Numerical Solution of Heat Equation
by Spectral Method

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Abstract
In this paper, we develop a spectral function method that allows $L_2$ projection of an operator onto a finite dimensional Hilbert space to solve heat equation numerically. Orthogonal basis are used to establish computational algorithm. Numerical results are presented. The accuracy and efficiency of proposed model are discussed.

Mathematics Subject Classification: 65M70

Keywords: Approximate solutions, Weak formulation, Spectral Function Method, Projection

1 Introduction
Finite difference and finite element methods have long been adopted as vital tools for efficient, accurate, and stable numerical solution of partial differential
equations. In the last two decades however, spectral methods have emerged as a viable numerical schemes for the solution of partial differential equations because of its high accuracy. It has been extensively used in quantum mechanics [2], [3], fluid dynamics [1], and weather prediction [11]. In this paper, we use spectral method to solve heat equation.

Let $\Omega = (0, L)$ be an open bounded set of $\mathbb{R}$. Let us consider the following parabolic partial differential equation

$$\frac{\partial u(x,t)}{\partial t} - \frac{\partial^2 u(x,t)}{\partial x^2} = f(x,t); \quad (x,t) \in Q$$

$$u(x,t) = 0, \quad x \in \partial \Omega \quad t \in (0, T)$$

$$u(x,0) = u_0(x), \quad x \in \Omega$$

(1)

where $u$ is a function of $x$ and $t$, $T > 0$, $Q = (0, T) \times \Omega$, $f \in L^2(Q)$, and $u_0 \in V = H^1(\Omega)$. The coefficient of $u_{xx}$ is chosen as unity. In this work, we develop a spectral function method that allows us $L^2$ projection of Laplace operator onto a finite dimensional Hilbert space spanned by a set of eigenfunctions of the operator. Such projection enables us to apply functional analytic theory as developed in [7]. The paper is organized as follows. In section 2, we present appropriate Hilbert spaces and establish eigenvalues and eigenfunctions of operator $A = -\Delta$. In section 3, we present weak formulation. Spectral method with computational algorithm is presented in section 4, and numerical results are presented in section 5.

2 Problem Setup

Let $H = L^2(\Omega)$ be a Hilbert space with following inner product and norm

$$(\phi, \psi) = \int_{\Omega} \phi(x) \psi(x) dx, \quad |\phi| = (\phi, \phi)^{\frac{1}{2}}$$

(2)

for all $\phi, \psi \in L^2(\Omega)$. Let $V = H^1_0(\Omega)$ be a Hilbert space with following inner product and norm

$$(\phi, \psi) = (\nabla \phi, \nabla \psi), \quad \|\phi\| = ((\phi, \phi))^{\frac{1}{2}}$$

(3)

for all $\phi, \psi \in H^1_0(\Omega)$. The dual $H'$ is identified with $H$ leading to $V \subset H \subset V'$ with compact, continuous, and dense injections [9]. Hence there exists a constant $K_1 = K_1(\Omega)$ such that

$$|w| \leq K_1 \|w\| \quad \text{for any} \quad w \in V.$$  

(4)

Let $< u, v >_{V', V'}$ denote the duality pairing between $V$ and $V'$. To use the variational formulation let us define the following bilinear form on $V \times V$

$$a(u, v) = \int_{\Omega} \nabla u \nabla v dx$$

(5)
for any $u, v \in H^1_0(\Omega)$. Clearly, $a(u, v)$ is bounded and coercive in $V$. Define a linear operator $A : D(A) = H^1_0(\Omega) \cap H^2(\Omega)$ into $H$ by $a(u, v) = (Au, v)$ for all $u \in D(A)$ and for all $v \in V$. Thapa [7] proved $A$ as an isomorphism between $D(A)$ and $H$. The operator $A : D(A) \subset H$ into $H$ is a self-adjoint. Enough to show that $A$ is symmetric and $R(A) = H$. For any $u, v \in D(A)$, we have $(Au, v) = a(u, v)$ and $(u, Av) = a(v, u)$ so $(Au, v) = (u, Av)$. Hence $A$ is symmetric bounded linear operator. Thapa [7] established $R(A) = H$. Therefore $A$ is self-adjoint operator. Since $A$ is bounded self-adjoint operator with $A^{-1}$ as an inverse, $A^{-1}$ is self-adjoint. Now it remains to show that $A^{-1}$ is compact. Let $B$ be any bounded set in $H$. $A^{-1}$ is bounded thus for any $h \in H$, $\|A^{-1}h\| \leq \|A^{-1}\||h|$. Hence the set $A^{-1}(B)$ is bounded in $V$. $A^{-1}$ is compact [9]. So there exist $\lambda_k$ for $k = 1, 2, ..., \infty$ such that $(\nabla w_k, \nabla v) = \lambda_k(w_k, v)$ for all $v \in V$. which shows that $\lambda_k$ and $w_k$ respectively are the nonzero eigenvalues and eigenfunctions for the operator $A$ defined in $V$ such that $\{w_k\}_{k=1}^{\infty}$ form an orthonormal basis in $H$.

### 3 Weak Formulation of the Problem

From now on the dependency on $x$ is suppressed, and $'$ stands for the time derivative. Let

$$W(0, T) = \{u : u \in L^2(0, T; V), u' \in L^2(0, T; H)\}. \tag{6}$$

$u'$ is the derivative in the distributional sense. That is, $u' \in L^2(0, T; H)$ is derivative of $u \in L^2(0, T; V)$ in the distributional sense if for any $\phi \in C^\infty_0(0, T)$ and $v \in V$

$$\int_0^T (u'(t), v)\phi(t)dt = -\int_0^T (u(t), v)\phi'(t)dt \tag{7}$$

For details see [10]. Let $\{w_j\}_{j=1}^{\infty}$ be the eigenfunctions of the operator $A$. The weak solution of (1) is a function $u \in W(0, T)$ satisfying

$$\langle u', w_j \rangle + a(u, w_j) = (f, w_j), \forall j \in N; \quad u(0) = u_0 \in V, \quad u'(0) = u_1 \in H, \tag{8}$$

where the equations in $t$ are satisfied in the distributional sense. Since the span $\{w_1, w_2, w_3, ...\}$ is dense in $V$, (8) is satisfied for any $v \in V$

$$\langle u' + Au, v \rangle = \langle f, v \rangle, \quad u(0) = u_0 \in V, \quad u'(0) = u_1 \in H. \tag{9}$$

Thus

$$u' + Au = f, \quad u(0) = u_0 \in V, \quad u'(0) = u_1 \in H \tag{10}$$

which is understood in the sense of distributions on $(0, T)$ with the values in $V'$. For more details see [8].
Let $\Omega = (0, L)$. To accommodate the boundary conditions in (1), non-normalized eigenfunctions $w_n = \sin(n\pi x)/L$, $n = 1, 2, 3, \ldots$, as a basis of $H = L_2(\Omega)$. Let $S_N = \text{span}(w_1, w_2, \ldots, w_N)$ be a finite dimensional subspace of $V$. Since $S_N$ is closed subspace of Hilbert space, $S_N$ itself is a Hilbert space. Thus we define a natural projection $P_N$ from $H$ onto $S_N$ in $H$. Let $u_N(t) = u_N(.,t) \in S_N$ that satisfies
\begin{equation}
\left( \frac{\partial u_N}{\partial t}, v \right) + ((u_N, v)) = (f, v); \quad t \in (0, T)
\end{equation}
\begin{equation}
(u_N(0) - u(0), v) = 0, \quad \forall v \in S_N.
\end{equation}

Now we can discuss the computational algorithm to find solutions $u_N$. Let $\{w_j\}_{j=1}^\infty = \{\sin(j\pi x)/L\}_{j=1}^\infty$ be eigenfunctions of $A$ that form a basis in $H$. Then $\{w_j\}_{j=1}^\infty$ is an basis on $V$. Fix $N \in \mathbb{N}$. Let $V_N = \text{span}\{w_1, w_2, \ldots, w_N\}$. Let $P_N : H \rightarrow V_N$ be the projection operator defined by $P_N v = \sum_{j=1}^N (v, w_j)w_j$ for any $v \in H$. Then the solution $u_N$ is given by
\begin{equation}
u_N(x, t) = \sum_{j=1}^N g_jN(t)w_j(x)
\end{equation}
that satisfies
\begin{equation}\frac{d}{dt}(u_N, w_j) + a((u_N, w_j)) = (f, w_j);
\end{equation}
\begin{equation}u_N(0) = P_N u_0, \quad \forall j.
\end{equation}

Let $\bar{g}_N = \{g_jN\}_{j=1}^N \in \mathbb{R}^N$. We can rewrite (13) as the following vector differential equation
\begin{equation}\bar{g}_N'(t) + \Lambda\bar{g}_N(t) = \bar{F}(t, x)
\end{equation}
with the initial data
\begin{equation}\bar{g}_N(0) = \begin{bmatrix}
(P_N u_0, w_1) \\
(P_N u_0, w_2) \\
\vdots \\
(P_N u_0, w_N)
\end{bmatrix} = \begin{bmatrix}
\int_0^L u_0 \sin(\pi x)/L \, dx \\
\int_0^L u_0 \sin(2\pi x)/L \, dx \\
\vdots \\
\int_0^L u_0 \sin(N\pi x)/L \, dx
\end{bmatrix}.
\end{equation}
Numerical solution of heat equation

where \( u_0 \in L^2(0,T;V), \) \( g_N(t) \in \mathbb{R}^N, \) \( F(t,g_N) \in \mathbb{R}^N, \) and

\[
\Lambda = \begin{bmatrix}
  (\pi/L)^2 & 0 & 0 & \cdots & 0 \\
  0 & (2\pi/L)^2 & 0 & \cdots & 0 \\
  0 & 0 & (3\pi/L)^2 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & \cdots & (N\pi/L)^2
\end{bmatrix}
\]

5 Numerical Results

In this section, we present numerical results based on the method defined in section 4. The \( L_2, \) and \( L_\infty \) errors are defined and given by

\[
\|u_{\text{exact}} - u_N\|_2 = \left( \sum_{i=1}^{N} (u_i - u_i^N)^2 \right)^{\frac{1}{2}}
\]

\[
\|u_{\text{exact}} - u_N\|_\infty = \max_{1 \leq i \leq N} \sum_{i=1}^{N} |(u_i - u_i^N)|
\]

Example: Let \( u(x,0) = x(1-x), \) and \( f(x,t) = 0 \) in (1),

| Table 1: The approximate solution for \( N = 16 \) |
|------------------|------------------|------------------|------------------|------------------|
| \( T \)    | \( X = 0.2 \) | \( X = 0.4 \) | \( X = 0.6 \) | \( X = 0.8 \) |
| 0.0 | 0.16004631 | 0.23998825 | 0.23998825 | 0.16004631 |
| 0.2 | 0.02106668 | 0.03408660 | 0.03408660 | 0.02106668 |
| 0.4 | 0.00292640 | 0.00473501 | 0.00473501 | 0.00292640 |
| 0.6 | 0.00040651 | 0.00065775 | 0.00065775 | 0.00040651 |
| 0.8 | 0.00005647 | 0.00009137 | 0.00009137 | 0.00005647 |
| 1.0 | 0.00000784 | 0.00001269 | 0.00001269 | 0.00000784 |
Analytical solution of sample problem

Approximate solution of sample problem

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Table 2: $L_2$ error for different values of $N$

<table>
<thead>
<tr>
<th>$T$</th>
<th>$N = 4$</th>
<th>$N = 8$</th>
<th>$N = 16$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>$5.5538 \times 10^{-17}$</td>
<td>$5.5538 \times 10^{-17}$</td>
<td>$5.5538 \times 10^{-17}$</td>
</tr>
<tr>
<td>0.2</td>
<td>$9.0205 \times 10^{-17}$</td>
<td>$1.2490 \times 10^{-16}$</td>
<td>$1.4571 \times 10^{-16}$</td>
</tr>
<tr>
<td>0.4</td>
<td>$9.5409 \times 10^{-18}$</td>
<td>$2.7755 \times 10^{-17}$</td>
<td>$2.4286 \times 10^{-17}$</td>
</tr>
<tr>
<td>0.6</td>
<td>$1.6263 \times 10^{-18}$</td>
<td>$8.3483 \times 10^{-18}$</td>
<td>$5.6378 \times 10^{-18}$</td>
</tr>
<tr>
<td>0.8</td>
<td>$3.7947 \times 10^{-19}$</td>
<td>$1.5449 \times 10^{-18}$</td>
<td>$1.6669 \times 10^{-18}$</td>
</tr>
<tr>
<td>1.0</td>
<td>$3.5745 \times 10^{-19}$</td>
<td>$3.6761 \times 10^{-19}$</td>
<td>$3.2865 \times 10^{-19}$</td>
</tr>
</tbody>
</table>
### Table 3: $L_\infty$ error for different values of $N$

<table>
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<th>$N = 8$</th>
<th>$N = 16$</th>
</tr>
</thead>
<tbody>
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<td>$5.5511 \times 10^{-17}$</td>
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<td>$3.2864 \times 10^{-19}$</td>
</tr>
</tbody>
</table>

### 6 Conclusion

In this paper, we used spectral method with orthogonal basis to solve heat equation. Numerical experiments showed that the result for $N = 16$ is relatively accurate than the other values of $N$. As expected, the spectral method is highly accurate on the order of $10^{-16}$ for even small values of $N$. We found that the increase in $N$ does not significantly and consistently affect the errors.

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**References**


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