On the Existence of Steady Solution for Flows between Rotating Conical Cylinders

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Abstract

The existence of steady solution for flows between two coaxial independently rotating conical cylinders (truncated cones) are considered under condition of small gap and neglecting the effect of the top and bottom boundary. It is proved mathematically that there doesn’t exist steady solution of the form:

\[ u = u_\theta(r,z)e_\theta + u_z(r,z)e_z \]

and

\[ u = u_r(r,z)e_r + u_\theta(r,z)e_\theta \]

with \( p = p(r,z) \), for any angular velocities of the conical cylinders. The results of numerical simulation suggest that there exists a statistical three-dimensional steady solution for this configuration with the inner cone rotating and outer one at rest.

Keywords: conical cylinder; steady solution; Taylor-Couette flow

1. Introduction

The flow between two concentric rotating cylinders is called Taylor-Couette (TC) flow, where the rotation of one or both of the cylinders results in fluid motion. It is well-known that TC flow admits one-dimensional steady basic flow:

\[ u = u_\theta(r)e_\theta, p = p(r) \]

the circular Couette flow. With increasing Reynolds number the basic flow becomes unstable and the flow undergoes a series of transition from basic flow to Taylor vortex flow\(^5\), then to chaotic and turbulent Taylor vortex flow. So far much has been done for the stability of the basic flow and the transition to Taylor vortices and turbulence(see the
review by S. Dong [1] and the references therein).

Here we consider an apparatus of two concentric conical cylinders (CC) with both the inner and outer cones rotating at angular velocity $\Omega_1$ and $\Omega_2$ respectively (see Fig.1). By $r_1$ and $R_1 (r_2, R_2)$ we denote the radii of the top and bottom surface of the inner(outer) cone respectively. The inclinations of the inner and outer cones are the same, denoted by $\theta_0$. The cones rotate independently. A viscous incompressible fluid is contained between two concentric conical cylinders. This apparatus is used in chemical laboratory of our university to precipitate nano-particles. A comparative experiment was carried out involving conventional precipitation in a flask. It is found that the nanoparticles obtained by using CC were smaller in size than that obtained in the flask. This motivates us to investigate CC flow theoretically.

The behavior of CC flow has been studied by some research workers, both numerically and experimentally. Most results are about the transition to Taylor vortices and turbulence (see Noui-Mehidi et al.[3,4] and Xu et al.[8]), the existence of steady solution of this problem has not received general concern. As TC flow is the limit case of CC flow, this stimulates us to investigate the existence of steady solution of CC flow mathematically.

The property of CC flow is quite different from that of TC flow. For TC flow the centrifugal forces are uniform along the axis. It generates a rotationally symmetric one-dimensional basic flow mentioned above. In changing from circular cylinder to conical one the radius changes linearly resulting in a linear change of the centrifugal forces in the annulus along the axis. In this case it is proved mathematically that one-dimensional basic flow (i.e. $u = u_r(r)e_r + u_\theta(r)e_\theta + u_z(r)e_z$) can not be generated[6]. In the case when the inner cone rotating and the outer one at rest, the top and bottom boundary being rigid, Wimmer[7] showed experimentally that the change of the centrifugal forces along axis drives a three-dimensional basic flow. As an approximation for CC flow Hoffmann and Busse[2] considered a configuration of two parallel plates moving relative to each other in a rotating system, in which the rotation axis is oriented at an arbitrary angle with respect to plates, but perpendicular to their relative motion. They applied this formulation to coaxial cones at large distance from the tip of the cones and found the general solution.

In this paper, for simplification, we assume that the effect of the top and bottom boundary can be neglected. It applies to the small-gap case when conical cylinders are very high in comparison with the gap width, or $\frac{R_2 - R_1}{H} \ll 1$, where $H$ is the height of the cylinders. We will prove that there does not exist steady solutions of the form: $u = u_\theta(r,z)e_\theta + u_z(r,z)e_z$ and $u = u_r(r,z)e_r + u_\theta(r,z)e_\theta$ with $p = p(r,z)$ for any angular velocities $\Omega_1$ and $\Omega_2$, where $e_r, e_\theta$ and $e_z$ are the unit vectors of the cylindrical coordinates system. And through numerical simulation we will confirm that there exists a three-dimensional statistical steady solution for CC flow. As far as we know,
there are not any papers dealing with this problem theoretically. One may say: it is clear that the geometry of apparatus would not allow the form of solutions above. But it need to verify whether these forms of solutions could be achieved for properly combined $\Omega_1$ and $\Omega_2$. This paper will give a purely mathematical answer of this question.

Section 2 is devoted to the mathematical formulation of the problem, the detailed mathematical treatment and numerical simulation are carried out in section 3 and 4. Finally, the main results of this paper are summarized in section 5.

Figure 1: Sketch of the geometry of the coaxial conical cylinders. $\Omega_i > 0 (i=1,2)$ denotes that the cone rotates counterclockwise whereas $\Omega_i < 0 (i=1,2)$ means cone’s rotation is clockwise.

2. Mathematical Formulation

The governing equations of the incompressible viscous fluid for CC flow in Fig. 1 are Navier-Stokes equations:

$$\frac{\partial u}{\partial t} - \nu \Delta u + u \cdot \nabla u + \frac{1}{\rho} \nabla p = 0$$  \hspace{1cm} (1)

$$\nabla \cdot u = 0$$  \hspace{1cm} (2)
where $\mathbf{u}$, $\rho$, $p$ and $\nu$ denote velocity, density, pressure, and kinematic viscosity of the fluid, respectively. And the boundary condition is given by

$$\omega|_{\Sigma_1} = \Omega_1, \quad \omega|_{\Sigma_2} = \Omega_2$$

(3)

$$u_r|_{\Sigma_1} = 0, \quad u_r|_{\Sigma_2} = 0, \quad u_\theta|_{\Sigma_1} = r_1(z)\Omega_1, \quad u_\theta|_{\Sigma_2} = r_2(z)\Omega_2, \quad u_z|_{\Sigma_1} = 0, \quad u_z|_{\Sigma_2} = 0$$

(4)

where $\omega$ refers to the angular velocity, $r_1(z) = R_1 - z \cot \theta_0$, $r_2(z) = R_2 - z \cot \theta_0$, $u_r$, $u_\theta$ and $u_z$ represent the components of the velocity in the cylindrical coordinates system, $\Sigma_1$ and $\Sigma_2$ denote the inner and the outer conical cylinders, respectively. There is no restriction on the boundary conditions for the end plates both at the top and the bottom. In cylindrical polar coordinates $\mathbf{x} = (r, \theta, z)$ and $\mathbf{u} = (u_r, u_\theta, u_z)$ the equations (1) and (2) take the forms:

$$\frac{Du_r}{Dt} - \frac{u_\theta^2}{r} = \frac{-1}{\rho} \frac{\partial p}{\partial r} + \nu (\Delta u_r - \frac{u_r}{r^2} - 2 \frac{\partial u_\theta}{\partial \theta})$$

(5)

$$\frac{Du_\theta}{Dt} + \frac{u_r u_\theta}{r} = \frac{-1}{\rho} \frac{\partial p}{\partial r} + \nu (\Delta u_\theta - \frac{u_\theta}{r^2} + 2 \frac{\partial u_r}{\partial \theta})$$

(6)

$$\frac{Du_z}{Dt} = \frac{-1}{\rho} \frac{\partial p}{\partial z} + \nu \Delta u_z$$

(7)

where

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}, \quad \frac{D}{Dt} = \frac{\partial}{\partial t} + u_r \frac{\partial}{\partial r} + \frac{u_\theta}{r} \frac{\partial}{\partial \theta} + u_z \frac{\partial}{\partial z}$$

The equation of continuity is given by

$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0.$$  

(8)

3. Non-existence of two-dimensional velocity

3.1 Non-existence of the velocity: $\mathbf{u} = u_\theta(r, z)e_\theta + u_z(r, z)e_z$

In this section we will show that for arbitrary angular velocities $\Omega_1$ and $\Omega_2$ there does not exist two-dimensional steady solution of the form $\mathbf{u} = u_\theta(r, z)e_\theta + u_z(r, z)e_z$, $p = p(r, z)$ for the flow between two rotating conical cylinders. For this case equations (5)-(8) reduce to

$$\frac{1}{\rho} \frac{\partial p}{\partial r} = \frac{u_\theta^2}{r}$$

(9)
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\[
\begin{align*}
  u_z \frac{\partial u_\theta}{\partial z} + \frac{1}{\rho} \frac{\partial p}{\partial \theta} &= \nu \left( \frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r} \frac{\partial u_\theta}{\partial r} + \frac{\partial^2 u_\theta}{\partial z^2} - \frac{u_\theta}{r^2} \right) \\
  u_z \frac{\partial u_z}{\partial z} + \frac{1}{\rho} \frac{\partial p}{\partial z} &= \nu \left( \frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r} \frac{\partial u_z}{\partial r} + \frac{\partial^2 u_z}{\partial z^2} \right) \\
  \frac{\partial u_z}{\partial z} &= 0.
\end{align*}
\]

By virtue of (11) and (12) we have

\[
\frac{\partial p}{\partial z} = \rho \nu \left( \frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r} \frac{\partial u_z}{\partial r} \right)
\]

From equation (12) it follows that \( u_z \) is independent on \( z \), so \( \frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r} \frac{\partial u_z}{\partial r} \) is only a function of \( r \), then equation (13) can be written in the form

\[
\frac{\partial p}{\partial z} = T(r)
\]

where \( T(r) = \rho \nu \left( \frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r} \frac{\partial u_z}{\partial r} \right) \).

Integrating on both sides of equation (14), we obtain

\[
p = T(r)z + C(r).
\]

By putting (15) into (9) we obtain velocity in azimuthal directions

\[
u^2 = \frac{r}{\rho} \left[ T'(r)z + C'(r) \right]
\]

where \( C(r) \) is an arbitrary differentiable function of \( r \) with \( T'(r)z + C'(r) \geq 0 \).

This solution must be considered together with the boundary conditions (see Fig.2). Let A and B be two points on the outer and inner conical cylinders respectively and \( r_0 \) be the distance of point A(B) from z-axis. We denote by \( z_A(z_B) \) the z-coordinate of point A(B). From (4) we have \( r_0 = R_1 - z_A \cot \theta_0 = R_2 - z_B \cot \theta_0 \). It follows that \( z_A \) and \( Z_B \) satisfy \((z_A - z_B) \cot \theta_0 = R_2 - R_1 \).
1st case: $\Omega_1 = \Omega_2$, i.e. two conical cylinders rotate at the same angular velocity.

From the boundary condition (3), we have

$$u_\theta|_A = \Omega_2 r_0 = \Omega_1 r_0 = u_\theta|_B.$$  \hspace{1cm} (17)

Substituting it into (16), we obtain

$$u_\theta|_A = \sqrt{\frac{r_A}{\rho}[z_A T'(r_A) + C'(r_A)]} = \sqrt{\frac{r_0}{\rho}[z_A T'(r_0) + C'(r_0)]},$$  \hspace{1cm} (18)

$$u_\theta|_B = \sqrt{\frac{r_B}{\rho}[z_B T'(r_B) + C'(r_B)]} = \sqrt{\frac{r_0}{\rho}[z_B T'(r_0) + C'(r_0)]}. \hspace{1cm} (19)$$

Equations (18) and (19) together with (17) lead to

$$\frac{r_0}{\rho}[z_A T'(r_0) + C'(r_0)] = \frac{r_0}{\rho}[z_B T'(r_0) + C'(r_0)]$$  \hspace{1cm} (20)

which implies that

$$(z_A - z_B)T'(r_0) = 0$$

and hence

$$T'(r_0) = 0.$$  

Let point A and point B slide along the border with $r_A = r_B$ then we get points $A', B', A'', B''$ (see Fig.3), where $A'B'$ and $A''B''$ are parallel to $AB$. 
Figure 3: The projection of coaxial conical cylinders on plane roz

Then we conclude

\[ T'(r_0) \equiv 0 \]

for any \( r_0 \in [r_2, R_1] \), then (16) becomes

\[ u_\theta^2 = \frac{r}{\rho} C'(r), \quad r \in [r_2, R_1] \]

Therefore \( u_\theta \) depends only on \( r \). Then the steady solution \( u = u_\theta(r, z)e_\theta + u_z(r, z)e_z \) of equations (5)–(8) is reduced to \( u = u_\theta(r)e_\theta + u_z(r)e_z \). It is proved in [6] that this type of one-dimensional steady solution does not exist for two rotating conical cylinders in the case of \( \theta_0 \neq \frac{\pi}{2} \). Therefore, the steady solution of the form: \( u = u_\theta(r, z)e_\theta + u_z(r, z)e_z \) does not exist for \( \Omega_1 = \Omega_2 \). Here and in what follows we require \( r_2 < R_1 \) (i.e. \( R_2 - R_1 < H \cot \theta_0 \)), which is satisfied with the small-case assumption.

2\textsuperscript{nd} case: \( \Omega_1 \neq \Omega_2 \), i.e. two conical cylinders rotate at different angular velocities.

In view of (3), we have

\[ u_\theta|_A = \Omega_2 r_0, \quad u_\theta|_B = \Omega_1 r_0 \] (21)

From the equation (16), we obtain

\[ u_\theta|_A = \sqrt{\frac{r_A}{\rho}} \left[ z_A T'(r_A) + C'(r_A) \right] = \sqrt{\frac{r_0}{\rho}} \left[ z_A T'(r_0) + C'(r_0) \right], \] (22)

\[ u_\theta|_B = \sqrt{\frac{r_B}{\rho}} \left[ z_B T'(r_B) + C'(r_B) \right] = \sqrt{\frac{r_0}{\rho}} \left[ z_B T'(r_0) + C'(r_0) \right]. \] (23)
By virtue of (21)–(23), we obtain
\[ T'(r_0) \neq 0. \]
Otherwise if \( T'(r_0) = 0 \), (22) and (23) are reduced to the form
\[ u_\theta|_A = \sqrt{\frac{\rho}{r}} C'(r_0), \quad u_\theta|_B = \sqrt{\frac{\rho}{r}} C'(r_0). \]
It follows from that \( u_\theta|_A = u_\theta|_B \). Together with (21) it turns out that \( \Omega_1 = \Omega_2 \). This contradicts to the assumption \( \Omega_1 \neq \Omega_2 \), and hence \( T'(r_0) \neq 0 \).

From (21) and (22) we obtain
\[ \sqrt{\frac{\rho}{r}} [z_A T'(r_0) + C'(r_0)] = \Omega_2 r_0, \]
which leads to
\[ z_A \cdot T'(r_0) = \rho \Omega_2^2 r_0 - C'(r_0). \tag{24} \]
Analogously, by (21) and (23) we have
\[ z_B \cdot T'(r_0) = \rho \Omega_1^2 r_0 - C'(r_0). \tag{25} \]
It follows from (24) and (25) that
\[ T'(r_0) = \frac{\rho r_0 (\Omega_2^2 - \Omega_1^2)}{z_A - z_B}, \quad C'(r_0) = \frac{\rho r_0 (z_A \Omega_1^2 - z_B \Omega_2^2)}{z_A - z_B}. \tag{26} \]

Since (26) is valid for any \( r_0 \in [r_2, R_1] \), together with \( z_A = (R_2 - r_0) \tan \theta_0 \), \( z_B = (R_1 - r_0) \tan \theta_0 \) we obtain
\[ T'(r) = \frac{\rho r (\Omega_2^2 - \Omega_1^2) \cot \theta_0}{R_2 - R_1}, \quad C'(r) = \frac{(R_2 \Omega_1^2 - R_1 \Omega_2^2) \rho}{R_2 - R_1} r + \frac{(\Omega_2^2 - \Omega_1^2) \rho}{3(R_2 - R_1)} r^2, \quad r \in [r_2, R_1]. \tag{27} \]

Combining with (16) we obtain
\[ u_\theta^2 = \frac{r^2}{R_2 - R_1} [z(\Omega_2^2 - \Omega_1^2) \cot \theta_0 + (R_2 \Omega_1^2 - R_1 \Omega_2^2) + (\Omega_2^2 - \Omega_1^2) r]. \tag{28} \]

By integrating two equations in (27) we have
\[ T(r) = \frac{\rho (\Omega_2^2 - \Omega_1^2) \cot \theta_0}{2(R_2 - R_1)} r^2 + C_0, \quad C(r) = \frac{(R_2 \Omega_1^2 - R_1 \Omega_2^2) \rho}{2(R_2 - R_1)} r^2 + \frac{(\Omega_2^2 - \Omega_1^2) \rho}{3(R_2 - R_1)} r^3 + C_1. \tag{29} \]

where \( r \in [r_2, R_1] \), \( C_0 \) and \( C_1 \) are arbitrary numbers. From definition of \( T(r) \) it follows that
\[ \rho \nu (\frac{d^2 u_\theta}{dr^2} + \frac{1}{r} \frac{du_\theta}{dr}) = \frac{\rho (\Omega_2^2 - \Omega_1^2) \cot \theta_0}{2(R_2 - R_1)} r^2 + C_0. \tag{30} \]
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By setting \( r = e^t \), (30) leads to

\[
\frac{d^2 u_z}{dt^2} = \frac{(\Omega_2^2 - \Omega_1^2) \cot \theta_0 - 2\nu (R_2 - R_1)}{2\nu(R_2 - R_1)} e^{4t} + C e^{2t}
\]

with \( C = \frac{C_0}{\rho \nu} \). Integrating this equation twice and taking account of \( t = \ln r \) we obtain

\[
u \frac{d^2 u_z}{dt^2} = -\frac{(\Omega_2^2 - \Omega_1^2) \cot \theta_0 r^4 + \frac{1}{4} C r^2 + A \ln r + B}{32\nu(R_2 - R_1)}
\]

where \( r \in [r_2, R_1] \), \( A, B \) are two arbitrary numbers.

By putting (29) into (15) we get the solution of the pressure

\[
p(r, z) = \frac{\rho(\Omega_2^2 - \Omega_1^2) \cot \theta_0}{2(R_2 - R_1)} z r^2 + C_0 z + \frac{(R_2 \Omega_1^2 - R_1 \Omega_2^2) \rho r^2}{2(R_2 - R_1)} + \frac{(\Omega_2^2 - \Omega_1^2) \rho}{3(R_2 - R_1)} r^3 + C_1.
\]

By substituting (28), (31) and (32) into (10), we obtain

\[
\begin{align*}
&\frac{(\Omega_2^2 - \Omega_1^2)^2 \cot \theta_0 - 2\nu (R_2 - R_1)}{2(R_2 - R_1)} r^2 + \frac{C_0(\Omega_2^2 - \Omega_1^2) \cot \theta_0 + A(\Omega_2^2 - \Omega_1^2) \cot \theta_0}{2(R_2 - R_1)} r^2 \ln r + \frac{64\nu(R_2 - R_1)^2}{2(R_2 - R_1)} r^2
\end{align*}
\]

\[
\begin{align*}
&\frac{B(\Omega_2^2 - \Omega_1^2) \cot \theta_0}{2(R_2 - R_1)} r^2 = \nu \left\{ \frac{7(\Omega_2^2 - \Omega_1^2)}{2(R_2 - R_1)} r + \frac{z \cot \theta_0 (\Omega_2^2 - \Omega_1^2)}{2(R_2 - R_1)} + \frac{R_2 \Omega_1^2 - R_1 \Omega_2^2}{R_2 - R_1} - \frac{2 z \cot \theta_0 (\Omega_2^2 - \Omega_1^2) r^2 + 2(R_2 \Omega_1^2 - R_1 \Omega_2^2) r + 3(\Omega_2^2 - \Omega_1^2) r^3}{4(R_2 - R_1)} \right\}.
\end{align*}
\]

If (28), (31) and (32) form the general solution of the problem for \( r \in [r_2, R_1] \), then (33) must be an identical relation.

Using cross-phase multiplication and rearrange the terms according to the descending power of \( r \), (33) correspondingly is reduced to

\[
\begin{align*}
&\frac{-128 A \rho \nu \cot \theta_0 (R_2 - R_1)^2 (\Omega_2^2 - \Omega_1^2) [z (\Omega_2^2 - \Omega_1^2) \cot \theta_0 + (R_2 \Omega_1^2 - R_1 \Omega_2^2) r] r^5 + 4 \rho \cot \theta_0 (R_2 - R_1)(\Omega_2^2 - \Omega_1^2)^3 r^6 + 4 \rho \cot \theta_0 (R_2 - R_1)(\Omega_2^2 - \Omega_1^2)^2 [z (\Omega_2^2 - \Omega_1^2) \cot \theta_0 + (R_2 \Omega_1^2 - R_1 \Omega_2^2) r] r^5 + 32 C_0 \rho \cot \theta_0 (R_2 - R_1)^2 (\Omega_2^2 - \Omega_1^2)^2 r^4 + 32 C_0 \cot \theta_0 (R_2 - R_1)^2 (\Omega_2^2 - \Omega_1^2) [z \cot \theta_0 (\Omega_2^2 - \Omega_1^2) + (R_2 \Omega_1^2 - R_1 \Omega_2^2)] r^3 + 128 B \rho \nu \cot \theta_0 (R_2 - R_1)^2 (\Omega_2^2 - \Omega_1^2)^2 r^2 + 64 \rho \nu (R_2 - R_1)^2 (\Omega_2^2 - \Omega_1^2) [z \cot \theta_0 (\Omega_2^2 - \Omega_1^2) + (R_2 \Omega_1^2 - R_1 \Omega_2^2)] r^2 + 2 B \cot \theta_0 (R_2 \Omega_1^2 - R_1 \Omega_2^2) - \nu (5 - \cot \theta_0 (\Omega_2^2 - \Omega_1^2) r] - 384 \rho \nu^2 (R_2 - R_1)^2 (\Omega_2^2 - \Omega_1^2) [z \cot \theta_0 (\Omega_2^2 - \Omega_1^2) + (R_2 \Omega_1^2 - R_1 \Omega_2^2)].
\end{align*}
\]
Since the left side of equality (34) includes a function \(\ln r\), and the right side is a polynomial of \(r\), this equality cannot be valid for any numbers \(A\), \(B\) and \(C_0\). Therefore, (28), (31) and (32) do not form the general solution of the problem for \(r \in [r_2, R_1]\). Collecting foregoing discussion our assertion is proved.

### 3.2 Non-existence of the velocity: \(u = u_r(r,z)e_r + u_\theta(r,z)e_\theta\)

In this section we will continue use the method of “proof by contradiction” to show that there does not exist two-dimensional steady solution of the form \(u = u_r(r,z)e_r + u_\theta(r,z)e_\theta\), \(p = p(r,z)\) for the flows between rotating conical cylinders. Assuming that there exists one, for which equations (5)-(8) reduce to

\[
\begin{align*}
  u_r \frac{\partial u_r}{\partial r} - \frac{u_\theta^2}{r} &= -\frac{1}{\rho} \frac{\partial p}{\partial r} + \nu \left( \frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r} \frac{\partial u_r}{\partial r} + \frac{\partial^2 u_r}{\partial z^2} - \frac{u_r}{r^2} \right), \\
  u_r \frac{\partial u_\theta}{\partial r} + u_r u_\theta \frac{1}{r} &= -\frac{1}{\rho r} \frac{\partial p}{\partial \theta} + \nu \left( \frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r} \frac{\partial u_\theta}{\partial r} + \frac{\partial^2 u_\theta}{\partial z^2} - \frac{u_\theta}{r^2} \right), \\
  \frac{\partial p}{\partial z} &= 0, \\
  \frac{\partial u_r}{\partial r} + \frac{u_r}{r} &= 0.
\end{align*}
\]

From the equation (36), we obtain

\[
u c''(z) + c'^2(z) + \frac{c^2(z)}{r^2} = 0.
\]

where \(c(z)\) is an arbitrary differentiable function of \(z\).

By substituting (37) into (33) we obtain the velocity in azimuthal direction

\[
u c''(z) + \frac{c^2(z)}{r^2} = \frac{r}{\rho} \frac{\partial p}{\partial r} - v c''(z) - \frac{c^2(z)}{r^2},
\]

which yields

\[
u c''(z) + \frac{c^2(z)}{r^2} = \frac{r}{\rho} \frac{\partial p}{\partial r} = F(r),
\]

From equation (35) it follows that \(p\) is independent on \(z\), thus \(u_\theta^2 + \nu c''(z) + \frac{c^2(z)}{r^2}\) is only a function of \(r\). Then equation (39) can be written in the form
where $F(r) = u_\theta^2 + \nu c''(z) + \frac{c^2(z)}{r^2}$.

By virtue of (38) and (40) we have

$$u_\theta^2 = F(r) - \nu c''(z) - \frac{c^2(z)}{r^2}. \tag{43}$$

From (37) and the boundary condition (4), we conclude $c(z) \equiv 0$. Then (37) and (41) become

$$u_r = 0, \quad u_\theta^2 = F(r). \tag{44}$$

Therefore, the steady solution $\mathbf{u} = u_r(r, z)\mathbf{e}_r + u_\theta(r, z)\mathbf{e}_\theta$ of equations (5)–(8) is reduced to $\mathbf{u} = u_\theta(r)\mathbf{e}_\theta$. It is proved in [6] that this type of one-dimensional steady solution does not exist for two rotating cones in the case of $\theta_0 \neq \frac{\pi}{2}$.

### 4. Existence of three-dimensional statistical steady solution by numerical simulation

In this section, the numerical simulation was conducted with the inner cone rotating and the outer one at rest. A viscous incompressible fluid is contained between the two conical cones with rigid boundary of the top and base surface. All the numerical results are described in terms of these non-dimensional parameters:

- Reynolds number: $Re = \Omega_1 L_1(L_2 - L_1)/\nu$;
- Aspect ratio: $\alpha = H/(L_2 - L_1)$;
- Radius ratio: $\beta = L_1/L_2$;

where $L_1 = (r_1 + R_1)/2(L_2 = (r_2 + R_2)/2)$ is the average radius of the inner (outer) conical cylinder (see figure 1).

#### 4.1 Convergence and validation

The grids used for the numerical simulations consist of tetrahedral elements. We have conducted extensive grid-refinement tests by varying the element order and the time step. In order to improve the accuracy and the convergence rate, a hybrid correction technique is used. In this technique, four grid levels are employed, which are refined sequentially in spatial dimensions. The solution obtained by a coarser grid is interpolated to initialize the solution on the finer grid. Figure 4 shows the convergence of the simulated results by comparing the profiles (averaged both in time and along radial and azimuthal directions) of pressure and velocity from four grids resolutions. The profiles with different resolutions essentially collapse into one curve, suggesting the independence of the grids on the results. It is worth mentioning that the fourth grid level refinement (case D) leads to insignificant changes in the obtained solution, which demonstrate the convergence of the present simulated results.
Figure 4: Convergence studies at $t=20s$ and $Re=20$ on the profiles of magnitude of (a) pressure, (b) velocity along axial direction at $\alpha=12.5$, $\beta=0.7574$, $\theta_0=82deg$. The number of grid points in case A~D are 170912, 395478, 562464, 736806 respectively. The magnitudes of pressure and velocity are normalized by the maximum value in the profiles. $\bar{\cdot}$ represents the quantities are averaged both in time and along the radial and azimuthal directions.

4.2 Statistics of kinetic energy

The present numerical result is concerned with kinetic energy of the flow between two rotating conical cylinders. A test was performed to confirm that there exists a stationary flow state. Figure 5 represents a time-based sequence of the total kinetic energy which was normalized by subtracting the mean value of the entire data set. The fluctuation intensity of the kinetic energy is considered close to zero. The total kinetic energy remaining approximately constant shows that a stationary flow state had been achieved at $Re = 20$. 
Existence of steady solution for flows

5. Conclusion

The flow between two rotating conical cylinders is considered. The viscous fluid contained between two cones is incompressible. No matter the angular velocities of the inner and outer conical cylinder are equal or not, we have proved that there does not exist steady solutions with the velocity field of the form:

\[ u = u_\theta(r, z)e_\theta + u_z(r, z)e_z \]
\[ u = u_r(r, z)e_r + u_\theta(r, z)e_\theta \]

with \( p = p(r, z) \). Through numerical simulation we showed that there exists a three-dimensional steady solution for CC flow with the inner cone rotating and the outer one at rest. This numerical result confirm, to a certain extent, Wimmer’s experimental observation.

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References


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