Coupled Fixed Point Theorems for a $\phi$-Contractive Mapping in Partial Metric Spaces

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Abstract

In this work, we prove some coupled fixed point theory for $\psi$-contractive in partial metric spaces which extend and generalize some results on fixed point theorems in ordered partial metric spaces. We also provide an example in support of our main result.

1 Introduction

The first result in the existence of a fixed point for contraction type of mappings in partially ordered metric spaces has been considered recently by Ran and Reurings [23] in 2004. Following this work, Nieto and lopez [21,22] extended the results in [23] for non-decreasing mapping. Later, Agarwal et al. [4] presented some new results for contractions in partially ordered metric spaces.

A notion of coupled fixed point theorem was defined by Guo and Lakshmikantham [16]. After that, Bhaskar and Lakshmikantham [15] introduced the concept of mixed monotone property. Furthermore, they proved the existence and uniqueness of a coupled fixed point theorems for mappings which satisfy the mixed monotone property in partially ordered metric space. Since 2006, many authors have studied coupled fixed point theorems in partially ordered metric space and their applications have been established. As a continuation of this work, several coupled fixed point and coupled coincidence point results have appeared in the recent literature. Work noted in [12–14,17,20,24,26,27] are some examples of these works.
In 1994, Matthews [19] introduced the concept of partial metric spaces which is a generalized metric space in which each object does not necessarily have to have a zero distance from itself. After this, many authors studied the existence and uniqueness of a fixed point for contractive conditions on partial metric spaces. See in [1–3, 5–11, 18, 28] for some example.

Recently, Wangkeeree and et al. [28] established some coupled fixed point for generalized weakly contractive mapping having the mixed monotone property in ordered partial metric spaces.

In this work, we prove some coupled fixed point theory for $\phi$-contractive in partial metric spaces.

2 Preliminaries

In this section, we recall some definitions, lemma and examples which are useful for main results in this paper.

**Definition 2.1** [19] A partial metric on a nonempty set $X$ is a function $p : X \times X \to \mathbb{R}^+_0$ such that for all $x, y, z \in X$:

(p1) $x = y \iff p(x, x) = p(x, y) = p(y, y)$,

(p2) $p(x, x) \leq p(x, y)$,

(p3) $p(x, y) = p(y, x)$,

(p4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

**Remark**

(i) A partial metric space is a pair $(X, p)$ such that $X$ is a nonempty set and $p$ is a partial metric on $X$.

(ii) If $p(x, y) = 0$, then $x = y$.

(iii) Each partial metric $p$ on $X$ generates a $T_0$ topology $\tau_p$ on $X$ which has a base of the family of open ball $p$-balls $\{B_p(x, \epsilon), x \in X, \epsilon > 0\}$, where

$$B_p(x, \epsilon) = \{y \in X : d(x, y) < d(x, x) + \epsilon\}$$

(iv) If $p$ is a partial metric on $X$, then the function $d_p : X \times X \to \mathbb{R}^+$ given by

$$d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y)$$

is a metric on $X$.

**Definition 2.2** [19] Let $(X, p)$ be a partial metric space.
(i) A sequence \( \{x_n\} \) in a partial metric space \((X, p)\) converges to a point \(x \in X\) if and only if \(p(x, x) = \lim_{n \to \infty} p(x, x_n)\).

(ii) A sequence \( \{x_n\} \) in a partial metric space \((X, p)\) is called a Cauchy sequence if and only if \(\lim_{n,m \to \infty} p(x_n, x_m)\) exists (and is finite).

(iii) A partial metric space \((X, p)\) is said to be complete if every Cauchy sequence \(\{x_n\}\) in \(X\) converges, with respect to \(\tau_p\), to a point \(x \in X\) such that \(p(x, x) = \lim_{n,m \to \infty} p(x_n, x_m)\).

(iv) A subset \(A\) of a partial metric space \((X, p)\) is closed if whenever \(\{x_n\}\) is a sequence in \(A\) such that \(\{x_n\}\) converges to some \(x \in X\), then \(x \in A\).

Remark The limit in a partial metric space is not unique.

Lemma 2.3 [19] Let \((X, p)\) be a partial metric space.

(a) \(\{x_n\}\) is a Cauchy sequence in \((X, p)\) if and only if it is a Cauchy sequence in the metric space \((X, d_p)\).

(b) A partial metric space \((X, p)\) is complete if and only if the metric space \((X, d_p)\) is complete. Furthermore,

\[
\lim_{n \to \infty} d_p(x_n, x) = 0 \iff p(x, x) = \lim_{n \to \infty} p(x_n, x) = \lim_{n,m \to \infty} p(x_n, x_m).
\]

Let \((X, p)\) be a partial metric. We endow the product space \(X \times X\) with the partial metric \(q\) defined as follows:

\[
(x, y), (u, v) \in X \times X, \quad q((x, y), (u, v)) = p(x, u) + p(y, v).
\]

A mapping \(F : X \times X \to X\) is said to be continuous at \((x, y) \in X \times X\) if for each \(\epsilon > 0\), there exists \(\delta > 0\) such that

\[
F(B_q((x, y), \delta)) \subset B_p((x, y), \epsilon).
\]

The concept of a mixed monotone property and a coupled fixed point have been introduced by Bhaskar and Lakshmikantham in [15].

Definition 2.4 [15] Let \((X, \leq)\) be a partially ordered set and \(F : X \times X \to X\). We say \(F\) has the mixed monotone property if for any \(x, y \in X\)

\[
x_1, x_2 \in X, \quad x_1 \leq x_2 \implies F(x_1, y) \leq F(x_2, y)
\]

and

\[
y_1, y_2 \in X, \quad y_1 \leq y_2 \implies F(x, y_1) \geq F(x, y_2).
\]
Definition 2.5 [15] An element \((x, y) \in X \times X\) is called a coupled fixed point of a mapping \(F : X \times X \rightarrow X\) if \(F(x, y) = x\) and \(F(y, x) = y\).

Lakshmikantham and Ćirić in [17] introduced the concept of a mixed g-monotone mapping and a coupled coincidence point.

Definition 2.6 [17] Let \((X, \leq)\) be a partially ordered set and \(F : X \times X \rightarrow X\) and \(g : X \rightarrow X\). We say \(F\) has the mixed g-monotone property if for any \(x, y \in X\)

\[
x_1, x_2 \in X, \quad gx_1 \leq gx_2 \text{ implies } F(x_1, y) \leq F(x_2, y)
\]

and

\[
y_1, y_2 \in X, \quad gy_1 \leq gy_2 \text{ implies } F(x, y_1) \geq F(x, y_2).
\]

Definition 2.7 [17] An element \((x, y) \in X \times X\) is called a coupled coincidence point of a mapping \(F : X \times X \rightarrow X\) and \(g : X \rightarrow X\) if \(F(x, y) = gx\) and \(F(y, x) = gy\).

Definition 2.8 [17] Let \(X\) be a non-empty set and \(F : X \times X \rightarrow X\) and \(g : X \rightarrow X\). We say \(F\) and \(g\) are commutative if \(gF(x, y) = F(gx, gy)\) for all \(x, y \in X\).

Let \(\Phi\) denote the set of functions \(\phi : [0, \infty) \rightarrow [0, \infty)\) satisfying

1. \(\phi\) is continuous and non-decreasing,
2. \(\phi(t) = 0\) if and only if \(t = 0\),
3. \(\phi(t + s) \leq \phi(t) + \phi(s)\),
4. \(\phi(\alpha t) \leq \alpha \phi(t)\) for all \(\alpha \in (0, \infty)\).

and let \(\Psi\) denote the set of functions \(\psi : [0, \infty) \rightarrow [0, \infty)\) satisfying

\[
\lim_{t \to r} \psi(t) > 0 \quad \text{for all } r > 0 \quad \text{and} \quad \lim_{t \to 0^+} \psi(t) = 0.
\]

Wangkeeree and et al. [28] prove coupled fixed point theorem which generalization of the result of Alsulami et al. [6] as follow.

Theorem 2.9 [28] Let \((X, \leq)\) be a partially ordered set and let there exist \(p\) be a metric on \(X\) such that \((X, p)\) is a complete partial metric space. Let \(F : X \times X \rightarrow X\) be mapping having the mixed monotone property on \(X\). Suppose that there exists \(\phi \in \Phi\) and \(\psi \in \Psi\), such that the following holds

\[
M_{F, \psi}^{\phi, \psi}(x, y, u, v) \leq \phi (p(x, u) + p(y, v)) - 2\psi \left( \frac{p(x, u) + p(y, v)}{2} \right)
\] (1)
for all $x, y, u, v \in X$ with $x \geq u$ and $y \leq v$, where
\[ M_F^\phi(x, y, u, v) = \phi(p(F(x, y), F(u, v))) + \phi(p(F(y, x), F(v, u))). \]

Suppose also either

(a) $F$ is continuous or

(b) $X$ has the following property:

(i) if a non-decreasing sequence $\{x_n\}$ is such that $x_n \to x$, then $x_n \leq x$
for all $n$.

(ii) if a non-increasing sequence $\{y_n\}$ is such that $y_n \to y$, then $y_n \geq y$
for all $n$.

If there exist $x_0, y_0 \in X$ such that
\[ x_0 \leq F(x_0, y_0) \text{ and } y_0 \geq F(y_0, x_0), \]
then there exist $(x, y) \in X \times X$ such that $x = F(x, y)$ and $y = F(y, x)$, that is $F$ has a coupled fixed point.

We shown our first result which generalizes Theorem (2.9).

3 Main results

Let $\Phi$ denote the set of functions $\phi : [0, \infty) \to [0, \infty)$ satisfying

1. $\phi(t) < t$ for all $t \in (0, \infty)$,

2. $\lim_{r \to t^{-}} \phi(r) < t$, for all $t \in (0, \infty)$.

**Theorem 3.1** Let $(X, \leq)$ be a partially ordered set and let there exist $p$
be a metric on $X$ such that $(X, p)$ is a complete partial metric space. Let $F : X \times X \to X$ be mapping having the mixed monotone property on $X$. Suppose that there exists $\phi \in \Phi$, such that the following holds
\[ M_F^\phi(x, y, u, v) \leq \phi(p(x, u) + p(y, v)) \] (2)
for all $x, y, u, v \in X$ with $x \geq u$ and $y \leq v$, where
\[ M_F^\phi(x, y, u, v) = p(F(x, y), F(u, v)) + p(F(y, x), F(v, u)). \]

Suppose also either
(a) $F$ is continuous or

(b) $X$ has the following property:

(i) if a non-decreasing sequence $\{x_n\}$ is such that $x_n \rightarrow x$, then $x_n \leq x$
for all $n$.

(ii) if a non-increasing sequence $\{y_n\}$ is such that $y_n \rightarrow y$, then $y_n \geq y$
for all $n$.

If there exist $x_0, y_0 \in X$ such that

$$x_0 \leq F(x_0, y_0) \text{ and } y_0 \geq F(y_0, x_0),$$

then there exist $(x, y) \in X \times X$ such that $x = F(x, y)$ and $y = F(y, x)$, that is
$F$ has a coupled fixed point.

**Proof** Let $x_0, y_0 \in X$. We can choose $x_1, y_1 \in X$ such that

$$x_1 = F(x_0, y_0) \text{ and } y_1 = F(y_0, x_0).$$

Again we can choose $x_2, y_2 \in X$ such that

$$x_2 = F(x_1, y_1) \text{ and } y_2 = F(y_1, x_1).$$

Continuing this process we can construct sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that

$$x_{n+1} = F(x_n, y_n) \text{ and } y_{n+1} = F(y_n, x_n) \quad \text{for all } n \geq 0. \hspace{1cm} (3)$$

Next, we show that

$$x_n \leq x_{n+1} \quad \text{and} \quad y_n \geq y_{n+1} \quad \text{for all } n \geq 0. \hspace{1cm} (4)$$

For $n = 0$. Since $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$ and (3), we have

$$x_0 \leq F(x_0, y_0) = x_1 \text{ and } y_0 \geq F(y_0, x_0) = y_1.$$

Thus (4) holds for $n = 0$. Now suppose that (4) hold for some fixed $n \geq 0$. Then

$$x_n \leq x_{n+1} \quad \text{and} \quad y_n \geq y_{n+1} \quad \text{for all } n \geq 0.$$

Since $F$ has mixed monotone property, we have

$$x_{n+2} = F(x_{n+1}, y_{n+1}) \geq F(x_n, y_n) = x_{n+1} \quad \text{and} \quad y_{n+2} = F(y_{n+1}, x_{n+1}) \leq F(y_n, x_n) = y_{n+1}.$$

By mathematical induction, we conclude that (4) holds for all $n \geq 0.$
If there exists \( k \in \mathbb{N} \) such that \((x_{k+1}, y_{k+1}) = (x_k, y_k)\) then \( x_k = x_{k+1} = F(x_k, y_k) \) and \( y_k = y_{k+1} = F(y_k, x_k)\). Thus, \((x_k, y_k)\) is a coupled fixed point of \( F \). This finishes the proof. Now we assume that \((x_{k+1}, y_{k+1}) \neq (x_k, y_k)\) for all \( n \geq 0 \). Thus, we have either \( x_{n+1} = F(x_n, y_n) \neq x_n \) or \( y_{n+1} = F(y_n, x_n) \neq y_n \) for all \( n \geq 0 \).

Consider now the sequence of nonnegative real number \( \{\delta_n\}_{n=1}^{\infty} \) given by
\[
\delta_n = p(x_{n+1}, x_n) + p(y_{n+1}, y_n) \quad \text{for all } n \geq 1
\]. Since \( x_n \geq x_{n-1} \) and \( y_n \leq y_{n-1} \), using (2) and (3), we have
\[
p(x_{n+1}, x_n) + p(y_{n+1}, y_n) = p(F(x_n, y_n), F(x_{n-1}, y_{n-1})) + p(F(y_n, x_n), F(y_{n-1}, x_{n-1}))
\]
\[
= M_F(x_n, x_{n-1}, y_n, y_{n-1})
\]
\[
\leq \phi(p(x_n, x_{n-1}) + p(y_n, y_{n-1})).
\]

Therefore, the sequence \( \{\delta_n\}_{n=1}^{\infty} \) satisfies
\[
\delta_n \leq \phi(\delta_{n-1}) \quad \text{for all } n \geq 1.
\]
(5)

From (5) and property of \( \phi \) it follows that the sequence \( \{\delta_n\}_{n=1}^{\infty} \) is non-increasing. Therefore, there exists some \( \delta \geq 0 \) such that
\[
\lim_{n \to \infty} \delta_n = \lim_{n \to \infty} [p(x_{n+1}, x_n) + p(y_{n+1}, y_n)] = \delta.
\]

We shall prove that \( \delta = 0 \). Assume, to the contrary, that \( \delta > 0 \). Then by letting \( n \to \infty \) in (5), from \( 0 = \phi(0) < \phi(t) < \phi(t) < t \) and \( \lim_{r \to t}, \phi(r) < t \) for each \( t > 0 \), we have
\[
\delta = \lim_{n \to \infty} \delta_n \leq \lim_{n \to \infty} \phi(\delta_{n-1}) = \lim_{\delta_{n-1} \to \delta^+} \phi(\delta_{n-1}) < \delta.
\]
which is a contradiction. Thus \( \delta = 0 \) and hence
\[
\lim_{n \to \infty} \delta_n = \lim_{n \to \infty} [p(x_{n+1}, x_n) + p(y_{n+1}, y_n)] = 0.
\]
(6)

Let
\[
\delta_n^p = d_p(x_{n+1}, x_n) + d_p(y_{n+1}, y_n) \quad \text{for all } n \geq 1.
\]
It is easy to see that \( \delta_n^p \leq 2\delta_n \) for all \( n \geq 1 \). Take \( \lim \) as \( n \to \infty \) and use (6), we get
\[
\lim_{n \to \infty} \delta_n^p = \lim_{n \to \infty} [d_p(x_{n+1}, x_n) + d_p(y_{n+1}, y_n)] = 0.
\]
(7)

We now prove that \( \{x_n\} \) and \( \{y_n\} \) are Cauchy sequence in \((X, d)\). Suppose, to the contrary, that is at least of \( \{x_n\} \) and \( \{y_n\} \) is not Cauchy sequence. Then
there exists an \(\varepsilon > 0\) for which we can find subsequences \(\{x_{m(k)}\}\) and \(\{x_{n(k)}\}\) of \(\{x_n\}\), \(\{y_{m(k)}\}\) and \(\{y_{n(k)}\}\) of \(\{y_n\}\) with \(n(k) > m(k) \geq K\) such that

\[
d_p(x_{n(k)}, x_{m(k)}) + d_p(y_{n(k)}, y_{m(k)}) \geq \varepsilon, \quad k \in \mathbb{N}.
\] (8)

Further, corresponding to \(m(k)\), we can choose \(n(k)\) in such a way that is the smallest integer with \(n(k) > m(k) \geq K\) and satisfying (8). Then

\[
d_p(x_{n(k) - 1}, x_{m(k)}) + d_p(y_{n(k) - 1}, y_{m(k)}) < \varepsilon.
\] (9)

Using (8) and (9) and the triangle inequality, we have

\[
\varepsilon \leq r_k^p := d_p(x_{n(k)}, x_{m(k)}) + d_p(y_{n(k)}, y_{m(k)}) \\
\leq d_p(x_{n(k)}, x_{n(k) - 1}) + d_p(x_{n(k) - 1}, x_{m(k)}) + d_p(y_{n(k)}, y_{n(k) - 1}) + d_p(y_{n(k) - 1}, y_{m(k)}) \\
\leq d_p(x_{n(k)}, x_{n(k) - 1}) + d_p(y_{n(k)}, y_{n(k) - 1}) + \varepsilon.
\] (10)

Letting \(k \to \infty\) in the above inequality and using (6), we get

\[
\varepsilon \leq \lim_{k \to \infty} r_k^p \leq \lim_{k \to \infty} \left[ d_p(x_{n(k)}, x_{n(k) - 1}) + d_p(y_{n(k)}, y_{n(k) - 1}) + \varepsilon \right] = \varepsilon,
\]

that is,

\[
\lim_{k \to \infty} r_k^p = \lim_{k \to \infty} \left[ d_p(x_{n(k)}, x_{m(k)}) + d_p(y_{n(k)}, y_{m(k)}) \right] = \varepsilon.
\] (11)

Using triangle inequality, we have

\[
r_k^p = d_p(x_{n(k)}, x_{m(k)}) + d_p(y_{n(k)}, y_{m(k)}) \\
\leq d_p(x_{n(k)}, x_{n(k) + 1}) + d_p(x_{n(k) + 1}, x_{m(k) + 1}) + d_p(x_{m(k) + 1}, x_{m(k)}) \\
+ d_p(y_{n(k)}, y_{n(k) + 1}) + d_p(y_{n(k) + 1}, y_{m(k) + 1}) + d_p(y_{m(k) + 1}, y_{m(k)}) \\
= \delta^p_{n(k)} + \delta^p_{m(k)} + d_p(x_{n(k) + 1}, x_{m(k) + 1}) + d_p(y_{n(k) + 1}, y_{m(k) + 1}).
\] (12)

Now, let

\[
r_k = p(x_{n(k)}, x_{m(k)}) + p(y_{n(k)}, y_{m(k)}).
\]

We have

\[
r_k^p = d_p(x_{n(k)}, x_{m(k)}) + d_p(y_{n(k)}, y_{m(k)}) \\
= 2p(x_{n(k)}, x_{m(k)}) - p(x_{n(k)}, x_{n(k)}) - p(x_{m(k)}, x_{m(k)}) \\
+ 2p(y_{n(k)}, y_{m(k)}) - p(y_{n(k)}, y_{n(k)}) - p(y_{m(k)}, y_{m(k)}) \\
= 2r_k - p(x_{n(k)}, x_{n(k)}) - p(x_{m(k)}, x_{m(k)}) - p(y_{n(k)}, y_{n(k)}) - p(y_{m(k)}, y_{m(k)}).
\] (13)
By Definition 2.1 and (6), we have

\[
\lim_{k \to \infty} p(x_{n(k)}, x_m(k)) = \lim_{k \to \infty} p(x_{m(k)}, x_m(k))
\]
\[
= \lim_{k \to \infty} p(y_{n(k)}, y_{m(k)})
\]
\[
= \lim_{k \to \infty} p(y_{m(k)}, y_m(k))
\]
\[
= 0.
\]

Taking \(k \to \infty\) in (13) and using (11), we have

\[
\lim_{k \to \infty} r_k = \frac{\varepsilon}{2}
\]

Since \(n(k) > m(k)\), by (4), we get \(x_{n(k)} \geq x_m(k)\) and \(y_{n(k)} \leq y_m(k)\). Hence we can use (2) one obtains

\[
r_k^p = d_p(x_{n(k)+1}, x_{m(k)+1}) + d_p(y_{n(k)+1}, y_{m(k)+1})
\]
\[
\leq 2p(x_{n(k)+1}, x_{m(k)+1}) + 2p(y_{n(k)+1}, y_{m(k)+1})
\]
\[
= 2p(F(x_{n(k)}, y_{n(k)}), F(x_{m(k)}, y_{m(k)})) + 2p(F(y_{n(k)}, x_{m(k)}), F(y_{m(k)}, x_{m(k)}))
\]
\[
= 2M_{F}^p(x_{n(k)}, y_{n(k)}, x_{m(k)}, y_{m(k)}
\]
\[
\leq 2\phi(p(x_{n(k)}, x_{m(k)})) + 2p(y_{n(k)}, y_{m(k)})
\]
\[
= 2\phi(r_k).
\]  \hspace{1cm} (14)

By (12), we have

\[
r_k^p \leq \delta_{n(k)}^p + \delta_{m(k)}^p + 2\phi(r_k).
\]  \hspace{1cm} (15)

Letting \(k \to \infty\) in (15), (7) and using \(\lim_{r\to t^+} \varphi(r) < t\) for each \(t > 0\), we have

\[
\varepsilon = \lim_{k \to \infty} r_k^p \leq 2 \lim_{k \to \infty} \phi(r_k) = 2 \lim_{r_k \to \infty^+} \phi(r_k) < 2\frac{\varepsilon}{2} = \varepsilon,
\]

which is a contradiction. This show that \(\{x_n\}\) and \(\{y_n\}\) are Cauchy sequence. Since \((x, p)\) is complete, by lemma 2.3 we have \((X, d_p)\) is a complete matric space and there exist \(x, y \in X\) such that

\[
\lim_{n \to \infty} d_p(x_n, x) = \lim_{n \to \infty} d_p(y_n, y) = 0.
\]  \hspace{1cm} (16)

which implies that

\[
\lim_{n \to \infty} F(x_n, y_n) = \lim_{n \to \infty} x_n = x
\]
\[
\lim_{n \to \infty} F(y_n, x_n) = \lim_{n \to \infty} y_n = y.
\]  \hspace{1cm} (17)
From Lemma 2.3, (6) and Definition 2.1, we have
\[ p(x, x) = \lim_{n \to \infty} p(x_n, x) = \lim_{n \to \infty} p(x_n, x_n) = 0 \]
\[ p(y, y) = \lim_{n \to \infty} p(y_n, y) = \lim_{n \to \infty} p(y_n, y_n) = 0. \]  

(18)

Using (18) and (2), we have
\[ p(F(x, y), F(x, y)) + p(F(y, x), F(y, x)) \leq \phi(p(x, x), p(y, y)) = \phi(0) = 0. \]  

(19)

Therefore, we get
\[ p(F(x, y), F(x, y)) = p(F(y, x), F(y, x)) = 0. \]  

(20)

We now prove that \( x = F(x, y) \) and \( y = F(y, x) \). Suppose that the assumption (a) holds. For any \( \varepsilon > 0 \), \( F \) is continue at a point \((x, y)\) implies that there exists \( \xi > 0 \) such that if \((u, v) \in X \times X \) with \( q((x, y), (x, y)) + \xi = \xi \), meaning that
\[ p(x, u) + p(y, v) < p(x, x) + p(y, y) + \xi = \xi, \]
by (18), we have
\[ p(F(x, y), F(u, v)) < p(F(x, y), F(x, y)) + \frac{\varepsilon}{2}. \]  

(21)

Since (18) and let \( \xi = \min\{\frac{\xi}{2}, \frac{\varepsilon}{2}\} > 0 \), there exists \( n_0 \in N \) such that for any \( n \geq n_0 \),
\[ p(x_n, x) + p(y_n, y) < 2\xi < \xi. \]

From (21), we have
\[ p(F(x, y), F(x, y)) < p(F(x, y), F(x, y)) + \frac{\varepsilon}{2}. \]  

(22)

For any \( n \geq n_0 \), we have
\[ p(F(x, y), x) \leq p(F(x, y), x_{n+1}) + p(x_{n+1}, x) \]
\[ = p(F(x, y), F(x_n, y_n)) + p(x_{n+1}, x) \]
\[ \leq p(F(x, y), F(x, y)) + \frac{\varepsilon}{2} + \xi \]
\[ \leq p(F(x, y), F(x, y)) + \varepsilon. \]  

(23)

Using (20) in above inequality, we have \( p(F(x, y), x) < \varepsilon. \) Since \( \varepsilon \) is arbitrary, we get
\[ p(F(x, y), x) = 0. \]  

(24)
Similarly, we also have \( p(F(y, x), y) = 0 \). From (20) and Definition 2.1, we have \( F(x, y) = x \) and \( F(y, x) = y \).

Next, suppose assumption (b) hold. From (4), (16) and (18), we have \( \{x_n\} \) is a non-decreasing sequence, \( x_n \to x \) and \( \{y_n\} \) is a non-increasing sequence, \( y_n \to y \) as \( n \to \infty \). By assumption (b), we have for all \( n \geq 0 \)

\[
x_n \leq x \quad \text{and} \quad y \leq y_n.
\]

We get

\[
p(F(x, y), x) \leq p(F(x, y), x_{n+1}) + p(x_{n+1}, x) = p(F(x, y), F(x, y_n)) + p(x_{n+1}, x)
\]

and

\[
p(F(y, x), y) \leq p(F(y, x), y_{n+1}) + p(y_{n+1}, y) = p(F(y, x), F(y, x_n)) + p(y_{n+1}, y).
\]

By (26) and (27), we have

\[
p(F(x, y), x) + p(F(y, x), y) \leq p(F(x, y), F(x, y_n)) + p(x_{n+1}, x) + p(F(y, x), F(y, x_n)) + p(y_{n+1}, y)
\]

\[
\leq \phi (p(x, x_n) + p(y, y_n)) + p(x_{n+1}, x) + p(y_{n+1}, y).
\]

Taking limit as \( n \to \infty \) in (28) and using (18) and property of \( \psi \) we get

\[
p(F(x, y), x) = 0 = p(F(y, x), y).
\]

Using (19), we have \( x = F(x, y) \) and \( y = F(y, x) \).

**Example 3.2** Let \( X = \mathbb{R}^+ \) and \( p(x, y) = \max\{x, y\} \), then \((X, p)\) is a partial metric space. Define the mapping \( F : X \times X \to X \) by \( F(x, y) = \frac{x - 2y}{4} \) for all \( x, y \in X \). Clearly, \( F \) has mixed monotone property. Consider for all \( x \geq u \) and \( y \leq v \), we have \( \phi (p(x, u) + p(y, v)) = \phi (x + v) \). Consider, we have

\[
x + v \geq -2u - 2y \iff 2x + 2v \geq x - 2u + v - 2y \iff \frac{x + v}{2} \geq \frac{x - 2u}{4} + \frac{v - 2y}{4}.
\]

Therefore, we get

\[
p(F(x, y), F(u, v)) + p(F(y, x), F(v, u)) = \max\left\{ \frac{x - 2y}{4}, \frac{u - 2v}{4} \right\} + \max\left\{ \frac{y - 2x}{4}, \frac{v - 2u}{4} \right\}
\]

\[
= \frac{x - 2y}{4} + \frac{v - 2u}{4} = \frac{x - 2u}{4} + \frac{v - 2y}{4}
\]

\[
\leq \frac{x + v}{2}
\]

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