Existence Result for Two-dimensional Kuramoto-Sivashinsky Equation with Nonlocal Source

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Abstract
We consider a nonlinear two-dimensional Kuramoto-Sivashinsky equation, with nonlocal source, and by a reduction method we determine a class of spatially periodic steady state solutions. Our analysis leads toward some computation, which can be easily automatized. By use of the symmetries of the problem we study the structure of the reduced equation, and we obtain an algebraic system of lower order, determining all the small solutions to the stationary problem.

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1 Introduction
We are concerned by the spatially periodic solutions of the Kuramoto-Sivashinsky (shortened as KS) equation
$$\frac{\partial u}{\partial t} = -\Delta^2 u - \alpha \Delta u + N(u, \alpha)$$
where $\Delta$ is the Laplace operator, and $\alpha$ is a real parameter. The KS equation is known as a model for pattern formation in different physical contexts [1].

In [8], the differential transformation method has been used for finding the solution of the KS equation in one-dimensional space. While in [3], the two-dimensional damped KS equation is studied by applying the symmetry group method.

Here we are interested in the bifurcating spatially periodic steady states of two-dimensional and nonlinear KS equation with nonlocal source

$$\frac{\partial u}{\partial t} (t, x, y) = -\Delta^2 u (t, x, y) - \alpha \Delta u (t, x, y) + N (u (t, x, y) , \alpha) \quad (1)$$

for $(t, x, y) \in \mathbb{R}^+ \times \mathbb{R}^2$, with initial value $u(0, x, y) = u_0(x, y)$, with the condition of periodicity

$$u(t, x, y) = u(t, x + 2\pi, y) = u(t, x, y + 2\pi), \quad (2)$$

where the nonlocal source is defined by

$$N (u (t, x, y) , \alpha) = \frac{\alpha}{2} \left( \left| \nabla u (t, x, y) \right|^2 - \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} | \nabla u (t, \xi, \eta) |^2 \, d\xi \, d\eta \right) \quad (3)$$

In this work we make use of the Liapunov-Schmidt reduction method, developed in [2], and also applied in [4] to stability studies. The method consists of splitting the time independent equation associated to (1) into two equations, to be solved separately and reduced to an algebraic equation, which is called the reduction equation. By considering the Taylor expansions of these equations, all the small stationary solutions of (1) can be computed. We focus our attention on the symmetries of the problem, which are the translations along $x$ and $y$ directions, and reflections. These symmetries will help us to determine the spatially periodic steady flow which appear when the trivial steady state ($u \equiv 0$) becomes unstable. They determine the structure of the reduced equation, and decrease significantly the computing effort needed to solve the stationary problem associated to (1).

In Section 2, the functional framework and properties of the steady state problem (1)-(3) are set. This allows us to apply the Liapunov-Schmidt method in Section 3, where the reduction process is presented. In Section 4, by a detailed analysis of the symmetries, we give our main result on the simplified form of the reduced algebraic equation. Thus we conclude by characterizing the families of steady state solutions, near the trivial state ($u \equiv 0$).
2 Functional framework

2.1 Destabilization of the trivial steady state

For all $\alpha > 0$, the equation (1) admits a trivial time independent solution: $u \equiv 0$. By linear stability, we can see that the trivial steady state becomes unstable to small spatial disturbances when the parameter $\alpha$ becomes larger than the critical value $\alpha_c = 2$, and a Fourier expansion shows that the first unstable mode corresponds to the critical wave vector $(s_c, r_c) = (1, 1)$.

2.2 Operational and functional setting

Firstly, we denote the Euclidean inner product in $\mathbb{R}^2$ by $\langle ., . \rangle$, and the related norm by $| . |$. With the change of variables $X = sx$ and $Y = ry$, where $(s, r) \in \mathbb{R}^2$, for every time independent solution $u(x, y)$ to (1) corresponds a solution $U(X, Y)$ to the following equation

$$
-L_{\alpha,s,r}^s U (X, Y) + \frac{\alpha}{2} \left| \nabla_{sr} U (X, Y) \right|^2 - \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} |\nabla_{sr} U (\xi, \eta)|^2 d\xi d\eta = 0
$$

where $L_{\alpha,s,r}^s = s^2 \frac{\partial^2}{\partial X^2} + r^2 \frac{\partial^2}{\partial Y^2}$, $\nabla_{sr} = (s \frac{\partial}{\partial X}, r \frac{\partial}{\partial Y})$ and $(X, Y) \in \Omega = ]0, 2\pi[ \times ]0, 2\pi[.

We rewrite (4) as an operational equation in $L^2(\Omega)$

$$
F(U, \alpha, s, r) = L_{\alpha,s,r}^s (U) + N_{\alpha,s,r}^s (U) = 0,
$$

where $L_{\alpha,s,r}^s (U) = -L_{\alpha,s,r}^s U (X, Y) - \alpha L_{\alpha,s,r}^s U (X, Y)$, is a linear operator, and $N_{\alpha,s,r}^s (U) = N(U, U)$ is a nonlinear (and nonlocal) operator, such that

$$
N(U, V) = \frac{\alpha}{2} \langle \nabla_{sr} U (X, Y), \nabla_{sr} V (X, Y) \rangle - \frac{\alpha}{8\pi^2} \int_0^{2\pi} \int_0^{2\pi} \langle \nabla_{sr} U (\xi, \eta), \nabla_{sr} V (\xi, \eta) \rangle d\xi d\eta.
$$

Taking into account that problem (5) is invariant under translations, and considering the conditions of periodicity (2), one can define the domain of $L_{\alpha,s,r}$ by $D = H^4(\Omega) \cap \dot{H}_{per}^4(\Omega)$, and the domain of $\Delta_{sr}$ by $D (\Delta_{sr}) = H^2(\Omega) \cap \dot{H}_{per}^2(\Omega)$, in the Hilbert space $H = \{ u \in L^2(\Omega) : \int_{\Omega} u(\xi, \eta) d\xi d\eta = 0 \}$. Where $H^m(\Omega)$ is the usual Sobolev space, and $\dot{H}_{per}^m(\Omega)$ is the space of functions $u$ in $H \cap H^1(\Omega)$ with periodic conditions $u(0, Y) = u(2\pi, Y)$ and $u(X, 0) = u(X, 2\pi)$. 
We endow $\mathcal{H}$ with the usual inner product $\langle u, v \rangle = \int_{\Omega} u(\xi, \eta)v(\xi, \eta) d\xi d\eta$, for all $(u, v) \in \mathcal{H}^2$, and $\parallel \cdot \parallel_{\mathcal{H}}$ denotes the induced norm of $\mathcal{H}$ (for further details on this type of Sobolev spaces see [6]).

### 2.3 Fredholm property of $L_{\alpha_c, s_c, r_c}$

The fact that $L_{\alpha_c, s_c, r_c}$ is a Fredholm operator of index zero is the key to apply the Liapunov-Schmidt reduction method (see [2, p.290]).

**Theorem 2.1.** $L_{\alpha_c, s_c, r_c}$ is a Fredholm operator of index zero.

**Proof.** On the one hand, we can see that $\Delta_{sr}$ and $(\Delta_{sr} + \alpha I)$ (where $I$ is the identity operator) are self-adjoint operators in $\mathcal{H}$, and since $\hat{\mathcal{H}}^1_{\text{per}}(\Omega)$ is compactly embedded in $\mathcal{H}$ (see [5, Theorem A.4]), they have compact resolvents in $\mathcal{H}$.

On the other hand, if $\lambda \in \mathbb{C}$ is such that $\text{Re}(\lambda - \alpha) > 0$, then we have

$$
(\Delta_{sr} - (\lambda - \alpha)I)^{-1}(\Delta_{sr} + \alpha I) = I + \lambda(\Delta_{sr} - (\lambda - \alpha)I)^{-1} = I + \lambda(\Delta_{sr} - (\lambda - \alpha)I)^{-1}.
$$

So, according to [7, Lemma 2.4] we deduce from (6) that $(\Delta_{sr} + \alpha I)$ is a Fredholm operator, and by the same argument $\Delta_{sr}$ is also a Fredholm operator. Since $\Delta_{sr}$ and $(\Delta_{sr} + \alpha I)$ are self-adjoint and Fredholm operators, they must have respectively a closed range and index zero (see [7]). Thus, noting that $L_{\alpha_c, s_c, r_c} = -\Delta_{sr}(\Delta_{sr} + \alpha_c I)$, from the above properties of $\Delta_{sr}$ and $(\Delta_{sr} + \alpha I)$, according to [7, Theorem 2.5], we obtain the result of the Theorem. \qed

### 2.4 Kernel of $L_{\alpha_c, s_c, r_c}$

To apply the Liapunov-Schmidt reduction method, we need a basis of the kernel of $L_{\alpha, s, r}$ at the critical values $(\alpha, s, r) = (\alpha_c, s_c, r_c) = (2, 1, 1)$.

**Theorem 2.2.** $\text{Ker}(L_{\alpha_c, s_c, r_c})$ is spanned by $w_1, w_2, w_3, w_4$, defined by

$$
w_1 = \cos(X + Y), w_2 = \sin(X + Y), w_3 = \cos(X - Y), w_4 = \sin(X - Y).
$$

**Proof.** To compute $\text{Ker}(L_{\alpha_c, s_c, r_c})$, we have to solve $L_{\alpha_c, s_c, r_c}(U) = 0$. Let us consider the Fourier expansion of the (periodic) functions in $D$

$$
U(X, Y) = \sum_{\sigma_1 \in \mathbb{Z}, \sigma_2 \in \mathbb{Z}} \alpha^{(\sigma_1, \sigma_2)} \exp(i\sigma_1 X + i\sigma_2 Y).
$$

where $\alpha^{(\sigma_1, \sigma_2)} \in \mathbb{C}$. 

Considering Theorem 2.1, $L_{\alpha,s,c}$ is closed in $H$, thus we obtain

$$L_{\alpha,s,c}(U) = \sum_{\sigma_1 \in \mathbb{Z}, \sigma_2 \in \mathbb{Z}}((\sigma_1^2 + \sigma_2^2)^2 + 2(\sigma_1^2 + \sigma_2^2))\alpha(\sigma_1, \sigma_2)exp(i\sigma_1 X + i\sigma_2 Y).$$

Hence, $U \in Ker(L_{\alpha,s,c})$ if and only if $(\sigma_1, \sigma_2) \in \{-1, 1\} \times \{-1, 1\}$, what implies the result of the theorem. \qed

### 2.5 Regularity of $N_{\alpha,s,r}$

The smoothness of the nonlinear part in (5) will be needed in Section 3. In order to state that $N_{\alpha,s,r} : D \mapsto H$ is of class $C^q$ (with $q$ as large as necessary), it is enough to note that the bilinear mapping $N$ is continuous from $D \times D$ to $H$. So $N_{\alpha,s,r}$ is analytic as a quadratic form associated to a continuous bilinear mapping.

### 3 The reduced equation

By setting $L_c = L_{\alpha,s,c}$, according to Theorem 2.1 $L_c$ is a Fredholm operator with index zero, and therefore $H$ may be split as a direct sum $H = Ker(L_c) \oplus R(L_c)$, where $R(L_c)$ indicates the range of $L_c$. Let $P$ be the projection of $H$ onto $Ker(L_c)$ which the kernel is equal to $R(L_c)$. Then $P$ and $(I - P)$ are bounded projections of $H$.

For every $(\alpha, s, r)$ in $\mathbb{R}^3$ we set $L_{\alpha,s,r} = L_c + L_{\mu,S,R}$ with $\mu = \alpha - \alpha_c = \alpha - 2$, $S = s - s_c = s - 1$, and $R = r - r_c = r - 1$, which will be considered in a neighborhood of $(0,0,0)$.

#### 3.1 The reduction principle

Every element in $H$ splits into $w + U$ with $w$ in $Ker(L_c)$ while $U$ is in $R(L_c)$. Equation (5) is equivalent to

$$L_c(U) + (I - P)(L_{\mu,S,R}(w + U) + N_{\mu+2,S+1,R+1}(w + U)) = 0 \quad (7)$$

$$P(L_{\mu,S,R}(w + U) + N_{\mu+2,S+1,R+1}(w + U)) = 0 \quad (8)$$

We will see that for every $w$ in $Ker(L_c)$ there exists exactly one $U$ in $R(L_c)$ such that (7) holds.

**Theorem 3.1.** For every $(w, \mu, S, R)$ in a neighborhood of the zero element $O \in Ker(L_c) \times \mathbb{R}^3$, (7) has exactly one smooth solution $U = U(w, \mu, S, R)$ in $R(L_c)$ such that $U(0,0,0,0) = 0$. 

Proof. We can rewrite equation (7) as \( \Phi(w, \mu, S, R, U) = 0 \), where
\[
\Phi(w, \mu, S, R, U) = L_c(U) + (I - P)(L_{\mu,S,R}(w + U) + N_{\mu+2,S+1,R+1}(w + U))
\]
is a smooth mapping from \( \text{Ker}(L_c) \times \mathbb{R}^3 \times (D \cap R(L_c)) \) to \( R(L_c) \). Endowing the spaces \( D \cap R(L_c) \) and \( R(L_c) \), respectively with the graph norm and the norm of \( \mathcal{H} \). We have \( \Phi(0,0,0,0,0) = 0 \). In view of the Fredholm property of \( L_c \) the mapping \( \partial_u \Phi(0,0,0,0) = L_c \) has an inverse which is bounded from \( R(L_c) \) to \( D \cap R(L_c) \). Hence by the implicit function theorem, there exists a unique smooth function \( U(w, \mu, S, R) \) defined from a neighborhood of \( 0 \) in \( \text{Ker}(L_c) \times \mathbb{R}^3 \) to a neighborhood of \( 0 \) in \( D \cap R(L_c) \), which satisfies \( \Phi(w, \mu, S, R, U(w, \mu, S, R)) = 0 \) and \( U(0,0,0,0) = 0 \).

3.2 The reduced equation

The elements in a neighborhood of the trivial element \( 0 \in D \), which solve (5) are of the form \( w + U(w, \mu, S, R) \) and satisfy the so called reduced equation
\[
P(L_{\mu,S,R}(w + U(w, \mu, S, R)) + N_{\mu+2,S+1,R+1}(w + U(w, \mu, S, R))) = 0 \quad (9)
\]
From now, we shall denote the left hand side of (9) by \( G_{\mu,S,R}(w) \). The solutions of (5), that are close to the trivial solution are those that take the form \( w + U(w, \mu, S, R) \) where \( w \) solves (9). Since \( U \) is smooth, \( G_{\mu,S,R} \) is smooth also. Now, we use the basis \( \{w_1, w_2, w_3, w_4\} \) of \( \text{Ker}(L_c) \), to describe the action of \( G_{\mu,S,R} \) on \( \text{Ker}(L_c) \) by means of the reduction function \( g = (g_1, g_2, g_3, g_4) \) defined from \( \mathbb{R}^4 \times \mathbb{R}^3 \) to \( \mathbb{R}^4 \), by
\[
g_j(a_1, a_2, a_3, a_4, \mu, S, R) = < L_{\mu,S,R}(w + U(w, \mu, S, R)) + N_{\mu+2,S+1,R+1}(w + U(w, \mu, S, R)), w_j >,
\]
where \( w = a_1w_1 + a_2w_2 + a_3w_3 + a_4w_4 \). Therefore (7) is equivalent to
\[
g(a_1, a_2, a_3, a_4, \mu, S, R) = 0. \quad (10)
\]

4 The symmetries of the problem

Equation (1) is equivariant with respect to the following symmetries

(i) the translation : \( (\tau_{x_0,y_0}U)(x, y) = U(x - x_0, y - y_0) \),

(ii) the reflection : \( (\kappa_1U)(x, y) = U(-x, -y) \),

(iii) the reflection : \( (\kappa_2U)(x, y) = U(x, -y) \).

These transformations leave \( \text{Ker}(L_c) \) and \( R(L_c) \) invariant. We can analyse their actions on the reduction function, through their action on \( \text{Ker}(L_c) \). To know more about the theory of equivariance see [2].
4.1 Consequences on the reduction function

According to [2], taking into account the uniqueness of the solution to (7), \( \tau_{x_0,y_0}, \kappa_1 \) and \( \kappa_2 \) commute with \( \mathcal{U} \), and with \( \mathbf{G}_{\mu,S,R} \) as well. Thus, since the inner product in \( \mathcal{H} \) is invariant under these symmetries, we deduce useful properties of the reduction function \( \mathbf{g} \).

4.1.1 Translations along the X and Y directions

Using polar coordinates: \((a_1, a_2, a_3, a_4) = (\rho_1 \cos \phi_1, \rho_1 \sin \phi_1, \rho_2 \cos \phi_2, \rho_2 \sin \phi_2)\), noting any \( w = a_1 w_1 + a_2 w_2 + a_3 w_3 + a_4 w_4 \) in \( \text{Ker}(\mathcal{L}_c) \) by \( w(\rho_1, \phi_1, \rho_2, \phi_2) \) and setting

\[
g_j(\rho_1, \phi_1, \rho_2, \phi_2) = g_j(\rho_1 \cos \phi_1, \rho_1 \sin \phi_1, \rho_2 \cos \phi_2, \rho_2 \sin \phi_2, \mu, S, R)
\]

for \( j = 1, 2, 3, 4 \), we get the following lemma

**Lemma 4.1.** For all \((\rho_1, \rho_2, \phi_1, \phi_2, x_0, y_0)\) in \((\mathbb{R}^+)^2 \times \mathbb{R}^4\) we have:

\[
\begin{pmatrix}
g_1(\rho_1, \phi_1 + \beta_1, \rho_2, \phi_2 + \beta_2) \\
g_2(\rho_1, \phi_1 + \beta_1, \rho_2, \phi_2 + \beta_2)
\end{pmatrix} =
\begin{pmatrix}
\cos \beta_1 & -\sin \beta_1 \\
\sin \beta_1 & \cos \beta_1
\end{pmatrix}
\begin{pmatrix}
g_1(\rho_1, \phi_1, \rho_2, \phi_2) \\
g_2(\rho_1, \phi_1, \rho_2, \phi_2)
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
g_3(\rho_1, \phi_1 + \beta_1, \rho_2, \phi_2 + \beta_2) \\
g_4(\rho_1, \phi_1 + \beta_1, \rho_2, \phi_2 + \beta_2)
\end{pmatrix} =
\begin{pmatrix}
\cos \beta_2 & -\sin \beta_2 \\
\sin \beta_2 & \cos \beta_2
\end{pmatrix}
\begin{pmatrix}
g_3(\rho_1, \phi_1, \rho_2, \phi_2) \\
g_4(\rho_1, \phi_1, \rho_2, \phi_2)
\end{pmatrix}
\]

with \( \beta_1 = x_0 + y_0 \) and \( \beta_2 = x_0 - y_0 \).

**Proof.** By use of the trigonometric formulae, the investigation of the action of \( \tau_{x_0,y_0} \) on \( \text{Ker}(\mathcal{L}_c) \) leads to

\[
\begin{align*}
(\tau_{x_0,y_0} w_1) &= w_1 \cos(x_0 + y_0) + w_2 \sin(x_0 + y_0), \\
(\tau_{x_0,y_0} w_2) &= -w_1 \sin(x_0 + y_0) + w_2 \cos(x_0 + y_0), \\
(\tau_{x_0,y_0} w_3) &= w_3 \cos(x_0 - y_0) + w_4 \sin(x_0 - y_0), \\
(\tau_{x_0,y_0} w_4) &= -w_3 \sin(x_0 - y_0) + w_4 \cos(x_0 - y_0).
\end{align*}
\]

Consequently, for any \( w \in \text{Ker}(\mathcal{L}_c) \) the linearity of \( \tau_{x_0,y_0} \) implies that

\[
(\tau_{x_0,y_0} \mathbf{G}_{\mu,S,R} w(\rho_1, \phi_1, \rho_2, \phi_2)) = \sum_{i=1}^{4} g_i(\rho_1, \phi_1, \rho_2, \phi_2) \tau_{x_0,y_0} w_i,
\]

and

\[
(\tau_{x_0,y_0} w)(\rho_1, \phi_1, \rho_2, \phi_2) = w(\rho_1, \phi_1 + x_0 + y_0, \rho_2, \phi_2 + x_0 - y_0),
\]

which induces that

\[
\mathbf{G}_{\mu,S,R}(\tau_{x_0,y_0} w(\rho_1, \phi_1, \rho_2, \phi_2)) = \sum_{i=1}^{4} g_i(\rho_1, \phi_1 + x_0 + y_0, \rho_2, \phi_2 + x_0 - y_0) w_i.
\]
According to the equivariance of $G_{\mu,S,R}$ with respect to $\tau_{x_0,y_0}$, by identification we obtain the following matrix equality

$$
\begin{pmatrix}
g_1(\rho_1, \phi_1 + \beta_1, \rho_2, \phi_2 + \beta_2) \\
g_2(\rho_1, \phi_1 + \beta_1, \rho_2, \phi_2 + \beta_2) \\
g_3(\rho_1, \phi_1 + \beta_1, \rho_2, \phi_2 + \beta_2) \\
g_4(\rho_1, \phi_1 + \beta_1, \rho_2, \phi_2 + \beta_2)
\end{pmatrix} = 
\begin{pmatrix}
cos\beta_1 & -sin\beta_1 & 0 & 0 \\
sin\beta_1 & cos\beta_1 & 0 & 0 \\
0 & 0 & cos\beta_2 & -sin\beta_2 \\
0 & 0 & sin\beta_2 & cos\beta_2
\end{pmatrix}
\begin{pmatrix}
g_1(\rho_1, \phi_1, \rho_2, \phi_2) \\
g_2(\rho_1, \phi_1, \rho_2, \phi_2) \\
g_3(\rho_1, \phi_1, \rho_2, \phi_2) \\
g_4(\rho_1, \phi_1, \rho_2, \phi_2)
\end{pmatrix}
$$

which is equivalent to the matricial equalities given in the Lemma.

**Remark 4.2.** Lemma 4.1 shows that the translations along the $x$ and $y$ directions act on $\text{Ker}(L_c)$ as rotations separately in the two linear manifolds spanned respectively by $\{w_1, w_2\}$ and by $\{w_3, w_4\}$. So for a complete study of (10) it suffices to take $\phi_1 = \phi_2 = 0$ and reduces (10) to

$$
g_i(\rho_1, 0, \rho_2, 0) = 0, \quad \text{for } i = 1, 2, 3, 4.
$$

### 4.1.2 Reflections

In order to obtain a more simple structure of the reduced function $g$, first we analyse the action of $\kappa_1$ and $\kappa_2$ on $\text{Ker}(L_c)$, then we carry the consequences to $G_{\mu,S,R}$. The results of this analysis are outlined in the following remark.

**Remark 4.3.** (i) Since for all $i$ we have $\kappa_1(w_i) = (-1)^{i+1}w_i$, then

$$
\kappa_1 w(\rho_1, \phi_1, \rho_2, \phi_2) = w(\rho_1, -\phi, \rho_2, -\phi_2) = -w(-\rho_1, -\phi, -\rho_2, -\phi_2) \quad (11)
$$

Thanks to the first equality of (11), if we proceed as above we obtain

$$
\sum_{i=1}^{4} g_i(\rho_1, \phi_1, \rho_2, \phi_2)(-1)^{i+1}w_i = \sum_{i=1}^{4} g_i(\rho_1, -\phi_1, \rho_2, -\phi_2)w_i.
$$

Hence $g_i(\rho_1, \phi_1, \rho_2, \phi_2)$ are even functions with respect to $(\phi_1, \phi_2)$ when $i = 1, 3$ and odd functions when $i = 2, 4$. While the second equality of (11) implies that

$$
\sum_{i=1}^{4} g_i(\rho_1, \phi_1, \rho_2, \phi_2)(-1)^{i+1}w_i = \sum_{i=1}^{4} -g_i(-\rho_1, -\phi_1, -\rho_2, -\phi_2)w_i.
$$

So, $g_i(\rho_1, \phi_1, \rho_2, \phi_2)$ are odd functions, with respect to $(\rho_1, \rho_2)$ for all $i$. 
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(ii) We have \( \kappa_2(w_i) = w_{i+2} \) for \( i = 1, 2 \) and \( \kappa_2(w_i) = w_{i-2} \) for \( i = 3, 4 \). Consequently \( \kappa_2w(\rho_1, \phi_1, \rho_2, \phi_2) = w(\rho_2, \phi_2, \rho_1, \phi_1) \) and therefore

\[
\sum_{i=1}^{4} g_i(\rho_1, \phi_1, \rho_2, \phi_2) w_i = \sum_{i=1}^{4} g_i(\rho_2, -\phi_2, \rho_1, -\phi_1) w_i.
\]

Thus

\[
g_1(\rho_1, \phi_1, \rho_2, \phi_2) = g_3(\rho_2, \phi_2, \rho_1, \phi_1),
\]

\[
g_2(\rho_1, \phi_1, \rho_2, \phi_2) = g_4(\rho_2, \phi_2, \rho_1, \phi_1).
\]

In view of the above remark we deduce the following Lemma.

**Lemma 4.4.**

(i) \( g_j(\rho_1, 0, \rho_2, 0) \) for \( j = 1, 3 \) are odd functions with respect to \( \rho_1 \) and \( \rho_2 \).

(ii) \( g_j(\rho_1, 0, \rho_2, 0) = 0 \) for \( j = 2, 4 \).

(iii) \( g_1(\rho_1, 0, \rho_2, 0) = g_3(\rho_2, 0, \rho_1, 0) \).

By setting \( f(\rho_1, \rho_2, \mu, S, R) = g_1(\rho_1, 0, \rho_2, 0) \), the meaning of the last two lemmas can be formulated in the following theorem.

**Theorem 4.5.** For any \((\mu, S, R) \in \mathbb{R}^3\), we have

(i) \((a_1, a_2, a_3, a_4)\) solves (10), if and only if \( \rho_1 = \sqrt{a_1^2 + a_2^2} \) and \( \rho_2 = \sqrt{a_3^2 + a_4^2} \) are solutions of one of the two equivalent equations

\[
f(\rho_1, \rho_2, \mu, S, R) = 0, \quad \text{or} \quad f(\rho_2, \rho_1, \mu, S, R) = 0.
\]

(ii) Every solution to (12) defines an infinite number of solutions to (10) formulated by \((\rho_1 \cos x_0, \rho_1 \sin x_0, \rho_2 \cos y_0, \rho_2 \sin y_0, \mu, S, R)\), where \( x_0 \) and \( y_0 \) are arbitrary. They correspond to solutions of (5) which are equal up to translations along \((x, y)\).

**4.2 Expansions of \( f \) in a neighborhood of \((0, 0, 0, 0, 0)\)**

We compute simultaneously the first terms in the Taylor expansions of \( f \) and \( \mathcal{U} \), by identifying in (7)-(8) the coefficients of same power of \( \rho_1, \rho_2, \mu, S \) and \( R \).

**4.2.1 General form of \( f \) and \( \mathcal{U} \)**

Since \( f \) is smooth enough, it has

\[
\sum_{\alpha+\beta+\gamma+\delta+\nu \geq 2} f_{\alpha \beta \gamma \delta \nu}^\mu S^\delta R^\gamma \rho_1^\alpha \rho_2^\beta \] as a Taylor expansion at \((0, 0, 0, 0, 0)\), for every \((\rho_1, \rho_2, \mu, S, R)\) close to \((0, 0, 0, 0, 0)\).

Similarly, if \( w = \rho_1 w_1 + \rho_2 w_3 \), \( \mathcal{U} \) is a function of \((\rho_1, \rho_2, \mu, S, R)\), and can be expanded as

\[
\mathcal{U}(\rho_1 w_1 + \rho_2 w_3, \mu, S, R) = \sum_{\alpha+\beta+\gamma+\delta+\nu \geq 2} \mathcal{U}_{\alpha \beta \gamma \delta \nu}^\mu S^\delta R^\gamma \rho_1^\alpha \rho_2^\beta.
\]

Substituting the expansions of \( f \) and \( \mathcal{U} \) into (7)-(8), and then by equating the coefficients of \( \mu^\alpha S^\delta R^\gamma \rho_1^\beta \rho_2^\nu \), we find the coefficients \( f_{\alpha \beta \gamma}^\mu \) and \( \mathcal{U}_{\alpha \beta \gamma}^\mu \).
4.2.2 The first terms in the expansion of $f$

To compute the dominant terms in the expansion of $f$ from (7)-(8), we decompose $L_{\mu,S,R} + N_{\mu+2,S+1,R+1}$ into a sum of maps $A_{\alpha\beta\gamma}$ associated to the powers of $\mu$, $S$ and $R$:

$$L_{\mu,S,R} + N_{\mu+2,S+1,R+1} = \sum \mu^\alpha S^\beta R^\gamma A_{\alpha\beta\gamma}. $$

The following results help in the calculations.

**Theorem 4.6.** The coefficients of $\rho_1$ and $\rho_2$ in the expansion of $U$ vanish.

**Proof.** Note that $U_{000}^{0} \in R(L_c)$, which is equal to $\partial_{\rho_1} U(\rho_1 w_1 + \rho_2 w_3, \mu, S, R) = (0, 0, 0, 0, 0)$, and satisfies

$$L_c(U(\rho_1 w_1 + \rho_2 w_3, \mu, S, R)) + (I - P) L_{\mu,S,R}(\rho_1 w_1 + \rho_2 w_3 + U(\rho_1 w_1 + \rho_2 w_3, \mu, S, R)) + (I - P) N_{\mu+2,S+1,R+1}(\rho_1 w_1 + \rho_2 w_3 + U(\rho_1 w_1 + \rho_2 w_3, \mu, S, R)) = 0.
$$

Differentiating this equation with respect to $\rho_1$, and taking into account that $U(0, 0, 0, 0, 0) = 0$, we get $L_c(w_1 + U_{000}^{0}) = 0$. Since $w_1 \in ker(L_c)$, we obtain $L_c(U_{000}^{0}) = 0$, and thus $U_{000}^{0} \in ker(L_c) \cap R(L_c) = \{0\}$. An identical reasoning for $\rho_2$ concludes the proof. \qed

**Corollary 4.7.** The first terms in the expansion of $f$ associated to powers of $\rho_1$ and $\rho_2$, are those of $\rho_1^3$ or $\rho_2^3$.

**Proof.** According to Lemma 4.4, by definition $f$ is an odd function with respect to $\rho_1$ and $\rho_2$. Thus, the result of the corollary is a direct consequence of Theorem 4.6. \qed

The calculations of the first terms yield the following expansion

$$f(\rho_1, \rho_2, \mu, S, R) = -\frac{1}{108} \pi^2 \rho_1^3 (18 + 21 \mu - 6 (S + R) + 144 SR + 288 \mu SR + 5 (S^2 + R^2)) - \frac{1}{1990650} \pi^2 \rho_1^5 (-576 - 1380 \mu + 3381 \mu S) + ...$$

where "+ ..." denotes the higher order terms.

Now, we know enough information about the reduced equation, to prove our main result concerning the problem (5).

**Theorem 4.8.** For every $(\mu, S, R)$ close to $(0, 0, 0)$ such that $\mu > 0$, $f(\rho_1, \rho_2, \mu, S, R) = 0$ has exactly one small solution. All the small solutions to (5) are of the form

$$\tau_{x_0,y_0}(\rho_1 w_1 + \rho_2 w_3 + U(\rho_1 w_1 + \rho_2 w_3, \mu, S, R))$$

where $x_0$ and $y_0$ are arbitrary.
Proof. In view of Theorem 3.1 and Theorem 4.5, the implicit function theorem applied to $f(\rho_1, \rho_2, \mu, S, R) \over \rho_1$ proves the theorem. \hfill \Box

5 Conclusion

Every solution of (5) represents a spatially periodic stationary solution to (1). Therefore, in a neighborhood of the conductive state of (1), any bifurcating spatially periodic steady states defined by (5), with wave numbers equal to $s = s_c + S, r = r_c + R,$ and amplitude equal to $\rho_1^2 + \rho_2^2$, are formulated by (13).

References


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