T \cdot \left( F^{-1} \right) \cdot (G) \cdot \left( F^{-1} \right) - \left( F^{-1} \right) \cdot (G) \cdot \left( F^{-1} \right) \cdot \left( h \right) = \left( \dot{h} \right) ^T \cdot \left( B^{-1} \right) - \left( B^{-1} \right) \cdot \left( h \right).
\]
Gathering representations (1.4), (1.5) and (1.6), we receive expression
\[
\left( g \right) = 2 \left[ (d) \cdot (h) ^T \cdot (A) - (A) \cdot (h) \right] - \left( h \right) ^T \cdot \left( B^{-1} \right) \cdot \left( B^{-1} \right) \cdot \left( h \right) =
\]
\[
= 2 \left( d \right) - \left( h \right) ^T \cdot \left[ 2 (A) + (B^4) \right] - \left[ 2 (A) + (B^4) \right] \cdot \left( h \right) = 2 \left( d \right) - \left( h \right) ^T \cdot \left( g \right) - \left( g \right) \cdot \left( h \right).
\]
That is parity completely similar (1.5).
Thus, representation for speed of change stress tensor (1.3) in global basis, that is in a kind
\[
\left( \dot{\sigma} \right) = \sum_{i,j} \dot{\theta}_i \left( \bar{e} \bar{e}_j \right)
\]
Suppose representation
\[ \dot{\sigma}_{ij} = 4G_{ij} - \dot{k}_{mm}h_{mn}. \]

Expression for easily \( G_{ij} \) is under construction by means of the above-stated parities, speed of change of stress is linear function from gradients of speeds. The received parities represent a theoretical basis of finite-element algorithm of research of finite deformations nonlinear elastic bodies at their force load.

2. Numerical algorithm

Calculation of thin-walled constructions with the account nonlinearity is based on step-by-step and iterative methods. The choice of a method and the algorithm realizing it depends on nonlinearity type. In the present work the method consecutive loads which can be naturally realized within the limits of FEM is used. Operating loading is reached consecutive load on each step, and the quantity of steps gets out so that on each of them the problem was quasilinear. At such statement the problem is reduced to search
\[ (k + 1) \text{conditions at already certain geometry and with the saved up stress } k \text{ conditions.} \]

Current configuration we will define in a kind.
\[ k_F = k x^i (\xi^1, \xi^2, \xi^3) \epsilon_i, \]
where
\[ k x^i (\xi^1, \xi^2, \xi^3) = \sum_{r=1}^{8} k_i \chi^i_n (\xi^1, \xi^2, \xi^3). \]  
\[ (2.1) \]

Accordingly we calculate basic a vector
\[ k_F^j = \frac{\partial k x^i}{\partial \xi^j} \epsilon_i = k_F^j \epsilon_i = \frac{\partial \xi^j}{\partial \epsilon_i} \epsilon_i = \frac{1}{2\sqrt{g}} \sum_{m} e_{mm} k_{m}^F = \epsilon_{ji} k_{m}^F, \]
– metric tensors
\[ (g)^k = g^j_i (k_F^j k_F^i) = g^j_i \epsilon_i \epsilon_j, \]
\[ (g)_{ji} = \{ k_F^j, k_F^i \} = \sum_{m} \frac{\partial \xi^m}{\partial \epsilon_i} \frac{\partial \xi^m}{\partial \epsilon_j}. \]

If to enter into consideration covariant components metric tensors deformations Almansi registers as follows
\[ (Z_{ji}) = \frac{1}{2} (g_{ji} - G_{ji}) = \frac{1}{2} \sum_{m} \left( \frac{\partial \xi^m}{\partial \epsilon_i} \frac{\partial \xi^m}{\partial \epsilon_j} - \frac{\partial X^m}{\partial \epsilon_i} \frac{\partial X^m}{\partial \epsilon_j} \right). \]

As tensor components deformations Cauchy-Grin and Almansi in curvilinear bases coincide among themselves, we receive tensor deformations Cauchy-Grin
\[ (E) = Z_{ji} (R^j R^i), \]
and tensor deformations Almansi
\[ (A) = Z_{ji} \epsilon_{ji} (R^j R^i). \]

Let's enter into consideration a vector of an increment of moving
\[ \Delta^i \tilde{U} = \epsilon_{ji} \tilde{F} - \epsilon_{ji} F = \Delta U' (\xi^1, \xi^2, \xi^3) \epsilon_i, \Delta^i U' (\xi^1, \xi^2, \xi^3) = \sum_{n=1}^{8} \Delta^i U'_n (\xi^1, \xi^2, \xi^3). \]

The analogue tensor a spatial gradient of speed will be presented in a kind
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\[(\Delta^t h_k) = \left( \sum_{m} \frac{\partial^t x^m}{\partial \xi_j} \frac{\partial^t U^m}{\partial \xi^j} \right) = \Delta^t \alpha_k (t^{p'} + t^{p'}) \].

(2.2)

The symmetric part of tensor looks like

\[\left(\Delta^t d_k\right) = \frac{1}{2} \sum_{m} \left( \frac{\partial^t x^m}{\partial \xi_j} \frac{\partial^t U^m}{\partial \xi^j} + \frac{\partial^t x^m}{\partial \xi^j} \frac{\partial^t U^m}{\partial \xi_j} \right) \left( t^{p'} + t^{p'} \right) = \Delta^t Z_{ij} \left( t^{p'} + t^{p'} \right) \].

We can similarly write down variations. We have

\[ (\delta^t h_k) = \sum_{m} \left( \frac{\partial^t x^m}{\partial \xi_j} \frac{\partial^t U^m}{\partial \xi^j} \right) \left( t^{p'} + t^{p'} \right) = \delta^t \alpha_k (t^{p'} + t^{p'}), (\delta^t d_k) = \delta^t Z_{ij} (t^{p'} + t^{p'}) \].

Here following parities take place

\[ \Delta^t Z_{ij} = \frac{1}{2} \left[ \Delta^t \alpha_{ij} + \Delta^t \alpha_{ ji} \right], \delta^t Z_{ij} = \frac{1}{2} \left[ \delta^t \alpha_{ij} + \delta^t \alpha_{ ji} \right] \]

Stress tensor Cauchy it is defined in a kind

\[ (t^r \sigma) = t^r \sigma \left( t^{r'} + t^{r'} \right) = \sigma^r (t^r \sigma) \]

where are entered covariant and contravariant components of stress tensor.

Let’s write down the known variation equation in speeds of stress without mass forces. This equation will look like

\[ \int_{V} \left[ \left( t^r \sigma \right) \cdot \left( \delta^t d \right) + \frac{\partial^t U^r}{\partial x^i} \left( \delta^t d \right) \right] dV_i = \int_{V} \left[ \left( t^r \sigma \right) \cdot \left( \delta^t d \right) \right] dV_i - \int_{S} \left[ \delta^t t^r \sigma \delta U dS_i \right] \]

Now we will make transition from and \( (t^r \sigma) \) to \( t^r_\alpha \) increments and, \( (\Delta^t \sigma) \) \( \Delta^t t^n \)

having accepted time increment equal \( \Delta t \) to unit. We will receive a following parity

\[ \int_{V} \left[ \left( \Delta^t \sigma \right) \cdot \left( \delta^t d \right) + \frac{\partial^t U^r}{\partial x^i} \left( \delta^t d \right) \right] dV_i = \int_{V} \left[ \Delta^t t^r_\alpha \cdot \delta U dS_i \right] - \int_{S} \left[ \delta^t t^r_\alpha \delta U dS_i \right] \]

(2.3)

As physical model we use material Seth for which law Hook for deformation tensors Almansi is fair

\[ (t^r \sigma) = 2\mu (t^r A) + \lambda (t^r A) t^r \]

Let’s paint for an increment of stress \( k \) conditions.

\[ (\Delta^t \sigma) = 2\mu (\Delta^t A) + \lambda (\Delta^r A) \] (2.4)

where

\[ (\Delta^t A) = (\Delta^t d_k) - (\Delta^t h_k) \cdot (t^r A) - (t^r A) \cdot (\Delta^t h_k). \]

Let’s paint the second and the third composed expressions (2.5), using a parity

(2.2)

\[ (\Delta^t h_k) = \Delta^t \alpha_k (t^{p'} + t^{p'}) \cdot (t^r A) = \Delta^t \alpha_k (t^{p'} + t^{p'}) \cdot (t^r A) \cdot (\Delta^t h_k) = \Delta^t \alpha_k \cdot (t^{p'} + t^{p'}) \cdot (t^r A) \cdot (\Delta^t h_k) \]

After digitization (2.4) it is received
\[
\Delta^k \sigma_y = 2 \mu \left( \Delta^k Z_{\eta} - b \sum_{m} g^m A_m \Delta^k \alpha_m - \sum_{m} g^m A_m \Delta^k \alpha_m \right) + \\
+ \Delta^k g_y \sum_{m} g^m \left( \Delta^k Z_{\nu} - b \sum_{m} g^m A_m \Delta^k \alpha_m - \sum_{m} g^m A_m \Delta^k \alpha_m \right),
\]

(2.6)

where

\[
\Delta^k Z_{\eta} = \sum_{m} \Delta^k U_{r} E_{r}^m, \quad \Delta^k \alpha_m = \frac{1}{2} \chi^m \left( \frac{\partial N_{r}}{\partial \xi} \frac{\partial N_{r}}{\partial \xi'} + \frac{\partial N_{r}}{\partial \xi'} \frac{\partial N_{r}}{\partial \xi} \right).
\]

(2.7)

By analogy to a linear parity we will enter in (2.6) simplified law Hook for increments of compression stress

\[
\Delta^k \sigma_y = E' \Delta^k \alpha_y.
\]

Using technology of truncation of deformations of cross-section shift, we receive

\[
\Delta^k \alpha_y = \frac{1}{2} \chi^m \left( \frac{\partial N_{r}}{\partial \xi} \frac{\partial N_{r}}{\partial \xi'} + \frac{\partial N_{r}}{\partial \xi'} \frac{\partial N_{r}}{\partial \xi} \right).
\]

(2.9)

(2.10)

If to enter designations

\[
\Delta^k \alpha_y = \sum_{m} \Delta^k U_{r} G_{r}^m,
\]

where for we accept \( \Delta^k Z_{\eta}, \) parities (2.9), (2.10) and the first parity (2.7), and we will present \( \Delta^k \omega_{\eta} \) in a kind

\[
\Delta^k \omega_{\eta} = \sum_{m} \Delta^k U_{r} G_{r}^m.
\]

With a view of specification of physical model, according to the theory, following parities are entered

\[
\Delta^k \alpha_{\eta} = \Delta^k Z_{\eta} + \Delta^k \omega_{\eta},
\]

where \( \Delta^k Z_{\eta} \), parities (2.9), (2.10) and the first parity (2.7), and we will present \( \Delta^k \omega_{\eta} \) in a kind

\[
\Delta^k \omega_{\eta} = \sum_{m} \Delta^k U_{r} G_{r}^m.
\]

Where

\[
G_{r}^m = \frac{1}{128} \chi^m \left[ \xi^3 \xi^3 - \xi^3 \xi^3 \right], G_{r}^m = \frac{1}{128} \chi^m \left[ \xi^3 \xi^3 - \xi^3 \xi^3 \right].
\]

That is equivalent to calculation of rotations in \( \Delta^k \omega_{\eta}, \) centre FE.

The matrix of geometrical rigidity of the second and third composed variation equation (2.3) will register in a kind

\[
\tilde{D}_{m}^w = k \sigma^w \left[ \frac{1}{2} \sigma^w \left[ A_{r}^w A_{r}^w + A_{r}^w A_{r}^w \right] \right] \sqrt{g}.
\]

As a result on each step on loading it is necessary to solve system of the linear algebraic equations.

\[
\left[ \tau \right] \Delta' \left[ u \right] = \left[ \tau \right] \Delta' \left[ p \right] - \left[ \tau \right] \left[ H \right],
\]

(2.11)

where \( \left\{ \Delta' \left[ u \right] \right\} \) a vector of an increment of central moving, \( \left[ \tau \right] \) a matrix of the left parts, \( \left\{ \Delta' \left[ p \right] \right\} \) a vector of an increment of central forces, \( \left[ \tau \right] \) a vector are nonviscous.
3. Numerical examples

**Problem 1.** The hemispherical cover with cut in a pole under the influence of the self-counterbalanced system of forces (fig. 1.a) with following mechanical and geometrical parameters is considered: radius of a median surface of a cover, \( R = 10.0 \, \text{cm} \) a thickness of a cover, \( h = 0.04 \, \text{cm} \) Young’s modulus of a material, \( E = 6.825 \times 10^7 \, \text{kg/cm}^2 \) factor Poisson, \( \mu = 0.3 \) size of operating force, \( F = 10 \cdot \lambda \, \text{kg} \) where - \( \lambda \) parameter load.

![Fig. 1](image1.png)  
On fig. 2.a the schedule of the maximum moving \( U_{\text{max}} \, (\text{cm}) \) and \( V_{\text{max}} \, (\text{cm}) \) on each step load for a grid in comparison with \( 20 \times 20 \) decisions of other authors [10-11] is resulted \( U_{\text{max}} \, V_{\text{max}} \, 20 \times 20 \). Also the deformed condition of a hemispherical cover is resulted at \( \lambda = 16 \): on fig. 2.b for a grid \( 10 \times 10 \) on fig. 2.c for a grid \( 13 \times 13 \) on fig. 2.d for a grid \( 20 \times 20 \).

**Problem 2.** The stretching of a cylindrical cover, by the appendix of the concentrated forces (fig. 1.b) with following mechanical and geometrical parameters is considered: radius of a median surface of a cover, \( R = 4.953 \, \text{cm} \) a thickness of a cover, \( h = 0.094 \, \text{cm} \) length of a cover, \( L = 10.35 \, \text{cm} \) Young’s modulus of a material, \( E = 10.5 \times 10^6 \, \text{kg/cm}^2 \) factor Poisson \( \mu = 0.3125 \). The deformed condition of a cylindrical cover at is presented \( P = 1000 \, \text{kg} \) on fig. 2.d.

4. Conclusions

In the present work the technique of construction offered three-dimensional isoparametrical FE the nonlinear theory of elasticity, material Seth allows to receive a special finite element at which help quite really to count the stress strain behavior of shells of an average thickness with use of single-layered approximation on a thickness are used. The received results of test examples show working capacity of the offered technique.

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References


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