Symmetric Properties for the Twisted $(h, q)$-Tangent Polynomials Associated with the $p$-Adic Integral on $\mathbb{Z}_p$

C. S. Ryoo

Department of Mathematics
Hannam University, Daejeon 306-791, Korea

Abstract

In [7], we studied the twisted $(h, q)$-tangent numbers and polynomials. By using these numbers and polynomials, we give some interesting relations between the power sums and the twisted Tangent polynomials.

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1 Introduction

Recently, many mathematicians have studied in the area of the $q$-analogues of the Bernoulli numbers, Euler numbers, Genocchi numbers, Tangent numbers. Tangent numbers and polynomials possess many interesting properties and arising in many areas of mathematics and physics. In [7], we constructed the twisted $(h, q)$-tangent numbers and polynomials. In this paper, by using the symmetry of $p$-adic $q$-integral on $\mathbb{Z}_p$, we give recurrence identities the twisted $(h, q)$-tangent polynomials and the power sums. Throughout this paper, we always make use of the following notations: $\mathbb{N}$ denotes the set of natural
numbers and \( \mathbb{Z}_+ = \mathbb{N} \cup \{0\} \), \( \mathbb{C} \) denotes the set of complex numbers, \( \mathbb{Z}_p \) denotes the ring of \( p \)-adic rational integers, \( \mathbb{Q}_p \) denotes the field of \( p \)-adic rational numbers, and \( \mathbb{C}_p \) denotes the completion of algebraic closure of \( \mathbb{Q}_p \). Let \( \nu_p \) be the normalized exponential valuation of \( \mathbb{C}_p \) with \( |p|_p = p^{-\nu_p(p)} = p^{-1} \). When one talks of \( q \)-extension, \( q \) is considered in many ways such as an indeterminate, a complex number \( q \in \mathbb{C} \), or \( p \)-adic number \( q \in \mathbb{C}_p \). If \( q \in \mathbb{C} \) one normally assume that \( |q| < 1 \). If \( q \in \mathbb{C}_p \), we normally assume that \( |q - 1|_p < p^{-\frac{1}{p-1}} \) so that \( q^x = \exp(x \log q) \) for \( |x|_p \leq 1 \).

g \in UD(\mathbb{Z}_p) = \{g : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\},

the fermionic \( p \)-adic invariant integral on \( \mathbb{Z}_p \) is defined by Kim as follows:

\[
I_{-1}(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x) = \lim_{N \to \infty} \sum_{x=0}^{p^{N-1}} g(x)(-1)^x, \quad \text{(see \([1, 2, 3]\))}. \quad (1.1)
\]

From (1.1), we obtain

\[
\int_{\mathbb{Z}_p} g(x + n) d\mu_{-1}(x) = (-1)^n \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x) + 2 \sum_{l=0}^{n-1} (-1)^{n-1-l} g(l). \quad (1.2)
\]

Let \( T_p = \cup_{N \geq 1} C_{p^N} = \lim_{N \to \infty} C_{p^N} \), where \( C_{p^N} = \{w | w^{p^N} = 1\} \) is the cyclic group of order \( p^N \). For \( w \in T_p \), we denote by \( \phi_w : \mathbb{Z}_p \to \mathbb{C}_p \) the locally constant function \( x \mapsto w^x \). In [7], we introduced the twisted \((h, q)\)-tangent numbers \( T_{n,q,\omega}^{(h)} \) and polynomials \( T_{n,q,\omega}^{(h)}(x) \) and investigate their properties. Let us define the twisted \((h, q)\)-tangent numbers \( T_{n,q,\omega}^{(h)} \) and polynomials \( T_{n,q,\omega}^{(h)}(x) \) as follows:

\[
I_{-1}(\phi_w(y)q^hye^{2yt}) = \int_{\mathbb{Z}_p} \phi_w(y)q^hye^{2yt} d\mu_{-1}(y) = \sum_{n=0}^{\infty} T_{n,q,\omega}^{(h)}(y) \frac{t^n}{n!}, \quad (1.3)
\]

\[
I_{-1}(\phi_w(y)q^hye^{(2y+x)t}) = \int_{\mathbb{Z}_p} \phi_w(y)q^hye^{(2y+x)t} d\mu_{-1}(y) = \sum_{n=0}^{\infty} T_{n,q,\omega}^{(h)}(x) \frac{t^n}{n!}. \quad (1.4)
\]

The following elementary properties of the twisted \((h, q)\)-tangent numbers \( E_{n,q,\omega}^{(h)} \) and polynomials \( T_{n,q,\omega}^{(h)}(x) \) are readily derived form (1.1), (1.2), (1.3) and (1.4)( see, for details, \([4, 5, 6, 7]\) ). We, therefore, choose to omit details involved.

**Theorem 1.1** ([7]) For \( \omega \in T_p \) and \( h \in \mathbb{Z} \), we have

\[
\int_{\mathbb{Z}_p} \phi_w(x)q^{hx}(2x)^n d\mu_{-1}(x) = T_{n,q,\omega}^{(h)},
\]

\[
\int_{\mathbb{Z}_p} \phi_w(y)q^{hy}(2y + x)^n d\mu_{-1}(y) = T_{n,q,\omega}^{(h)}(x).
\]
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**Theorem 1.2** ([7]) For any positive integer \(n\), we have

\[
T_{n, q, \omega}^{(h)}(x) = \sum_{k=0}^{n} \binom{n}{k} T_{k, q, \omega}^{(h)} x^{n-k}.
\]

2 Alternating sums of powers of consecutive \((h, q)\)-even integers

In this section, we assume that \(q \in \mathbb{C}\), with \(|q| < 1\) and \(h \in \mathbb{Z}\). Let \(\omega\) be the \(p^n\)-th root of unity. By using (1.4), we give the alternating sums of powers of consecutive \((h, q)\)-even integers as follows:

\[
\sum_{n=0}^{\infty} T_{n, q, \omega}^{(h)} \frac{t^n}{n!} = \frac{2}{\omega q^h e^{2t} + 1} = 2 \sum_{n=0}^{\infty} (-1)^n \omega^n q^{nh} e^{2nt}.
\]

By using (1.3) and (1.4), we obtain

\[
-\frac{1}{2} \sum_{j=0}^{\infty} T_{j, q, \omega}^{(h)} (2k) \frac{t^j}{j!} + \frac{1}{2} (-1)^{-k} \omega^{-k} q^{-kh} \sum_{j=0}^{\infty} T_{j, q, \omega}^{(h)} \frac{t^j}{j!}
= \sum_{j=0}^{\infty} \left( (-1)^{-k} \omega^{-k} q^{-kh} \sum_{n=0}^{k-1} (-1)^n \omega^n q^{nh} (2n)^j \right) \frac{t^j}{j!}.
\]

By comparing coefficients of \(\frac{t^j}{j!}\) in the above equation, we obtain the following theorem:

**Theorem 2.1** Let \(k\) be a positive integer and \(q \in \mathbb{C}\) with \(|q| < 1\) and \(\omega\) be the \(p^n\)-th root of unity. Then we obtain

\[
T_{j, q, \omega}^{(h)} (k-1) = \sum_{n=0}^{k-1} (-1)^n \omega^n q^{nh} (2n)^j = \frac{(-1)^{k+1} \omega^{k} q^{kh} T_{j, q, \omega}^{(h)} (2k) + T_{j, q, \omega}^{(h)}}{2}.
\]

**Remark 2.2** For the alternating sums of powers of consecutive odd integers, we have

\[
\lim_{q \to 1} T_{j, q, \omega}^{(h)} (k-1) = \sum_{n=0}^{k-1} (-1)^n \omega^n (2n)^j = \frac{(-1)^{k+1} \omega^{k} T_{j, \omega}^{(h)} (2k) + T_{j, \omega}}{2},
\]

where \(T_{j, \omega}(x)\) and \(T_{j, \omega}\) denote the twisted tangent polynomials and the twisted tangent numbers, respectively (see [6]).
3 Symmetry property of the twisted $\,(h, q)$-tangent polynomials $T_{n,q,\omega}(x)$

In this section, we assume that $q \in \mathbb{C}_p$ and $\omega \in T_p$. In [3], Kim investigated interesting properties of symmetry $p$-adic invariant integral on $\mathbb{Z}_p$ for Bernoulli polynomials and Euler polynomials. By using same method of [3], expect for obvious modifications, we obtain recurrence identities the twisted $(h, q)$-polynomials and Euler polynomials. By using same method of (3), expect tangent polynomials and the alternating sums of powers of consecutive $(h, q)$-even integers. Let $n$ be odd number. Substituting $g(x) = \omega^x q^h e^{2xt}$ into (1.2), we have

$$\int_{\mathbb{Z}_p} \omega^{x+n} q^{h(x+n)} e^{(2x+2n)t} d\mu_{-1}(x) + \int_{\mathbb{Z}_p} \omega^x q^h e^{2xt} d\mu_{-1}(x) = 2 \sum_{j=0}^{n-1} (-1)^j \omega^j q^j h e^{(2j)t}. \quad (3.1)$$

After some elementary calculations, we get

$$\int_{\mathbb{Z}_p} \omega^{x+n} q^{h(x+n)} e^{(2x+2n)t} d\mu_{-1}(x) + \int_{\mathbb{Z}_p} \omega^x q^h e^{2xt} d\mu_{-1}(x) = \frac{2 \int_{\mathbb{Z}_p} \omega^x q^h e^{2xt} d\mu_{-1}(x)}{\int_{\mathbb{Z}_p} \omega^{nx} q^{hnx} e^{2ntx} d\mu_{-1}(x)}. \quad (3.2)$$

By using (3.1) and (3.2), we arrive at the following theorem:

**Theorem 3.1** Let $n$ be odd positive integer. Then we obtain

$$\frac{2 \int_{\mathbb{Z}_p} \omega^x q^h e^{2xt} d\mu_{-1}(x)}{\int_{\mathbb{Z}_p} \omega^{nx} q^{hnx} e^{2ntx} d\mu_{-1}(x)} = \sum_{m=0}^{n} \frac{(2T_{m,q,\omega}^{(h)}(n-1)) t^m}{m!}. \quad (3.3)$$

Let $w_1$ and $w_2$ be odd positive integers. By using (3.3), we have

$$\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} \omega^{(w_1 x_1 + w_2 x_2)} q^{h(w_1 x_1 + w_2 x_2)} e^{(w_1 x_1 + w_2 x_2 + w_1 w_2 x_2)t} d\mu_{-1}(x_1) d\mu_{-1}(x_2) = \frac{2 e^{w_1 x_1 t} (\omega^{w_1} q^{w_1} e^{2w_1 x_1 t} + 1)}{(\omega^{w_1} q^{w_1} e^{2w_1 t} + 1)(\omega^{w_2} q^{w_2} e^{2w_2 t} + 1)}. \quad (3.4)$$

By using (3.3) and (3.4), after elementary calculations, we obtain

$$a = \left( \frac{1}{2} \sum_{m=0}^{\infty} T_{m,q,w_1,\omega w_1}^{(h)} (w_2 x_1) w_1^m \frac{t^m}{m!} \right) \left( \frac{2}{2} \sum_{m=0}^{\infty} T_{m,q,w_2,\omega w_2}^{(h)} (w_1 - 1) w_2^m \frac{t^m}{m!} \right). \quad (3.5)$$

By using Cauchy product in the above, we have

$$a = \sum_{m=0}^{\infty} \left( \sum_{j=0}^{m} \binom{m}{j} T_{j,q,w_1,\omega w_1}^{(h)} (w_2 x_1) w_1^j T_{m-j,q,w_2,\omega w_2}^{(h)} (w_1 - 1) w_2^{m-j} \right) \frac{t^m}{m!}. \quad (3.6)$$
By using the symmetry in (3.5), we obtain

\[ a = \left( \frac{1}{2} \sum_{m=0}^{\infty} T_{m,q}^{(h)} \omega_{2} w_{2}^{m} \frac{t^{m}}{m!} \right) \left( 2 \sum_{m=0}^{\infty} T_{m,q}^{(h)} w_{2} - 1 \right) w_{1}^{m} \frac{t^{m}}{m!} . \]

Thus we have

\[ a = \sum_{m=0}^{\infty} \left( \sum_{j=0}^{m} \left( \begin{array}{c} m \\ j \end{array} \right) T_{j,q}^{(h)} \omega_{2} w_{2}^{j} w_{1}^{m-j} \frac{t^{m}}{m!} \right) \frac{t^{m}}{m!} . \]  

(3.7)

By comparing coefficients \( \frac{t^{m}}{m!} \) in the both sides of (3.6) and (3.7), we arrive at the following theorem:

**Theorem 3.2** Let \( w_{1} \) and \( w_{2} \) be odd positive integers. Then we obtain

\[ \sum_{j=0}^{m} \left( \begin{array}{c} m \\ j \end{array} \right) T_{j,q}^{(h)} \omega_{2} w_{2}^{j} w_{1}^{m-j} \frac{t^{m}}{m!} \]

\[ = \sum_{j=0}^{m} \left( \begin{array}{c} m \\ j \end{array} \right) T_{j,q}^{(h)} w_{2} x \frac{t^{m}}{m!} \]

where \( T_{k,q}^{(h)}(x) \) and \( T_{m,q}^{(h)}(k) \) denote the twisted \((h,q)\)-tangent polynomials and the alternating sums of powers of consecutive \((h,q)\)-even integers, respectively (see [7]).

By using Theorem 1.2, we have the following corollary:

**Corollary 3.3** Let \( w_{1} \) and \( w_{2} \) be odd positive integers. Then we obtain

\[ \sum_{j=0}^{m} \sum_{k=0}^{j} \left( \begin{array}{c} m \\ j \end{array} \right) \left( \begin{array}{c} j \\ k \end{array} \right) w_{1}^{m-k} w_{2}^{j-k} \frac{t^{m}}{m!} \]

\[ = \sum_{j=0}^{m} \sum_{k=0}^{j} \left( \begin{array}{c} m \\ j \end{array} \right) \left( \begin{array}{c} j \\ k \end{array} \right) w_{1}^{j} w_{2}^{m-k} \frac{t^{m}}{m!} . \]

By using (3.4), we have

\[ a = \sum_{j=0}^{w_{1}-1} (-1)^{j} \omega^{w_{1}j} q^{w_{2}h} \int_{\mathbb{Z}_{p}} \omega^{w_{1}x_{1}} q^{h} \frac{2x_{1} + w_{2}x}{w_{1}} \frac{t^{n}}{n!} d\mu_{-1}(x_{1}) \]

\[ = \sum_{n=0}^{\infty} \left( \sum_{j=0}^{w_{1}-1} (-1)^{j} \omega^{w_{1}j} q^{w_{2}h} T_{n,q}^{(h)} w_{2} + \frac{2j w_{2}}{w_{1}} \right) \frac{t^{n}}{n!} . \]  

(3.8)
By using the symmetry property in (3.8), we also have

\[ a = \sum_{j=0}^{w_2-1} (-1)^j \omega w_1^j q^{w_1 h_j} \int_{\mathbb{Z}_p} \omega^{w_2 x} q^{h w_2 x e} \left( \frac{2x_2 + w_1 x + 2j w_1}{w_2} \right) (w_2 t) d\mu_{-1}(x_1) \]

\[ = \sum_{n=0}^{\infty} \left( \sum_{j=0}^{w_2-1} (-1)^j \omega w_1^j q^{w_1 h_j} T_{n, q^{w_2}, \omega^{w_2}}^{(h)} \left( w_1 x + \frac{2j w_1}{w_2} \right) \right) \frac{t^n}{n!}. \]

(3.9)

By comparing coefficients \( \frac{t^n}{n!} \) in the both sides of (3.8) and (3.9), we have the following theorem.

**Theorem 3.4** Let \( w_1 \) and \( w_2 \) be odd positive integers. Then we obtain

\[ \sum_{j=0}^{w_1-1} (-1)^j \omega w_1^j q^{w_2 h_j} T_{n, q^{w_1}, \omega^{w_1}}^{(h)} \left( w_2 x + \frac{2j w_2}{w_1} \right) \]

\[ = \sum_{j=0}^{w_2-1} (-1)^j \omega w_1^j q^{w_1 h_j} T_{n, q^{w_2}, \omega^{w_2}}^{(h)} \left( w_1 x + \frac{2j w_1}{w_2} \right) \]

(3.10)

Substituting \( w_1 = 1 \) into (3.10), we arrive at the following corollary.

**Corollary 3.5** Let \( w_2 \) be odd positive integer. Then we obtain

\[ T_{n, q, \omega}^{(h)}(x) = w_2^n \sum_{j=0}^{w_2-1} (-1)^j \omega^j q^{h_j} T_{n, q^{w_2}, \omega^{w_2}}^{(h)} \left( \frac{x + 2j}{w_2} \right). \]

**References**


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