Soft Abel-Grassman’s Ring

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Abstract

In this paper we discuss the notions of soft AG-ring, soft AG-subring, soft ideal of soft AG-ring and idealistic soft AG-ring and and to investigate its properties.

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1 Introduction

D. Molotsov, P.K. Maji and L.A. Zadeh [1] published detailed theoretical studies on operations of soft set and their algebraic properties. A pair \((F, A)\) is called soft set over \(U\), where \(F\) is a mapping \(F : A \rightarrow P(U)\). In other word, a soft set over \(U\) is a parameterized family of subset of the universe \(U\). For \(x \in A\), \(F(x)\) may be considered as the set of \(x\)-approximate elements of the soft set \((F, A)\) i.e. \((F, A) = \{F(x) \in P(U) : x \in A \subseteq E\}\). For two soft sets \((F, A)\) and \((G, B)\) over a common universe \(U\), we say that \((F, A)\) is a soft subset of \((G, B)\) if \(A \subseteq B\) and for all \(e \in A\), \(F(e)\) and \(G(e)\) are identical approximations.

M.A. Kazim and MD. Naseeruddin [2] have introduced the concept of an AG-groupoid. Let \(G\) be any nonempty sets. \(G\) is called an AG-groupoid if \(G\) satisfies the identity \((ab)c = (cb)a\) for all \(a, b, c \in G\). Moreover every LA-semigroups \(G\) a medial law hold

\[(a \cdot b) \cdot (c \cdot d) = (a \cdot c) \cdot (b \cdot d), \quad \forall a, b, c, d \in G.\]
Q. Mushtaq and M. Khan [4, p.322] asserted that, in every LA-semigroups $G$ with left identity
\[(a \cdot b) \cdot (c \cdot d) = (d \cdot c) \cdot (b \cdot a), \quad \forall a, b, c, d \in G.\]

Further M. Khan, Faisal, and V. Anjum [3], asserted that, if a LA-semigroup $G$ with left identity the following law holds
\[a \cdot (b \cdot c) = b \cdot (a \cdot c), \quad \forall a, b, c \in G.\]

M. Sarwar (Kamran) [5, p.112] defined LA-group as the following; a groupoid $G$ is called a left almost group, abbreviated as LA-group, if (i) there exists $e \in G$ such that $ea = a$ for all $a \in G$, (ii) for every $a \in G$ there exists $a' \in G$ such that, $a'a = e$, (iii) $(ab)c = (cb)a$ for every $a, b, c \in G$.

S.M. Yusuf in [6, p.211] introduces the concept of a left almost ring (LA-ring). That is, a non-empty set $R$ with two binary operations “+” and “·” is called a left almost ring, if $\langle R, + \rangle$ is an LA-group, $\langle R, \cdot \rangle$ is an LA-semigroup and distributive laws of “·” over “+” holds. T. Shah and I. Rehman [6, p.211] asserted that a commutative ring $\langle R, +, \cdot \rangle$, we can always obtain an LA-ring $\langle R, +, \cdot \rangle$ by defining, for $a, b, c \in R$, $a \oplus b = b - a$ and $a \cdot b$ is same as in the ring. We can not assume the addition to be commutative in an LA-ring. An LA-ring $\langle R, +, \cdot \rangle$ is said to be LA-integral domain if $a \cdot b = 0$, $a, b \in R$, then $a = 0$ or $b = 0$. Let $\langle R, +, \cdot \rangle$ be an LA-ring and $S$ be a non-empty subset of $R$ and $S$ is itself and LA-ring under the binary operation induced by $R$, the $S$ is called an LA-subring of $R$, then $S$ is called an LA-subring of $\langle R, +, \cdot \rangle$. If $S$ is an LA-subring of an LA-ring $\langle R, +, \cdot \rangle$, then $S$ is called a left ideal of $R$ if $RS \subseteq S$. Right and two-sided ideals are defined in the usual manner. An ideal $I$ of $R$ is called prime if $AB \in I$ implies $A \in I$ or $B \in I$.

In this note we prefer to called left almost rings (LA-rings) as Abel-Grassmann’s rings (abbreviated as an “AG-rings”).

T. Shah, I.Rehman and A. Razzaque [7, p.6121] defined soft AG-groupoid as the followig. Let $S = (S, \cdot)$ be set and $A$ a nonempty set. Define a mapping $F : A \rightarrow P(S)$ as $F(x) = \{ y \in S \text{ such that } xRy \}$ for all $x \in A$, where $R \subseteq A \times S$ is a binary relation between the elements of $A$ and $S$. The call the pair $(F, A)$ as a soft set over $S$. Let the pair $(F, A)$ be a soft set over AG-groupoid $(S, \cdot)$ . Then $(F, A)$ is called a soft AG-groupoid over $S$ if for all $x \in A$, $F(x) \neq \emptyset$ imply that $F(x)$ is a AG-subgroupoid of $S$.

### 2 Main Results

In this paper, we define soft of AG-ring and study properties of soft AG-ring.

**Definition 2.1.** Let $R$ be a AG-ring and let $(F, A)$ be a soft set over $R$. Then $(F, A)$ is called a soft AG-ring over $R$ if $F(x)$ is a AG-subring of $R$, denoted by $F(x) \leq R$ for all $x \in A$. 

Theorem 2.2. Let \((F, A)\) and \((G, B)\) be soft AG-rings over \(R\). Then

(1) \((F, A) \land (G, B)\) is a soft AG-ring, if it is non-null.

(2) \((F, A) \impliedby (G, B)\) is a soft AG-ring, if it is non-null.

(3) \((F, A) \lor (G, B)\) is a soft AG-ring, if \(A \cap B = \emptyset\).

Proof. (1) Let \((F, A) \land (G, B) = (H, C)\), where \(C = A \times B\) and \(H(x, y) = F(x) \cap G(y)\) for all \((x, y) \in C\). If \((H, C)\) is a non-null, then \(H(x, y) = F(x) \cap G(y) \neq \emptyset\). If \(F(x)\) and \(G(y)\) are AG-subrings of \(R\), then \(H(x, y)\) is a AG-subrings of \(R\), since \(H(x, y)\) is a family AG-subrings. Therefore \((H, C)\) is a soft AG-ring over \(R\).

(2) Let \((F, A) \impliedby (G, B) = (H, C)\), where \(C = A \cap B \neq \emptyset\), and \(H(x) = F(x) \cap G(x)\) for all \(x \in C\). Also \(H(x) = F(x) \cap G(x) \neq \emptyset\) for all \(x \in \text{Supp}(H, C)\), since \(F(x)\) and \(G(x)\) are sub AG-rings of \(R\). Then, \(H(x) < R\) Therefore \((H, C)\) is a soft AG-ring over \(R\).

(3) Let \((F, A) \lor (G, B) = (H, C)\), where \(C = A \cup B\), and:

\[
H(x) = \begin{cases} 
F(x), & x \in A - B, \\
G(x), & x \in B - A, \\
F(x) \cup G(x) & x \in A \cap B.
\end{cases}
\]

Since \(A \cup B \neq \emptyset\), \(F(x) \cup G(x) = \emptyset\). Therefore for all \(x \in C\).

\[
H(x) = \begin{cases} 
F(x), & x \in A - B, \\
G(x), & x \in B - A.
\end{cases}
\]

Also \(H(x) < R\) since \(F(x)\) and \(G(x)\) are AG-subrings of \(R\). Therefore \((H, C)\) is a soft AG-ring over \(R\). Generalizing the above theorem, we have the following:

\[
\square
\]

Theorem 2.3. Let \((F_i, A_i)_{i \in I}\), where \(I\) is an index set, be a non empty family of soft AG-rings over \(R\). Then

(1) \(\land_{i \in I}(F, A)\) is a soft AG-ring, if it is non-null.

(2) \(\impliedby_{i \in I}(F, A)\) is a soft AG-ring, if it is non-null.

(3) \(\lor_{i \in I}(F, A)\) is a soft AG-ring, if \(A_i \cap A_j = \emptyset, i \neq j, i, j \in I\).

Proof. (1) \(\land_{i \in I}(F, A)\) is a soft AG-ring, where \(C = \Pi_{i \in I} A_i\) and \(H(x_i) = \cap_{i \in I} F_i(x_i)\) for all \((x_i)_{i \in I} \in C\). If \(H(C)\) is a non-null and \((x_i)_{i \in I} \in \text{supp}(H, C)\) then \(H((x_i)_{i \in I}) = \cap_{i \in I} F_i(x_i) \neq \emptyset\) and \(F_i(x_i) < R, \forall i \in I\). Hence \(H((x_i)_{i \in I}) < R\) for all \((x_i)_{i \in I} \in \text{supp}(H, C)\). Therefore \((H, C)\) is a soft AG-ring over \(R\).
(2) $\cap_{i \in I}(F, A)$ is a soft AG-ring, where $C = \cap_{i \in I}A_i$ and $H(x) = \cap_{i \in I}F_i(x_i)$ for all $(x_i)_{i \in I} \in C$. If $H(C)$ is a non-null and $x \in supp(H, C)$ then $H(x) = \cap_{i \in I}F_i(x_i) \neq \emptyset \Rightarrow F_i(x) < R, \forall i \in I \Rightarrow H(x) < R. \forall x \in supp(H, C)$. Therefore $(H, C)$ is a soft AG-ring over $R$.

(3) $\tilde{\cap}_{i \in I}(F, A)$ is a soft AG-ring, where $C = \cup_{i \in I}A_i$ and for all $x \in C, H(x) = \cup_{i \in I}(x)F_i(x)$ where $I(x) = \{i \in I | x \in A_i\}$. $(H, C)$ is a non-null since $supp(H, C) = \cup_{i \in I} supp(F_i, A_i) \neq \emptyset$. Therefore $H(x) \cup_{i \in I} (x)F_i(x) \neq \emptyset$ implies $F_{i_o}(x) \cup_{i \in I} \neq \emptyset$ for some $i_o \in I(x)$ since $Ai \cap Aj = \emptyset, i \neq j$, then $i_o$ is unique. Therefore $H(x) = F_{i_o}(x) < R$. Therefore $H(x) < R$. Hence $(H, C)$ is a soft AG-ring over $R$.

$\blacksquare$

**Definition 2.4.** Let $(F, A)$ and $(G, B)$ be soft AG-rings over $R$. Then $(G, B)$ is called a soft AG-subring of $(F, A)$, if it satisfies the following:

1. $B \subset A$
2. $G(x)$ is a AG-subring of $F(x)$, for all $x \in supp(G, B)$.

**Theorem 2.5.** Let $(F, A)$ and $(G, B)$ be soft AG-rings over $R$. Then

1. If $G(x) \subset F(x), \forall x \in B \subset A$, then $(G, B)$ is a soft AG-subring of $(F, A)$.
2. $(F, A) \cap (G, B)$ is a soft AG-subring of both $(F, A)$ and $(G, B)$ if it is non-null.

*Proof.* 1. Since $B \subset A$ and $G(x) < F(x) \forall x \in B$, then $G(x) < F(x) \forall x \in B$. Therefore $(G, B)$ is a soft AG-sub ring of $(F, A)$.

2. Let $(F, A) \cap (G, B) = (H, C)$, where $C = A \cap B \subset A$ and $H(x) = F(x) \cap G(x) \neq \emptyset$ is a AG-subring of $F(x)$ for all $x \in supp(H, C)$. Hence $(H, C)$ is a soft AG-subring of $(F, A)$.

$\blacksquare$

**Definition 2.6.** Let $(F, A)$ be a soft AG-ring over $R$. A non-null soft set $(I, B)$ over $R$ is called a soft ideal of $(F, A)$ denoted by $(I, B) \triangleleft (F, A)$, if it satisfies the following conditions:

1. $B \subset A$
2. $I(x)$ is an ideal of $F(x)$ for all $x \in supp(I, B)$.

**Theorem 2.7.** Let $(I_1, A_1)$ and $(I_2, A_2)$ be soft ideals of a soft AG-ring $(F, A)$ over $R$. Then

1. $(I_1, A_1) \cap (I_2, A_2)$ is a soft ideal of $(F, A)$ if it is non-null.
(2) \( A_1 \cap A_2 \neq \emptyset \), then \((I_1, A_1)\tilde{\cup}(I_2, A_2)\) is a soft ideal of \((F, A)\).

**Proof.** Let \((I_1, A_1)\cap(I_2, A_2) = (I, B)\), where \(B = A_1 \cap A_2 \subset A\). But \(I_1(x) \triangleleft F(x)\) and \(I_2(x) \triangleleft F(x)\) for all \(x \in \text{supp}(I, B)\). Therefore \(I(x) = I_1(x) \cap I_2(x) \neq \emptyset\) since \((I, B)\) is non-null. This implies that \(I(x) \triangleleft F(x)\). Hence \((I, B) \triangleleft (F, A)\).

(2) Let \((I_1, A_1)\tilde{\cup}(I_2, A_2) = (I, C)\) where \(C = A_1 \cap A_2 \subset A\) and for all \(x \in C \subset A\). Therefore \(I(x) = I_1(x)\) is a non-empty ideal of \(F(x)\) for all \(x \in \text{supp}(I, C)\). Also \(I(x) = I_2(x)\) is a non-empty ideal of \(F(x)\) for all \(x \in \text{supp}(I, C)\). Therefore \(I(x)\) is a non-empty ideal of \(F(x)\) for all \(x \in \text{supp}(I, C)\). Hence \((I, C)\) is a soft ideal of \((F, A)\).

**Theorem 2.8.** Let \((I_1, A_1)\) and \((I_2, A_2)\) be soft ideals of soft AG-rings, \((F, A)\) and \((G, B)\) over \(R\) respectively. Then \((I_1, A_1)\cap(I_2, A_2)\) is a soft ideal of \((F, A)\cap(G, B)\), if it is non-null.

**Proof.** Let \((I_1, A_1)\cap(I_2, A_2) = (I, D)\), where \(D = A_1 \cap A_2 \subset A\) and for all \(x \in D\). \(I(x) = I_1(x) \cap I_2(x)\). Let \((F, A)\cap(G, B) = (H, C)\) where \(C = A \cap B\) and for all \(x \in C\) then \(H(x) = F(x) \cap G(x)\). Since \((I, D)\) is non-null we have \(I(x) = I_1(x) \cap I_2(x) \neq \emptyset\) since \(A_1 \subset A\) and \(A_2 \subset B\) we have \(A_1 \cap A_2 \subset A \cap B\) i.e \(D \subset C\). Since \(I_1(x), F(x)\) and \(I_2(x), G(x)\) for all \(x \in D\) we have \(I_1(x) \cap I_2(x) \subset F(x) \cap G(x)\) for all \(x \in D\) implies \(I(x) \subset H(x)\) so \(I(x) < H(x)\). Finally we need to show that \(ar \in I(x)\) for all \(r \in H(x)\) and for all \(a \in I(x)\). Since \(I_1(x), F(x), r \in H(x) = F(x) \cap G(x)\) and \(a \in I(x) = I_1(x) \cap I_2(x)\). We have \(ar \in I_1(x)\) and \(ar \in I_2(x)\). Hence \(ar \in I(x)\). Therefore \((I, D) \triangleleft (H, C)\).

**Definition 2.9.** Let \((F, A)\) and \((G, B)\) be a soft AG-ring over \(R\) and \(R'\) respectively. Let \(f : R \to R'\) and \(g : A \to B\) be two mappings. The pair \((f, g)\) is called a soft AG-ring homomorphism if the following conditions are satisfied;

1. \(F\) is a AG-ring epimorphism
2. \(g\) is surjective
3. \(f(F(x) = G(g(x))\) for all \(x \in A\).

If we have a soft AG-ring homomorphism between \((F, A)\) and \((G, B)\), \((F, A)\) is said to be soft AG-ring homomorphism to \((G, B)\), denoted by \((F, A) \sim (G, B)\). In addition, if \(f\) is an AG-ring isomorphism and \(g\) is a bijective, then \((f, g)\) is called a soft ring isomorphism and we say that \((F, A)\) is softly isomorphic to \((G, B)\) denoted by \((F, A) \cong (G, B)\).

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