On the Classical Solutions of the Extended Magnetostatic Born-Infeld System

Junichi Aramaki

Division of Science, Faculty of Science and Engineering,
Tokyo Denki University
Hatoyama-machi, Saitama 350-0394, Japan

Abstract

We consider the Born-Infeld system in a bounded domain. In order to get nontrivial solution, we extend the original Born-Infeld system. For extended Born-Infeld system under the Dirichlet condition or the Neumann condition, we shall prove the existence of weak solution and its regularity.

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1 Introduction

The Born-Infeld theory is an extension of the Maxwell theory. In order to overcome the infinity problem associated with a point charge source in the original Maxwell theory, the introduction of the Born-Infeld electromagnetic field theory is well recognized. See Born [2, 3], Born-Infeld [4, 5] and Yang [16]. In the Maxwell theory the action function is given by

\[ \mathcal{L} = \frac{1}{2}(|E|^2 - |B|^2) \]
where \( \mathbf{E} \) and \( \mathbf{B} \) are electric and magnetic field, respectively. In the Born-Infeld theory, the action density is replaced by

\[
\mathcal{L}_B = b^2 \left( 1 - \sqrt{1 - \frac{1}{b^2}(|\mathbf{E}|^2 - |\mathbf{B}|^2)} \right)
\]

where \( b > 0 \) is a scale parameter. See [2, 3]. Moreover taking the invariance principle into consideration, The author of [2, 3] introduced the other action density

\[
\mathcal{L}_{BI} = b^2 \left( 1 - \sqrt{1 - \frac{1}{b^2}(|\mathbf{E}|^2 - |\mathbf{B}|^2) + \frac{1}{b^4}(\mathbf{E} \cdot \mathbf{B})^2} \right).
\]

In the magnetostatic case where \( \mathbf{B}(x, t) = \text{curl} \mathbf{A}(x) \) and \( \mathbf{E}(x, t) = 0 \), we see that

\[
\mathcal{L} = \mathcal{L}_B = \mathcal{L}_{BI} = -S(|\text{curl} \mathbf{A}|^2)
\]

where

\[
S(t) = b^2 \left( \sqrt{1 + \frac{1}{b^2}t} - 1 \right), \quad t \geq 0.
\] (1.1)

We consider the functional

\[
S_{BI}[\mathbf{A}] = \int_{\mathbb{R}^3} -\mathcal{L} \, dx = \int_{\mathbb{R}^3} S(|\text{curl} \mathbf{A}|^2) \, dx.
\]

The Euler-Lagrange equation of this functional is

\[
\text{curl} \left( S'(|\text{curl} \mathbf{A}|^2) \text{curl} \mathbf{A} \right) = 0 \quad \text{in} \quad \mathbb{R}^3. \quad (1.2)
\]

The author of [16] showed that the solution \( \mathbf{A} \) of (1.2) with finite energy satisfies \( \text{curl} \mathbf{A} = 0 \). If the solution \( \mathbf{A} \) satisfies \( \text{curl} \mathbf{A} \equiv 0 \), we say that the solution \( \mathbf{A} \) is trivial and if not so, we say that the solution is nontrivial. The fact that the solution of (1.2) is trivial means that in the vacuum space \( \mathbb{R}^3 \) the magnetic monopole without an external effect does not exist. For reason of this triviality, we change the equation (1.2) into a new equation with a lower order term in a bounded domain \( \Omega \) in \( \mathbb{R}^3 \), and with some boundary condition. Therefore our energy functional is of the form:

\[
\int_{\Omega} (S(|\text{curl} \mathbf{A}|^2) + F(x, \mathbf{A})) \, dx.
\] (1.3)

The Euler-Lagrange equation of this functional is

\[
\text{curl} \left( S'(|\text{curl} \mathbf{A}|^2) \text{curl} \mathbf{A} \right) + \frac{1}{2} \nabla_x F(x, \mathbf{A}) = 0 \quad \text{in} \quad \Omega. \quad (1.4)
\]
The boundary condition is to prescribe the tangential component of $A$:

$$A_T = A_T^0 \text{ on } \partial \Omega$$

or a natural condition

$$S'(|\text{curl } A|^2)(\text{curl } A)_T = D_T^0 \text{ on } \partial \Omega.$$ (1.6)

The problem of such setting was considered by Chen and Pan [6]. They proved that if $F = 0$, the solutions $A$ of (1.4) with the boundary condition (1.6) are trivial, and if $F = 0$ and $\Omega$ is simply connected, without holes and if $\nu \cdot \text{curl } A_T^0 = 0$ on $\partial \Omega$ where $\nu$ is the outer normal unit vector to $\partial \Omega$, then the solutions $A$ of (1.4) with the boundary condition (1.5) are trivial. Thus we modify (1.2) to get nontrivial solutions. In order to do so, we consider the extended Born-Infeld model in the magnetostatic case by adding a lower order term $F(x, A)$ as in (1.3).

In this paper we consider the case where the lower order term $F(x, A)$ is of the form

$$F(x, A) = \langle M(x)A, A \rangle + 2b \cdot A + c(x)$$

where $M(x) = (M_{ij}(x))$ is a given positively definite symmetric $3 \times 3$ matrix, $b(x)$ is a given vector field and $c(x)$ is a given function. The authors of [6] considered the case where $F(x, A) = a(x)|A|^2$ where $a(x)$ is a positive scalar function, and got some interesting results. Our purpose is to extend their results to the case where a lower order term is of the form (1.7). When $F(x, A) = a(x)|A|^2$, the authors of [6] observed that if the boundary data is small, there exists a classical solution of (1.4) with (1.5) or (1.6). However, in the case (1.7), it will be seen that we can not ignore the effect of the vector field $b$.

Throughout this paper, we impose that the following assumptions hold.

(A1) $\Omega$ is simply connected bounded domain in $\mathbb{R}^3$ without holes and with a $C^4$ boundary $\partial \Omega$.

(A2) $M(x) = (M_{ij}(x))_{i,j=1,2,3}$ is a symmetric and positively definite matrix with $M_{ij} \in C^{1,1}(\bar{\Omega})$ and, that is, $M_{ij}(x) = M_{ji}(x)$ and there exists a constant $m_0 > 0$ such that

$$\sum_{i,j=1}^3 M_{ij}(x)\xi_i \xi_j \geq m_0 |\xi|^2 \text{ for all } \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \text{ and } x \in \bar{\Omega},$$

$$b(x) = (b_1(x), b_2(x), b_3(x)) \text{ with } b_i \in C^{1,1}(\bar{\Omega}) \text{ and } c(x) \in C^0(\bar{\Omega}).$$

(A3) $A_T^0 \in C^{2,\alpha}(\partial \Omega, \mathbb{R}^3)$ where $A_T$ is a tangent vector field to $\partial \Omega$, and $\nu \cdot \text{curl } A_T^0 = 0$ on $\partial \Omega$ where $\nu$ is the outer unit vector field to $\partial \Omega$.

(A4) $D_T \in C^{2,\alpha}(\partial \Omega, \mathbb{R}^3)$ where $D_T$ is a tangent vector field to $\partial \Omega$.

(A2') (A2) holds with $M_{ij} \in C^{2,\alpha}(\bar{\Omega})$ and $b_i \in C^{2,\alpha}(\bar{\Omega}).$
In the following for any vector field $A$, we denote the tangential component of $A$ by $A_T$, that is, $A = A - (\nu \cdot A)\nu$.

We call the following system the Dirichlet problem.

\[
\begin{align*}
\text{curl } (S'(|\text{curl } A|^2)\text{curl } A) + M(x)A + b &= 0 \quad \text{in } \Omega, \\
A_T &= A_T^0 \quad \text{on } \partial \Omega.
\end{align*}
\tag{1.8}
\]

Then we get the following result.

**Theorem 1.1.** Assume that (A1), (A2') and (A3) with $0 < \alpha < 1$ hold. Then there exists $R_1 > 0$ such that if

\[
\|A_T^0\|_{C^{2,\alpha}(\partial \Omega)} + \|b\|_{C^{1,\alpha}(\overline{\Omega})} \leq R_1,
\]

then the system (1.8) has a classical solution $A \in C^{2,\alpha}(\overline{\Omega}, \mathbb{R}^3)$.

**Remark 1.2.** If $A_T^0 \neq 0$, the above solution is nontrivial.

Next, we call the following system the Neumann problem.

\[
\begin{align*}
\text{curl } (S'(|\text{curl } A|^2)\text{curl } A) + M(x)A + b &= 0 \quad \text{in } \Omega, \\
S'(|\text{curl } A|^2)(\text{curl } A)_T &= D_T^0 \quad \text{on } \partial \Omega.
\end{align*}
\tag{1.9}
\]

Then we get the following result.

**Theorem 1.3.** Assume that (A1), (A2) and (A4) with $0 < \alpha < 1$ hold. Then there exists $R_2 > 0$ such that if

\[
\|D_T^0\|_{C^{2,\alpha}(\partial \Omega)} + \|b\|_{C^{1,\alpha}(\overline{\Omega})} \leq R_2,
\]

then the system (1.9) has a solution $A$, and $A$ can be written in the form:

\[A = v + \nabla \phi\]

where $v \in C^{2,\alpha}(\overline{\Omega}, \mathbb{R}^3)$ satisfies $\nu \cdot v = 0$ on $\partial \Omega$ and $\text{div } v = 0$ in $\Omega$, and $\phi \in C^{2,\alpha}(\overline{\Omega})$. In addition to the above hypotheses, if $M_{ij} \in C^{2,\alpha}(\overline{\Omega})$ and $\nu \cdot \text{curl } D_T \in C^{2,\alpha}(\partial \Omega)$, then we get $v \in C^{3,\alpha}(\overline{\Omega}, \mathbb{R}^3)$ and $\phi \in C^{3,\alpha}(\overline{\Omega})$.

This paper consists of the following sections. In section 2, we give some preliminaries. We attempt to modify the given function $S(t)$ in (1.1) for $t > K$ with $0 < K < b^2$ so that the modified function $S_K$ has quadratic growth of $|\text{curl } A|$, and we see some properties of $S_K$ as in [6]. In section 3, we consider the Dirichlet problem (1.8) and give the proof of Theorem 1.1. Section 4 is devoted to the Neumann problem (1.9) and to the proof of Theorem 1.3.


2 Preliminaries

First, we consider the Dirichlet problem (1.8). Since the corresponding functional

\[ S^+[A] = \int_\Omega (S(|\text{curl } A|^2) + \langle M(x)A, A \rangle + 2b \cdot A + c(x))dx \]  

(2.1)

does not have weak compactness in the admissible space which we treat, we modify \( S(t) \) for \( t > K \) with \( 0 < K < b^2 \) in the functional (2.1) to get a strictly increasing function \( S_K(t) \) which has quadratic growth in \( |\text{curl } A| \) at infinity. Then we consider the modified functional

\[ S_K^+[A] = \int_\Omega (S_K(|\text{curl } A|^2) + \langle M(x)A, A \rangle + 2b \cdot A + c(x))dx. \]  

(2.2)

If we take a minimizer \( A_K \) of \( S_K^+ \) in some admissible space, and if \( \|\text{curl } A_K\|_{L^\infty(\Omega)} \leq \sqrt{K} \), then we will be able to see that \( A_K \) is a critical point of the original functional (2.1).

Next, for the Neumann problem (1.9) we consider the modified functional

\[ H_K^+[A] = \int_\Omega (S_K(|\text{curl } A|^2) + \langle M(x)A, A \rangle + 2b \cdot A + c(x))dx + 2\int_{\partial\Omega} (D_T \times A_T) \cdot \nu dS \]  

(2.3)

where \( dS \) denotes the surface element. The Euler-Lagrange equations of (2.2) and (2.3) are following, respectively:

\begin{align*}
\{ \text{curl } (S_K'(|\text{curl } A|^2)\text{curl } A) + M(x)A + b = 0 \quad \text{in } \Omega, \\
A_T = A_T^0 \quad \text{on } \partial\Omega. \}
\end{align*}

(2.4)

and

\begin{align*}
\{ \text{curl } (S_K'(|\text{curl } A|^2)\text{curl } A) + M(x)A + b = 0 \quad \text{in } \Omega, \\
S'(|\text{curl } A|^2)(\text{curl } A)_T = D_T^0 \quad \text{on } \partial\Omega. \}
\end{align*}

(2.5)

Now we introduce some function spaces. First we define subspaces of \( L^2(\Omega, \mathbb{R}^3) \)

\[ \mathbb{H}_1(\Omega) = \{ v \in L^2(\Omega, \mathbb{R}^3); \text{curl } v = 0, \text{div } v = 0 \text{ in } \Omega, \nu \cdot v = 0 \text{ on } \partial\Omega \}, \]

\[ \mathbb{H}_2(\Omega) = \{ v \in L^2(\Omega, \mathbb{R}^3); \text{curl } v = 0, \text{div } v = 0 \text{ in } \Omega, \nu \times v = 0 \text{ on } \partial\Omega \}. \]

We say that \( \Omega \) is simply connected if \( \dim \mathbb{H}_1(\Omega) = 0 \), and \( \Omega \) has no holes if \( \dim \mathbb{H}_2(\Omega) = 0 \). Next we define

\[ \mathcal{H}^2(\Omega, \text{curl}) = \{ u \in L^2(\Omega, \mathbb{R}^3); \text{curl } u \in L^2(\Omega, \mathbb{R}^3) \}, \]

\[ \mathcal{H}^2(\Omega, \text{div}) = \{ u \in L^2(\Omega, \mathbb{R}^3); \text{div } u \in L^2(\Omega) \}, \]

\[ \mathcal{H}^2(\Omega, \text{curl}, \text{div}) = \mathcal{H}(\Omega, \text{curl}) \cap \mathcal{H}(\Omega, \text{div}). \]
It is easy to show that these spaces are Banach spaces with respect to the norm
\[
\|u\|_{\mathcal{H}^2(\Omega,\text{curl})} = \|u\|_{L^2(\Omega)} + \|\text{curl } u\|_{L^2(\Omega)},
\]
\[
\|u\|_{\mathcal{H}^2(\Omega,\text{div})} = \|u\|_{L^2(\Omega)} + \|\text{div } u\|_{L^2(\Omega)},
\]
\[
\|u\|_{\mathcal{H}^2(\Omega,\text{curl,div})} = \|u\|_{L^2(\Omega)} + \|\text{curl } u\|_{L^2(\Omega)} + \|\text{div } u\|_{L^2(\Omega)},
\]
respectively.

The next lemma was shown by Dautray and Lions [8, p. 204].

**Lemma 2.1.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^3 \) with a \( C^2 \) boundary. Then the following holds.

(i) The normal trace map \( u \mapsto \nu \cdot u \) is a continuous map from \( \mathcal{H}^2(\Omega,\text{div}) \) to \( H^{-1/2}(\Omega) \).

(ii) The tangential trace map \( u \mapsto u_T = u - (u \cdot \nu)\nu = \nu \times (u \times \nu) \) is a continuous map from \( \mathcal{H}^2(\Omega,\text{curl}) \) to \( H^{-1/2}(\Omega,\mathbb{R}^3) \).

Next lemma is followed from [8, Proposition 6] and Pan [15].

**Lemma 2.2.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^3 \) with \( C^2 \) boundary. Then the following holds.

(i) If \( \Omega \) has no holes, \( u \in \mathcal{H}^2(\Omega,\text{curl},\text{div}) \) and \( \nu \times u \in H^{1/2}(\partial \Omega) \), then \( u \in H^1(\Omega,\mathbb{R}^3) \) and there exists a constant \( C(\Omega) > 0 \) such that
\[
\|u\|_{H^1(\Omega)} \leq C(\Omega)(\|\text{curl } u\|_{L^2(\Omega)} + \|\text{div } u\|_{L^2(\Omega)} + \|\nu \times u\|_{H^{1/2}(\partial \Omega)}).
\]

(ii) If \( \Omega \) has simply connected, \( u \in \mathcal{H}^2(\Omega,\text{curl},\text{div}) \) and \( \nu \cdot u \in H^{1/2}(\partial \Omega) \), then \( u \in H^1(\Omega,\mathbb{R}^3) \) and there exists a constant \( C(\Omega) > 0 \) such that
\[
\|u\|_{H^1(\Omega)} \leq C(\Omega)(\|\text{curl } u\|_{L^2(\Omega)} + \|\text{div } u\|_{L^2(\Omega)} + \|\nu \cdot u\|_{H^{1/2}(\partial \Omega)}).
\]

Here we introduce the notion of weak solution of the modified system.

For the Dirichlet problem (2.4), we call \( A \in H^1(\Omega,\mathbb{R}^3, A^0_T) := \{A \in H^1(\Omega,\mathbb{R}^3); A_T = A^0_T\} \) is a weak solution of (2.4) if
\[
\int_{\Omega} (S_K(|\text{curl } A|^2)\text{curl } A \cdot \text{curl } H + (MA) \cdot H + b \cdot H)dx = 0 \tag{2.6}
\]
for any \( H \in H^1_0(\Omega,\mathbb{R}^3) := \{H \in H^1(\Omega,\mathbb{R}^3), H_T = 0 \text{ on } \partial \Omega\} \).

For the Neumann problem (2.5), we call \( A \in \mathcal{H}^2(\Omega,\text{curl}) \) is a weak solution of (2.5) if
\[
\int_{\Omega} (S_K(|\text{curl } A|^2)\text{curl } A \cdot \text{curl } H + (MA) \cdot H + b \cdot H)dx
+ \int_{\partial \Omega} (D_T \times H_T) \cdot \nu dS = 0 \tag{2.7}
\]
for any $\mathbf{H} \in C^1(\overline{\Omega}, \mathbb{R}^3)$.

Finally following [6], we construct a modified function $S_K(t)$ from $S(t)$ in (1.1). First we note that $S(t) \in C^\infty([0, \infty))$ is a positive, strictly increasing, and strictly concave function. If we define

$$\Phi(t) = t(S'(t))^2, \quad t \geq 0,$$

then $\Phi(t) \in C^\infty([0, \infty))$ is a positive, strictly increasing, and strictly concave function on $(0, \infty)$. Moreover,

$$2\Phi'(t) - \frac{\Phi(t)}{t} > 0 \text{ for all } t \in (0, b^2).$$

We can construct the modified function $S_K(t)$ as follows (cf. [6]).

**Lemma 2.3.** For any $K > 0$, we can construct a function $S_K(t) \in C^3([0, \infty))$ such that for some small $\delta > 0$,

(i) $0 < S_K(t) = \begin{cases} 
S(t) & t \in [0, K], \\
C^3\text{-connected} & t \in [K, K+\delta], \\
a_Kt + b_1 & t \in [K+\delta, \infty) 
\end{cases}$ \quad (2.8)

where $a_K$ is a positive constant and $b_1$ is also a positive constant satisfying

$$b_1 > b^2 \left( \sqrt{1 + \frac{1}{b^2}K - 1} \right) - a_K(K+\delta).$$

(ii) $0 < S'_K(t) = \begin{cases} 
S'(t) & t \in [0, K], \\
C^2\text{-connected} & t \in [K, K+\delta], \\
a_K & t \in [K+\delta, \infty) 
\end{cases}$ \quad (2.9)

If we define $\Phi_K(t) = t(S'_K(t))^2$, then $\Phi_K(t) \in C^2([0, \infty))$ is strictly increasing and

$$\Phi_K(t) = \begin{cases} 
\Phi(t) & t \in [0, K], \\
\text{concave} & t \in [K, K+\delta], \\
a_Kt & t \in [K+\delta, \infty) 
\end{cases}$$ \quad (2.10)

(ii) Since $\rho = \Phi_K(t)$ is strictly increasing, the inverse function $t = \Phi^{-1}_K(\rho)$ is defined. Define a function $f_K$ as

$$f_K(\rho) = \frac{1}{S'_K(\Phi^{-1}_K(\rho))}. \quad (2.11)$$

Then $f_K \in C^2([0, \infty))$ and

$$0 < f_K(\rho) = \begin{cases} 
\frac{1}{S'(\Phi^{-1}_K(\rho))} & \rho \in \left[0, \frac{K}{4(1+K/b^2)}\right], \\
C^2\text{-connected} & \rho \in \left[\frac{K}{4(1+K/b^2)}, a_K^2(K+\delta)\right], \\
\frac{1}{a_K} & \rho \in \left[a_K^2(K+\delta), \infty\right) 
\end{cases} \quad (2.12)$$
(iii) If $0 < K < b^2$, we can choose the small $\delta > 0$ such that

$$2\Phi'_K(t) - \frac{\Phi_K(t)}{t} \geq l(K) > 0 \text{ for all } t \in [K, K + \delta],$$

and

$$f_K(\rho) - 2f'_K(\rho) \rho \geq \lambda(K, \delta) > 0 \text{ for all } \rho \in [0, \infty).$$

Moreover, we can show that

$$S'_K(t) + 2tS''_K(t) > 0 \text{ for all } t \in [0, \infty).$$

From the definition of $S_K$, we can easily show that the functional

$$\int_{\Omega} S_K(|\text{curl } \nu|^2)dx$$

is a strictly convex functional on $\mathcal{H}^2(\Omega, \text{curl})$. Moreover, we can see that

$$0 < \inf_{0 \leq t < \infty} S'_K(t) < \sup_{0 \leq t < \infty} S'_K(t) < \infty.$$

## 3 The Dirichlet problem

In this section we shall prove the existence of the minimizer of the functional $S_K^+$ and its regularity of the minimizer for the Dirichlet problem (1.8). Since $S_K^+$ has not the term $\text{div } A$, the functional has not compactness in the natural space $H^1_t(\Omega, \mathbb{R}^3, A^0_T)$. To overcome this, we decompose the minimizer problem into two steps. First we define

$$\mathcal{X}^2 = H^1_t(\Omega, \text{div } 0, A^0_T) \oplus \text{grad} H^1_0(\Omega)$$

where

$$H^1_t(\Omega, \text{div } 0, A^0_T) = \{ \nu \in H^1(\Omega, \mathbb{R}^3); \text{div } \nu = 0 \text{ in } \Omega, \nu_T = A^0_T \text{ on } \partial \Omega \}.$$

Define

$$a_t(A^0_T, \mathcal{X}^2) = \inf_{\nu + \nabla \phi \in \mathcal{X}^2} S^+_K[\nu + \nabla \phi].$$

Then we can prove the existence of the unique minimizer.

**Lemma 3.1.** Let $\Omega$ be a bounded domain in $\mathbb{R}^3$ with $C^4$ boundary and $A^0_T \in H^{1/2}(\partial \Omega, \mathbb{R}^3)$, and let $0 < K < \infty$. Then $S_K^+$ has a unique minimizer $\nu^*_K + \nabla \phi^*_K \in \mathcal{X}^2$, that is

$$a_t(A^0_T, \mathcal{X}^2) = S^+_K[\nu^*_K + \nabla \phi^*_K].$$
Proof. Since we can construct the divergence-free lifting $v \in H^1(\Omega, \mathbb{R}^3)$ of $A^0_T$ (for example see Aramaki [1]), we can see $X^2 \neq \emptyset$. Let $\{v_n + \nabla \phi_n\} \subset X^2$ be a minimizing sequence of $S^+_K$. Let

$$\Omega_n = \{x \in \Omega; |\text{curl } v_n(x)| \geq K + \delta\}.$$  

We will use an elementary inequality

$$ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2\text{ for any } \varepsilon > 0 \text{ and } a, b \geq 0. \hspace{1cm} (3.1)$$

Then using (A2) and (3.1), we have

$$S^+_K[v_n + \nabla \phi_n] = \int_\Omega (S_K(|\text{curl } v_n|^2) + (M(v_n + \nabla \phi_n), v_n + \nabla \phi_n)
+ b \cdot (v_n + \nabla \phi_n) + c)dx
\geq \int_{\Omega_n} S_K(|\text{curl } v_n|^2)dx + \int_{\Omega} ((M(v_n + \nabla \phi_n), v_n + \nabla \phi_n)
+ b \cdot (v_n + \nabla \phi_n) + c)dx
\geq \int_{\Omega_n} (a_K|\text{curl } v_n|^2 + b_1)dx + \int_{\Omega} (m_0|v_n + \nabla \phi_n|^2
- |b||v_n + \nabla \phi_n| - ||c||_{C^0(\Omega)}\Omega
\geq a_K \int_{\Omega} |\text{curl } v_n|^2dx - \int_{\Omega\setminus\Omega_n} (a_K|\text{curl } v_n|^2 + b_1)dx
\geq \int_{\Omega} \{(m_0 - \varepsilon)|v_n + \nabla \phi_n|^2 - \frac{1}{4\varepsilon} |b|^2\} dx - ||c||_{C^0(\Omega)}\Omega
\geq a_K\|\text{curl } v_n\|_{L^2(\Omega)}^2 + (m_0 - \varepsilon)\|v_n + \nabla \phi_n\|^2_{L^2(\Omega)}
\geq -(a_K(K + \delta)^2 + b_1 + ||c||_{C^0(\Omega)}\Omega$$

where $|\Omega|$ denotes the volume of $\Omega$. Since $v_n$ and $\nabla \phi_n$ are orthogonal in $L^2(\Omega, \mathbb{R}^3)$ to each other, we see that $\|v_n + \nabla \phi_n\|^2_{L^2(\Omega)} = \|v_n\|^2_{L^2(\Omega)} + \|\nabla \phi_n\|^2_{L^2(\Omega)}$. If we put $\varepsilon = m_0/2$, we see that $\{\text{curl } v_n\}, \{v_n\}$ and $\{\nabla \phi_n\}$ are bounded in $L^2(\Omega, \mathbb{R}^3)$. From the Poincaré inequality, $\{\phi_n\}$ is bounded in $H^1_0(\Omega)$. Since div $v_n = 0$ in $\Omega$ and $v_{n,T} = A^0_T$ on $\partial \Omega$, it follows from Lemma 2.2 that $\{v_n\}$ is bounded in $H^1(\Omega, \mathbb{R}^3)$. Passing to subsequences, we may assume that $v_n \rightharpoonup v^*_K$ weakly in $H^1(\Omega, \mathbb{R}^3)$ and $\phi_n \rightarrow \phi^*_K$ weakly in $H^1_0(\Omega)$. Then we can see that div $v^*_K = 0$ in $\Omega$ and $v^*_{K,T} = A^0_T$ on $\partial \Omega$. Thus $v^*_K \in H^1(\Omega, \text{div} 0, A^0_T)$. Since $S^+_K$ is convex functional and lower semi continuous on $X^2$, it is weakly lower semi-continuous in $X^2$. Thus $v^*_K + \nabla \phi^*_K$ is a minimizer of $S^+_K$. Since $S^+_K$ is strictly convex, the uniqueness follows. \hfill $\square$

Secondly we define

$$a_t(A^0_T, H^1) = \inf_{A \in H^1(\Omega, \mathbb{R}^3, A^0_T)} S^+_K[A].$$
Then we have $a_t(A^0_T, H^1) = a_t(A^0_T, X^2)$. In fact, let $A \in H^1_t(\Omega, \mathbb{R}^3, A^0_T)$. Then the following Dirichlet problem
\[
\begin{cases}
\Delta \phi = \text{div} A & \text{in } \Omega, \\
\phi = 0 & \text{on } \partial \Omega
\end{cases}
\]has a unique solution $\phi \in H^2(\Omega) \cap H^1_0(\Omega)$. If we define $v = A - \nabla \phi \in H^1(\Omega, \mathbb{R}^3)$, we have $\text{div} v = 0$ in $\Omega$ and $v_T = A_T - (\nabla \phi)_T = A_T - A^0_T$. Hence $v \in H^1(\Omega, \text{div} 0, A^0_T)$, so $A = v + \nabla \phi \in X^2$. This implies that $a_t(A^0_T, X^2) \leq a_t(A^0_T, H^1)$. On the other hand, since $A^K = v^K + \nabla \phi^K \in H^1_t(\Omega, \mathbb{R}^3, A^0_T)$, we see that
\[a_t(A^0_T, X^2) = S^+_K[A^K] \geq a_t(A^0_T, H^1).
\]
This fact is a reason of decomposition of the above minimizing problem into two steps.

Summing up, since $A^K = v^K + \nabla \phi^K \in H^1_t(\Omega, \mathbb{R}^3, A^0_T)$, we obtain the following.

**Proposition 3.2.** Let $\Omega$ be a bounded domain in $\mathbb{R}^3$ with a $C^4$ boundary, $A^0_T \in H^{1/2}(\partial \Omega, \mathbb{R}^3)$ and $0 < K < \infty$. Then $S^K_+$ has a unique minimizer $A^K \in H^1_t(\Omega, \mathbb{R}^3, A^0_T)$.

In the following we shall give the regularity of the minimizer $A^K = v^K + \nabla \phi^K$. First we examine the $H^2$ regularity of $\phi^K$.

**Lemma 3.3.** Let $\Omega$ be a bounded domain in $\mathbb{R}^3$ with a $C^2$ boundary and $0 < K < \infty$. Then we see that $\phi^K \in H^2(\Omega) \cap H^1_0(\Omega)$ and
\[
\|\phi^K\|_{H^2(\Omega)} \leq C(\|\phi^K\|_{L^2(\Omega)} + \|v^K\|_{H^1(\Omega)} + \|b\|_{H^1(\Omega)})
\]where $C$ depends on $\Omega, m_0$ and $\|M\|_{C^0(\overline{\Omega})}$.

**Proof.** Since $\phi^K$ is the minimizer of the functional $S^K_+[v^K + \nabla \phi]$ on $H^1_0(\Omega)$, the Euler-Lagrange equation becomes
\[
\begin{cases}
\text{div} (M(x)(v^K + \nabla \phi) + b) = 0 & \text{in } \Omega, \\
\phi = 0 & \text{on } \partial \Omega.
\end{cases}
\]
We rewrite (3.3) into the Dirichlet system for the second order elliptic linear equation with respect to $\phi$:
\[
\begin{cases}
-\sum_{i,j=1}^3 \partial_i (M_{ij}(x) \partial_j \phi) = \sum_{i,j=1}^3 \partial_i (M_{ij} v^K_{i,j}) + \text{div } b & \text{in } \Omega, \\
\phi = 0 & \text{on } \partial \Omega
\end{cases}
\]
where $v^K = (v^K_{1,1}, v^K_{1,2}, v^K_{1,3})$. Since $M_{ij} \in W^{1,\infty}(\Omega)$ and the right hand side of the first equation of (3.4) belongs to $L^2(\Omega)$, it follows from Chen and Wu [7, Chapter 1, Theorem 5.2] that $\phi^K \in H^2(\Omega)$ and the estimate (3.2) holds. \qed
Remark 3.4. In the right hand side of the estimate (3.2), we can remove the term $\|\phi_K^*\|_{L^2(\Omega)}$. In fact, if we multiply the first equation of (3.3) by $\phi_K^*$, and then integrate by parts, we see that

$$
\int_\Omega M(x) \nabla \phi_K^* \cdot \nabla \phi_K^* dx = \int_\Omega \text{div} (M(x) v_K^* + b) \phi_K^* dx \\
\leq C(\|v_K^*\|_{H^1(\Omega)} + \|b\|_{H^1(\Omega)}) \|\phi_K^*\|_{L^2(\Omega)}
$$

where $C$ depends on $\|M\|_{C^1(\overline{\Omega})}$. If we use the Poincaré inequality:

$$
\|\phi_K^*\|_{L^2(\Omega)} \leq C(\Omega) \|\nabla \phi_K^*\|_{L^2(\Omega)}
$$

and positivity of the matrix $M(x)$, we easily see that

$$
\|\phi_K^*\|_{L^2(\Omega)} \leq C(\Omega) \|\nabla \phi_K^*\|_{L^2(\Omega)} \leq C(\|v_K^*\|_{H^1(\Omega)} + \|b\|_{H^1(\Omega)})
$$

where $C$ depends on $\Omega, m_0$ and $\|M\|_{C^1(\overline{\Omega})}$. Thus we can remove $\|\phi_K^*\|_{L^2(\Omega)}$ from (3.2).

We shall give the $C^{2,\alpha}$ estimate of $A_K^* = v_K^* + \nabla \phi_K^*$ and prove that if the boundary data $A_T^0$ and $b$ are small, then $\|\text{curl} A_K^*\|_{C^0(\overline{\Omega})}$ is small. Then we will see that $A_K^*$ is a classical solution of the extended magnetostatic Born-Infeld system (1.8). First the minimizer $A_K^*$ of $a_t(A_T^0, H^1)$ can be written into the form $A_K^* = v_K^* + \nabla \phi_K^*$ where $v_K^* \in H^1_t(\Omega, \text{div} 0, A_T^0)$ and $\phi_K^* \in H^2(\Omega) \cap H^1_0(\Omega)$. Moreover, $v_K^*$ and $\phi_K^*$ are weak solutions of the following equations, respectively.

$$
\left\{ \begin{array}{ll}
\text{curl} (S_K^t(|\text{curl} v|^2)\text{curl} v) + M(x)(v + \nabla \phi_K^*) + b = 0 & \text{in } \Omega, \\
v_T = A_T^0 & \text{on } \partial \Omega
\end{array} \right. \tag{3.5}
$$

and

$$
\left\{ \begin{array}{ll}
\text{div} (M(x)(v_K^* + \nabla \phi) + b) = 0 & \text{in } \Omega, \\
\phi = 0 & \text{on } \partial \Omega.
\end{array} \right. \tag{3.6}
$$

Lemma 3.5. Let $\Omega$ be a bounded domain in $\mathbb{R}^3$ with a $C^4$ boundary, $A_T^0 \in H^{1/2}(\partial \Omega)$ and $0 < K < \infty$. Then we get the estimate

$$
\|v_K^*\|_{H^1(\Omega)} + \|\phi_K^*\|_{H^1(\Omega)} \leq C(\|A_T^0\|_{H^{1/2}(\partial \Omega)} + \|b\|_{L^2(\Omega)}) \tag{3.7}
$$

where $C$ depends on $\Omega, m_0, \|M\|_{C^0(\overline{\Omega})}$ and $S_K^t$.

Proof. For brevity of notations, we write $v_K^*$ and $\phi_K^*$ by $v$ and $\phi$, respectively. It is well known that there exists a divergence-free extension $A^e \in H^1(\Omega, \mathbb{R}^3)$ of $A_T^0$ such that $\|A^e\|_{H^1(\Omega)} \leq C(\Omega)\|A_T^0\|_{H^{1/2}(\partial \Omega)}$. For example, see [1]. Define
where $\nabla$. Then $\text{div } u = 0$ in $\Omega$ and $u_T = 0$ on $\partial \Omega$. If we choose $u$ as a test field of (2.4), we see that

$$0 = \int_{\Omega} (S_K'(|\text{curl } v|^2)\text{curl } v \cdot \text{curl } u + M(x)(v + \nabla \phi) \cdot u + b \cdot u) dx$$

$$= \int_{\Omega} (S_K'(|\text{curl } v|^2)\text{curl } v \cdot (\text{curl } v - \text{curl } A^e)$$

$$+ M(x)(v + \nabla \phi) \cdot (v + \nabla \phi - \nabla \phi - A^e)$$

$$+ b \cdot (v + \nabla \phi - \nabla \phi - A^e) dx.$$
Next we give the regularity of \( v^*_K \) and \( \phi^*_K \).

**Proposition 3.6.** Assume that \( \Omega, M(x) \) and \( b(x) \) satisfy (A1), (A2') and (A3) with \( 0 < \alpha < 1 \), respectively, and \( 0 < K < b^2 \). Let \( A^*_K = v^*_K + \nabla \phi^*_K \) be a minimizer of \( S'_K \). Then \( v^*_K \in C^{2,\alpha}(\Omega, \mathbb{R}^3) \), \( \phi^*_K \in C^{3,\alpha}(\Omega) \) and

\[
\|v^*_K\|_{C^{2,\alpha}(\Omega)} + \|
abla \phi^*_K\|_{C^{2,\alpha}(\Omega)} \leq C
\]

where \( C \) depends on \( \Omega, \alpha, m_0, \|M\|_{C^{1,1}(\Omega)}, \|b\|_{C^{1,1}(\Omega)}, K, \delta \) and \( \|A^0_T\|_{C^{2,\alpha}(\partial \Omega)} \).

The proof of Proposition 3.6 consists of following several lemmas.

First we consider the following div-curl system.

\[
\begin{cases}
\text{curl } Q = -M(x)(v^*_K + \nabla \phi^*_K) - b(x) & \text{in } \Omega, \\
\text{div } Q = 0 & \text{in } \Omega, \\
\nu \cdot Q = 0 & \text{on } \partial \Omega.
\end{cases}
\]  

(3.8)

We define a space \( H^m(\Omega, \text{div } 0, \mathbb{R}^3) = \{ u \in H^m(\Omega, \mathbb{R}^3); \text{div } u = 0 \text{ in } \Omega \} \) for \( m = 1, 2 \). First we have the following lemma.

**Lemma 3.7.** The above system (3.8) has a unique solution \( Q_K \in H^2(\Omega, \text{div } 0, \mathbb{R}^3) \) and

\[
\|Q\|_{H^2(\Omega)} \leq C(\|A^0_T\|_{H^{1/2}(\partial \Omega)} + \|b\|_{H^1(\Omega)})
\]

where \( C \) depends on \( \Omega, m_0, \|M\|_{C^{1}(\Omega)} \) and \( S'_K \).

**Proof.** Since \( M(v^*_K + \nabla \phi^*_K) + b \in H^1(\Omega, \text{div } 0, \mathbb{R}^3) \) and \( \Omega \) has no holes, it follows from [15, Lemma 5.7] or [1] that (3.8) has a solution \( Q_K \in H^2(\Omega, \text{div } 0, \mathbb{R}^3) \). Since \( \Omega \) is simply connected, \( \dim \mathbb{H}_1(\Omega) = \{0\} \), so the uniqueness follows. Moreover, we have

\[
\|Q\|_{H^2(\Omega)} \leq C(\Omega)\|M(v^*_K + \nabla \phi^*_K) + b\|_{H^1(\Omega)}
\]

\[
\leq C(\|v^*_K\|_{H^1(\Omega)} + \|
abla \phi^*_K\|_{H^1(\Omega)} + \|b\|_{H^1(\Omega)})
\]

where \( C \) depends on \( \Omega \) and \( \|M\|_{C^{1}(\Omega)} \). From Lemma 3.3, Remark 3.4 and Lemma 3.5, we get the conclusion. \( \square \)

From (2.4) and (3.8), we have

\[
\text{curl } (S'_K(|\text{curl } v^*_K|^2)\text{curl } v^*_K) = \text{curl } Q_K
\]

in \( \Omega \). Since \( \Omega \) is simply connected, there exists \( \psi_K \in H^1(\Omega) \) such that

\[
S'_K(|\text{curl } v^*_K|^2)\text{curl } v^*_K = Q_K + \nabla \psi_K.
\]  

(3.9)
From (2.11), we can write
\[
\text{curl } A_K^* = \text{curl } v_K^* = \frac{Q_K + \nabla \psi_K}{S'K(\Phi^{-1}(Q_K + \nabla \psi_K)^2)} = f_K((Q_K + \nabla \psi_K^2)(Q_K + \nabla \psi_K)).
\]
Since \( \nu \cdot \text{curl } v_K = \nu \cdot \text{curl } A_T^* = 0 \) by the hypothesis (A3), and \( \nu \cdot Q_K = 0 \) on \( \partial \Omega \) by (3.8), it follows from (3.9) that \( \partial \psi_K / \partial \nu = 0 \) on \( \partial \Omega \). Here we used the fact \( \nu \cdot \text{curl } v = \nu \cdot \text{curl } v_T \) according to Monneau [12]. Thus \( \psi_K \in H^1(\Omega) \) is a weak solution of the following system.

\[
\begin{cases}
\text{div } (f_K((Q_K + \nabla \psi_K^2)(Q_K + \nabla \psi))) = 0 & \text{in } \Omega, \\
\frac{\partial \psi_K}{\partial \nu} = 0 & \text{on } \partial \Omega.
\end{cases}
\]

(3.10)

**Lemma 3.8.** Let \( \psi_K \in H^1(\Omega) \) be a weak solution of (3.10). Then for any \( 1 < q < \infty \), \( \psi_K \in W^{1,q}(\Omega) \) and
\[
\| \psi_K \|_{W^{1,q}(\Omega)} \leq C(a_K + \| A_T^0 \|_{H^{1/2}(\partial \Omega)} + \| b \|_{H^1(\Omega)})
\]
where \( C \) depends on \( \Omega, q, K, \delta \) and \( f_K \).

**Proof.** We rewrite the system (3.10) as a linear equation of \( \psi_K \).

\[
\begin{cases}
a_K^{-1} \Delta \psi_K = \text{div } f & \text{in } \Omega, \\
\frac{\partial \psi_K}{\partial \nu} = 0 & \text{on } \partial \Omega
\end{cases}
\]

where \( f = a_K^{-1} \nabla \psi_K - f_K((Q_K + \nabla \psi_K^2)(Q_K + \nabla \psi_K)). \) Note that
\[
\int_{\Omega} \text{div } f dx = \int_{\partial \Omega} f \cdot \nu dS = 0
\]
because of (3.8) and (3.10), and by the definition of \( f_K \) we see that if \( |Q_K(x) + \nabla \psi_K(x)|^2 \geq a_K^2(K + \delta) \), then \( f(x) = -f_K((|Q_K + \nabla \psi_K^2|)(Q_K + \nabla \psi_K)). \) From Lemma 3.7 and the Sobolev embedding theorem, we see that \( Q_K \in C^{0,1/2}(\overline{\Omega}, \mathbb{R}^3) \). Since \( |\nabla \psi_K| \leq |Q_K + \nabla \psi_K| + |Q_K| \), we see that
\[
|f(x)| \leq C(1 + a_K^{-1}|Q_K(x)|)
\]
where \( C \) depends on \( K, \delta \) and \( f_K \). In particular, \( f \in L^\infty(\Omega, \mathbb{R}^3). \) By the classical \( L^q \) Schauder estimate (cf. Morrey [13, Theorem 5.5.5’ and the remarks in p.157], Lions and Magenes [11]), we see that \( \psi_K \in W^{1,q}(\Omega) \) for any \( 1 < q < \infty \) and
\[
\| \psi_K \|_{W^{1,q}(\Omega)} \leq C(\Omega, q) \| f \|_{L^q(\Omega)} \leq C(a_K + \| Q_K \|_{L^q(\Omega)})
\]
where \( C \) depends on \( \Omega, q, K, \delta \) and \( f_K \). Since \( H^2(\Omega) \hookrightarrow C^{0,1/2}(\overline{\Omega}) \hookrightarrow L^q(\Omega) \), using Lemma 3.7
\[
\| Q_K \|_{L^q(\Omega)} \leq C(\Omega) \| Q_K \|_{H^2(\Omega)} \leq C(\| A_T^0 \|_{H^{1/2}(\partial \Omega)} + \| b \|_{H^1(\Omega)}).
\]
Since $W^{1,q}(\Omega) \hookrightarrow C^{\tau}(\overline{\Omega})$ for $\tau = 1 - 3/q > 0$ by the Sobolev embedding theorem, we see that $\psi_K \in C^{0,\tau}(\overline{\Omega})$ for such $\tau$. Since $1 < q < \infty$ is arbitrary, it follows that $\psi_K \in C^{0,\tau}(\overline{\Omega})$ for any $\tau \in (0, 1)$. Next we examine the regularity of $\mathbf{v}_K^*$.

**Lemma 3.9.** For any $1 < q < \infty$, $\mathbf{v}_K^* \in W^{1,q}(\Omega, \mathbb{R}^3)$ and

$$
\|\mathbf{v}_K^*\|_{W^{1,q}(\Omega)} \leq C(\Omega, q)\|\text{curl} \mathbf{A}_K\|_{L^q(\Omega)} \leq C(\Omega, q)(d_0\|\mathbf{Q}_K + \nabla \psi_K\|_{L^q(\Omega)} + \|\mathbf{A}_T^0\|_{W^{1-1/q,q}(\partial\Omega)})
$$

where $d_0 = \|f_K\|_{C^q(0, \infty)}$. In particular, $\mathbf{v}_K^* \in C^{0,\tau}(\overline{\Omega}, \mathbb{R}^3)$ for any $\tau \in (0, 1)$. 

**Proof.** We note that $\mathbf{Q}_K + \nabla \psi_K \in L^q(\Omega, \mathbb{R}^3)$ for any $1 < q < \infty$, and so

$$
\text{curl} \mathbf{A}_K^* = f_K(|\mathbf{Q}_K + \nabla \psi_K|^2)(\mathbf{Q}_K + \nabla \psi_K) \in L^q(\Omega, \mathbb{R}^3).
$$

Here we remember that $\mathbf{v}_K^*$ is a solution of the following system

$$
\begin{cases}
\text{curl} \mathbf{v} = \text{curl} \mathbf{A}_K^* = f_K(|\mathbf{Q}_K + \nabla \psi_K|^2)(\mathbf{Q}_K + \nabla \psi_K) & \text{in } \Omega, \\
\text{div} \mathbf{v} = 0 & \text{in } \Omega, \\
\mathbf{v}_T = \mathbf{A}_T^0 & \text{on } \partial\Omega.
\end{cases}
\tag{3.11}
$$

By the hypothesis (A3), $\mathbf{v} \cdot \text{curl} \mathbf{A}_T^0 = 0$ on $\partial\Omega$. Since $\Omega$ has no holes, it follows from [1] that (3.11) has a unique solution $\mathbf{v}_K^* \in W^{1,q}(\Omega, \mathbb{R}^3)$ satisfying

$$
\|\mathbf{v}_K^*\|_{W^{1,q}(\Omega)} \leq C(\Omega, q)(\|f_K(|\mathbf{Q}_K + \nabla \psi_K|^2)(\mathbf{Q}_K + \nabla \psi_K)\|_{L^q(\Omega)} + \|\mathbf{A}_T^0\|_{W^{1-1/q,q}(\partial\Omega)}) \leq C(\Omega, q)(d_0\|\mathbf{Q}_K + \nabla \psi_K\|_{L^q(\Omega)} + \|\mathbf{A}_T^0\|_{W^{1-1/q,q}(\partial\Omega)}).
$$

Since $q$ is arbitrary, it follows from the Sobolev embedding theorem that $\mathbf{v}_K^* \in C^{\tau}(\overline{\Omega}, \mathbb{R}^3)$ for any $\tau \in (0, 1)$. 

We examine the $C^{1,\tau}$ regularity of $\phi_K^*$. 

**Lemma 3.10.** For any $\tau \in (0, 1)$, $\phi_K^* \in C^{1,\tau}(\overline{\Omega})$ and

$$
\|\phi_K^*\|_{C^{1,\tau}(\overline{\Omega})} \leq C(\|\mathbf{v}_K^*\|_{W^{1,q}(\Omega)} + \|\mathbf{b}\|_{W^{1,q}(\Omega)})
$$

where $C$ depends on $\Omega, q, \tau$ and $\|M\|_{C^1(\overline{\Omega})}$.

**Proof.** Since $\phi_K^*$ is a solution of (3.3), we can rewrite (3.3) into the form

$$
\begin{cases}
-\sum_{i,j=1}^{3} M_{ij}(x) \partial_i \partial_j \phi_K^* - \sum_{j=1}^{3} (\sum_{i=1}^{3} (\partial_i M_{ij}(x))) \partial_j \phi_K^* \\
\quad = \sum_{i,j=1}^{3} \partial_i (M_{ij}(x) v_{K,j}^*) + \text{div} \mathbf{b} & \text{in } \Omega, \\
\phi_K^* = 0 & \text{on } \partial\Omega
\end{cases}
\tag{3.12}
$$
where we denote $\mathbf{v}_K^* = (v^*_{K,1}, v^*_{K,2}, v^*_{K,3})$. Since $M_{ij} \in C^0(\Omega)$ satisfies (A2) and the right hand side of the first equation in (3.8) belongs to $L^q(\Omega)$ for any $1 < q < \infty$, it follows from Gilbarg and Trudinger [9, Theorem 9.15] or [7, Chapter 3, Theorem 6.3] that $\phi^*_K \in W^{2,q}(\Omega)$ and

$$
\|\phi^*_K\|_{W^{2,q}(\Omega)} \leq C(\Omega,q)(\|\nabla (M\mathbf{v}_K^*)\|_{L^q(\Omega)} + \|\text{div } \mathbf{b}\|_{L^q(\Omega)})
\leq C(\|\mathbf{v}_K^*\|_{W^{1,q}(\Omega)} + \|\mathbf{b}\|_{W^{1,q}(\Omega)})
$$

where $C$ depends on $\Omega, q$ and $\|M\|_{C^1(\Omega)}$. By the Sobolev embedding theorem: $W^{2,q}(\Omega) \hookrightarrow C^{1,1-3/q}(\Omega)$ for any $1 < q < \infty$. Thus the conclusion holds.

We examine $C^{1,\tau}$ regularity of $\mathbf{Q}_K$.

**Lemma 3.11.** $\mathbf{Q}_K \in C^{1,\tau}(\overline{\Omega}, \mathbb{R}^3)$ for any $\tau \in (0,1)$ and

$$
\|\mathbf{Q}_K\|_{C^{1,\tau}(\overline{\Omega})} \leq C(\|\mathbf{v}_K^*\|_{C^{0,\tau}(\overline{\Omega})} + \|\nabla \phi^*_K\|_{C^{0,\tau}(\overline{\Omega})} + \|\mathbf{b}\|_{C^{0,\tau}(\overline{\Omega})})
$$

where $C$ depends on $\Omega, \tau$ and $\|M\|_{C^{0,\tau}(\overline{\Omega})}$.

**Proof.** From Lemma 3.9, 3.10 and (A2), the right hand side of the first equation in (3.8) belongs to $C^{0,\tau}(\overline{\Omega})$. Since $\Omega$ has no holes, it follows from the regularity of div-curl system (cf. [15, Lemma 5.7 and Corollary 5.4]) that $\mathbf{Q}_K \in C^{1,\tau}(\overline{\Omega}, \mathbb{R}^3)$ and

$$
\|\mathbf{Q}_K\|_{C^{1,\tau}(\overline{\Omega})} \leq C(\Omega,q)\|\mathbf{v}_K^*\|_{C^{0,\tau}(\overline{\Omega})} + \|\nabla \phi^*_K\|_{C^{0,\tau}(\overline{\Omega})} + \|\mathbf{b}\|_{C^{0,\tau}(\overline{\Omega})}.
$$

We examine $C^{1,\theta}$ estimate of $\psi_\xi$ for some $\theta \in (0,1)$.

**Lemma 3.12.** $\psi_\xi \in C^{1,\theta}(\overline{\Omega})$ for some $\theta \in (0,1)$ and $\|\psi_\xi\|_{C^{1,\theta}(\overline{\Omega})} \leq C$ where $C$ depends on $\Omega, \|\mathbf{Q}_K\|_{C^{1,\tau}(\overline{\Omega})}, \|\psi_\xi\|_{C^{0,\tau}(\overline{\Omega})}, m_0$ and $\|M\|_{C^{0,\tau}(\overline{\Omega})}$.

**Proof.** Define

$$
\mathbf{A}(x, z) = (A_1(x, z), A_2(x, z), A_3(x, z)) = f_K(|\mathbf{Q}_K(x) + z|^2)(\mathbf{Q}_K + z).
$$

Then by Lemma 3.11, we know $\mathbf{Q}_K \in C^{1,\tau}(\overline{\Omega}, \mathbb{R}^3)$, so $\mathbf{A}(x, z) \in C^{1,\tau}(\overline{\Omega} \times \mathbb{R}^3 \times \mathbb{R}^3)$. From (2.14), we have

$$
\sum_{i,j=1}^3 \frac{\partial A_i}{\partial z_j}(x, z)\xi_i \xi_j \geq \lambda(K, \delta)|\xi|^2 \text{ for all } \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3.
$$

Moreover, we have

$$(1 + |z|^2) \left| \frac{\partial A_i}{\partial z_j}(x, z) \right| + (1 + |z|) \left( \left| \frac{\partial A_i}{\partial x_j}(x, z) \right| + |A_i(x, z)| \right) \leq \Lambda(1 + |z|^2)$$
for some constant $\Lambda > 0$. Therefore $\psi_K$ is a solution of the Neumann problem
\[
\begin{cases}
\text{div} \mathbf{A}(x, \nabla \psi) = 0 & \text{in } \Omega, \\
\frac{\partial \psi}{\partial \mathbf{n}} = 0 & \text{on } \partial \Omega.
\end{cases}
\]

Applying Ladyzhenskaya and Ural’tzeva [10, Chapter 10, Theorem 2.1], there exists $\theta \in (0, 1)$ and $C > 0$ depending on $\Omega$, $\|Q\|_{C^1(\Omega)}$, $\|\psi_K\|_{C^0(\Omega)}$, $\lambda(K, \delta)$ and $\Lambda$ such that $\psi_K \in C^{1,\theta}(\Omega)$ and $\|\psi_K\|_{C^{1,\theta}(\Omega)} \leq C$. \hfill \Box

We examine $C^{1,\theta}$ estimate of $A_K^*$.\hfill \Box

**Lemma 3.13.** We can see that $A_K^* = v_K^* + \nabla \phi_K^* \in C^{1,\theta}(\overline{\Omega}, \mathbb{R}^3)$ and $\|A_K^*\|_{C^{1,\theta}(\Omega)} \leq C$ where $C$ depends on $\Omega, \theta, \|f_K\|_{C^{1,\theta}(\Omega)}, \|Q_K\|_{C^{0,\theta}(\Omega)}, \|\nabla \psi_K\|_{C^{0,\theta}(\Omega)}, \|A_T^0\|_{H^{1/2}(\partial \Omega)}$ and $\|b\|_{C^{1,\theta}(\Omega)}$.

**Proof.** First we return to the system (3.11). From Lemma 3.11 we see that $Q_K \in C^{1,\theta}(\overline{\Omega}, \mathbb{R}^3)$. Thus the right hand side of the first equation of (3.11) belongs to $C^{0,\theta}(\overline{\Omega}, \mathbb{R}^3)$ according to Lemma 3.12. We can easily see that
\[
\text{div} (f_K(|Q_K + \nabla \psi_K|^2)(Q_K + \nabla \psi_K)) = \text{div} (\text{curl} A_K^*) = 0 \text{ in } \Omega,
\]

$\mathbf{\nu} \cdot A_T^0 = 0$ on $\partial \Omega$, and using (A3)
\[
\mathbf{\nu} \cdot f_K(|Q_K + \nabla \psi_K|^2)(Q_K + \nabla \psi_K) = 0 = \mathbf{\nu} \cdot \text{curl} ((\mathbf{\nu} \times A_T^0) \times \mathbf{\nu}) = \mathbf{\nu} \cdot \text{curl} A_T^0 \text{ on } \partial \Omega.
\]

If we take the lifting $\mathbf{H}$ of $-\mathbf{\nu} \times A_T^0$, then $\mathbf{\nu} \times \mathbf{H} = A_T^0$. Since $\Omega$ is simply connected, it follows from [15, Corollary 5.6] that $v_K^* \in C^{1,\theta}(\overline{\Omega}, \mathbb{R}^3)$ and
\[
\|v_K^*\|_{C^{1,\theta}(\Omega)} \leq C(\Omega, \theta)(\|\text{curl} A_K^*\|_{C^{0,\theta}(\Omega)} + \|A_T^0\|_{C^{1,\theta}(\partial \Omega)}) \leq C
\]
where $C$ depends on $\Omega, \theta, \|f_K\|_{C^{1,\theta}(\Omega)}, \|Q_K\|_{C^{0,\theta}(\Omega)}$ and $\|\nabla \psi_K\|_{C^{0,\theta}(\Omega)}$. Here we return to the system (3.4). Since the right hand side of the first equation of (3.4) belongs to $C^{0,\theta}(\overline{\Omega})$, by the Schauder theory we have $\phi_K^* \in C^{2,\theta}(\overline{\Omega})$. Thus $A_K^* = v_K^* + \nabla \phi_K^* \in C^{1,\theta}(\overline{\Omega}, \mathbb{R}^3)$ and we get the concluding estimate. \hfill \Box

We examine $C^{2,\theta}$ estimate of $\psi_K$.\hfill \Box

**Lemma 3.14.** We can see that $\psi_K \in C^{2,\theta}(\overline{\Omega})$ and $\|\psi_K\|_{C^{2,\theta}(\Omega)} \leq C$ where $C$ depends on $\Omega, \theta, \lambda(\Omega, \delta), \|f_K\|_{C^{1,\theta}(\Omega)}, \|Q_K\|_{C^{1,\theta}(\Omega)}$, and $\|\psi_K\|_{C^{1,\theta}(\Omega)}$.

**Proof.** First we consider the system (3.8). From Lemma 3.13, the right hand side of the first equation of (3.8) belongs to $C^{1,\theta}(\overline{\Omega}, \mathbb{R}^3)$. Therefore it follows from the regularity of the div-curl system that $Q_K \in C^{2,\theta}(\overline{\Omega}, \mathbb{R}^3)$ and
\[
\|Q_K\|_{C^{2,\theta}(\Omega)} \leq C(\|M(v_K^* + \nabla \phi_K^*) + b\|_{C^{1,\theta}(\Omega)}) \\
\leq C(\|v_K^*\|_{C^{1,\theta}(\Omega)} + \|
abla \phi_K^*\|_{C^{1,\theta}(\Omega)} + \|b\|_{C^{1,\theta}(\Omega)})
\]
where $C$ depends on $\Omega, \theta$, and $\|M\|_{C^{1,\theta}(\overline{\Omega})}$.

Next we rewrite (3.10) into the following linear equation

\[
\begin{cases}
\sum_{i,j=1}^{3} a_{ij}(x) \frac{\partial^2 \psi_k}{\partial x_i \partial x_j} = h & \text{in } \Omega, \\
\frac{\partial \psi_k}{\partial \nu} = 0 & \text{on } \partial \Omega
\end{cases}
\]  
(3.13)

where

\[
a_{ij} = f_K(|Q_K + \nabla \psi_K|^2) \delta_{ij} + 2f'_K(|Q_K + \nabla \psi_K|^2)(Q_{Ki,j} + \partial_i \psi_K)(Q_{K,j} + \partial_j \psi_K),
\]

\[
h = -2f'_K(|Q_K + \nabla \psi_K|^2)\langle \nabla Q_K(Q_K + \nabla \psi_K), Q_K + \nabla \psi_K \rangle - f_K(|Q_K + \nabla \psi_K|^2)\text{div} Q_K
\]

where $Q_K = (Q_{K,1}, Q_{K,2}, Q_{K,3})$. We note that $a_{ij} \in C^{0,\theta}(\overline{\Omega})$ and the system is uniformly elliptic according to (14.1). Since $h \in C^{0,\theta}(\overline{\Omega})$ and $f_K \in C^2([0, \infty))$, we see that $\psi_K \in C^{2,\theta}(\overline{\Omega})$ and $\|\psi_K\|_{C^{2,\theta}(\overline{\Omega})} \leq C \|h\|_{C^{1,\theta}(\overline{\Omega})} \leq C_1$ where $C_1$ depends on $\Omega, \theta, \lambda(K, \delta), \|f_K\|_{C^{1,\theta}(\overline{\Omega})}, \|Q_K\|_{C^{1,\theta}(\overline{\Omega})}$ and $\|\psi_K\|_{C^{1,\theta}(\overline{\Omega})}$.

Finally we examine $C^{2,\theta}$ estimate of $v^*_K$.

**Lemma 3.15.** It follows that $v^*_K \in C^{2,\theta}(\overline{\Omega}, \mathbb{R}^3)$ and

\[
\|v^*_K\|_{C^{2,\theta}(\overline{\Omega})} \leq C(\Omega, \theta)\|\nabla A^*_K\|_{C^{1,\theta}(\overline{\Omega})}.
\]

**Proof.** Since $Q_K + \nabla \psi_K \in C^{1,\theta}(\overline{\Omega}, \mathbb{R}^3)$ by Lemma 3.11 and 3.14, it follows from the regularity of the div-curl system (3.11) that $v^*_K \in C^{2,\theta}(\overline{\Omega}, \mathbb{R}^3)$,

\[
\text{curl } A^*_K = f_K(|Q_K + \nabla \psi_K|^2)(Q_K + \nabla \psi_K) \in C^{1,\theta}(\overline{\Omega}, \mathbb{R}^3)
\]

and $\|v^*_K\|_{C^{2,\theta}(\overline{\Omega})} \leq C\|\text{curl } A^*_K\|_{C^{1,\theta}(\overline{\Omega})}$.

Since the right hand side of the first equation of (3.12) belongs to $C^{1,\theta}(\overline{\Omega}) \subset C^{0,\alpha}(\overline{\Omega})$, we see that $\phi^*_K \in C^{2,\alpha}(\overline{\Omega}) \cap C^{3,\theta}(\overline{\Omega})$, and so $A^*_K = v^*_K + \nabla \phi^*_K \in C^{2,\theta}(\overline{\Omega}, \mathbb{R}^3) + \text{grad} C^{2,\alpha}(\overline{\Omega}) \subset C^{1,\alpha}(\overline{\Omega}, \mathbb{R}^3)$.

**End of the proof of Proposition 3.6.**

When $\{\theta, \alpha\} := \min\{\theta, \alpha\} = \alpha$, since $v^*_K \in C^{2,\alpha}(\overline{\Omega}, \mathbb{R}^3)$ and $\phi^*_K \in C^{3,\alpha}(\overline{\Omega})$, the proof is done.

When $\{\theta, \alpha\} = \theta$, since $A^*_K = v^*_K + \nabla \phi^*_K \in C^{1,\alpha}(\overline{\Omega}, \mathbb{R}^3)$, it follows from (3.8) that $Q_K \in C^{2,\alpha}(\overline{\Omega}, \mathbb{R}^3)$ and

\[
\|Q_K\|_{C^{2,\alpha}(\overline{\Omega})} \leq C(\|v^*_K\|_{C^{1,\alpha}(\Omega)} + \|\nabla \phi^*_K\|_{C^{1,\alpha}(\overline{\Omega})} + \|b\|_{C^{1,\alpha}(\overline{\Omega})}).
\]

By Lemma 3.14, $\psi_K \in C^{2,\theta}(\overline{\Omega}) \subset C^{1,\alpha}(\overline{\Omega})$. We consider the system (3.13). We note that $a_{ij} \in C^{0,\alpha}(\overline{\Omega})$ and $h \in C^{0,\alpha}(\overline{\Omega})$. Thus $\psi_K \in C^{2,\alpha}(\overline{\Omega})$ and

\[
\|\psi_K\|_{C^{2,\alpha}(\overline{\Omega})} \leq C(\|\psi_K\|_{C^{0,\alpha}(\Omega)} + \|h\|_{C^{0,\alpha}(\overline{\Omega})}) \leq C_2
\]
where $C_2$ depends on $\Omega, \alpha, \|f_K\|_{C^1([0, \infty))}, \|Q_K\|_{C^{1,\alpha}(\overline{\Omega})}$ and $\|\psi_K\|_{C^{1,\alpha}(\overline{\Omega})}$. Therefore we have $Q_K + \nabla \psi_K \in C^{1,\alpha}(\overline{\Omega}, \mathbb{R}^3)$. So the right hand side of the first equation of (3.11) belongs to $C^{1,\alpha}(\overline{\Omega}, \mathbb{R}^3)$. Hence from the regularity of the div-curl system (3.11), we see that $v_K^* \in C^{2,\alpha}(\overline{\Omega}, \mathbb{R}^3)$ and

$$\|v_K^*\|_{C^{2,\alpha}(\overline{\Omega})} \leq C(\|\text{curl} A_K^*\|_{C^{1,\alpha}(\overline{\Omega})} + \|A_T^0\|_{C^{2,\alpha}(\overline{\Omega})}).$$

Therefore, from (3.12) we find $\phi_K^* \in C^{3,\alpha}(\overline{\Omega})$ and we can write

$$A_K^* = v_K^* + \nabla \phi_K^* \in C^{2,\alpha}(\overline{\Omega}, \mathbb{R}^3) + \text{grad} C^{3,\alpha}(\overline{\Omega}).$$

This completes the proof of Proposition 3.6.

**Proposition 3.16.** Let $0 < \alpha < 1$, $0 < K < b^2$ and $A_K^* = v_K^* + \nabla \phi_K^* \in C^{2,\alpha}(\overline{\Omega}, \mathbb{R}^3) + \text{grad} C^{3,\alpha}(\overline{\Omega}) \subset C^{2,\alpha}(\overline{\Omega}, \mathbb{R}^3)$ be the minimizer of $S_K^*$ in $X^2$. Then for any $0 < \sigma < \alpha$, we have

$$\lim_{n \to \infty} \|A_{K,n}^*\|_{C^{2,\alpha}(\overline{\Omega})} = 0.$$  

**Proof.** Assume that the conclusion is false. Then there exist $0 < \sigma < \alpha$, $A_{0,n,T}^* \in C^{2,\alpha}(\partial \Omega, \mathbb{R}^3)$ and $b_n \in C^{2,\alpha}(\overline{\Omega}, \mathbb{R}^3)$ such that $A_{0,n,T}^* \to 0$ in $C^{2,\alpha}(\partial \Omega, \mathbb{R}^3)$, $b_n \to 0$ in $C^{2,\alpha}(\overline{\Omega}, \mathbb{R}^3)$ as $n \to \infty$ and

$$\lim_{n \to \infty} \|A_{K,n}^*\|_{C^{2,\alpha}(\overline{\Omega})} > 0 \quad (3.14)$$

where $A_{K,n}^*$ are corresponding solutions. By Proposition 3.6, $A_{K,n}^* \in C^{2,\alpha}(\overline{\Omega}, \mathbb{R}^3)$ and $\{A_{K,n}^*\}$ is bounded in $C^{2,\alpha}(\overline{\Omega}, \mathbb{R}^3)$. Therefore there exist a subsequence $\{A_{K,n_j}^*\}$ and $A^0 \in C^{2,\alpha}(\overline{\Omega}, \mathbb{R}^3)$ such that $A_{K,n_j}^* \to A^0$ in $C^{2,\sigma}(\overline{\Omega}, \mathbb{R}^3)$, $A^0 \neq 0$ in $\Omega$ from (3.14) and $A_{0,T}^* = 0$ on $\partial \Omega$. From (2.6), for any $H \in C^1(\overline{\Omega}, \mathbb{R}^3) := \{H \in C^0(\overline{\Omega}, \mathbb{R}^3); H_T = 0 \text{ on } \partial \Omega\}$, we have

$$\int_{\Omega} S_K^* (|\text{curl} A_{K,n_j}^*|^2) \text{curl} A_{K,n_j}^* \cdot \text{curl} H \, dx + \int_{\Omega} (M A_{K,n_j}^*) \cdot H \, dx + \int_{\Omega} b_{n_j} \cdot H \, dx = 0.$$

Letting $j \to \infty$, it follows that

$$\int_{\Omega} S_K^* (|A^0|^2) \text{curl} A^0 \cdot \text{curl} H \, dx + \int_{\Omega} (M A^0) \cdot H \, dx = 0.$$

If we put $H = A^0$ and note that $S_K^* > 0$ and the matrix $M$ is positive definite, we obtain $A^0 = 0$. This leads to a contradiction. \qed
Proof of Theorem 1.1

Fix $K$ such that $0 < K < b^2$, and construct the function $S_K$ as in section 2. By Lemma 3.1, for any $A_0^T \in C^{2,\alpha}(\partial \Omega, \mathbb{R}^3)$ and $b \in C^{1,1}(\Omega, \mathbb{R}^3)$, the system (2.4) has a unique weak solution $A_K^* = v_K^* + \nabla \phi_K^* \in X^2$. By Proposition 3.6, we see that $A_K^* \in C^{2,\alpha}(\Omega, \mathbb{R}^3)$. Moreover, by Proposition 3.16, there exists $R_0 > 0$ such that if $\|A_0^T\|_{C^{2,\alpha}(\partial \Omega)} + \|b\|_{C^{1,1}(\Omega)} \leq R_0$, then $\|\text{curl } A_K^*\|_{C^0(\Omega)} \leq \sqrt{K}$.

Thus $S'_K(|\text{curl } A_K^*|^2) = S'(\text{curl } A_K^*)$. Hence $A_K^*$ is a classical solution of (1.8). The uniqueness of classical solution satisfying $\|\text{curl } A_K^*\|_{C^0(\Omega)} \leq \sqrt{K}$ follows from the uniqueness of weak solution of the system (2.4).

Remark 3.17. When $F(x, A)$ is of the form $F(x, A) = a(x)|A|^2$ where $a(x) > 0$ is a scalar function, [6] showed that if the boundary data is small, then $A_K^*$ is a classical solution of (1.8). However, in our case where $F(x, A)$ is of the form (1.7), in order to get a classical solution of (1.8), not only the boundary data but also the term $b$ in (1.7) must be small.

4 The Neumann problem.

In this section, we consider the Neumann problem (2.5). Remember that the corresponding extended functional is

$$
\mathcal{H}^+_K[A] = \int_\Omega (S_K(|\text{curl } A|^2) + \langle M(x)A, A \rangle + b(x) \cdot A + c(x))dx + 2 \int_{\partial \Omega} (D_T \times A_T) \cdot \nu ds
$$

and the Euler-Lagrange equation is

$$
\begin{cases}
\text{curl } (S_K'(|\text{curl } A|^2)\text{curl } A) + M(x)A + b(x) = 0 & \text{in } \Omega, \\
S_K'(|\text{curl } A|^2)(\text{curl } A)_T = D_T & \text{on } \partial \Omega.
\end{cases}
$$

Define

$$
a(\mathcal{H}^+_K, D_T) = \inf_{A \in \mathcal{H}^2(\Omega, \text{curl})} \mathcal{H}^+_K[A].
$$

We can prove the following proposition as similar as [6].

Proposition 4.1. Let $\Omega$ be a bounded domain in $\mathbb{R}^3$ with a $C^2$ boundary and $D_T \in H^{1/2}(\partial \Omega)$. Then $\mathcal{H}^+_K$ has a unique minimizer $A_K \in \mathcal{H}^2(\Omega, \text{curl})$.

Therefore the minimizer $A_K$ is a weak solution of (4.2) in the sense of (2.7). We shall examine the regularity of the minimizer.
Proposition 4.2. Assume that $\Omega, M(x), b(x), c(x)$ and $D_T$ satisfy (A1), (A2) and (A4) with $0 < \alpha < 1$, respectively, and let $0 < K < b^2$ and $A_K \in \mathcal{H}^2(\Omega, \text{curl})$ be a minimizer given in Proposition 4.1. Then

$$A_K = v_K + \nabla \phi_K \in C^{2,\alpha}_0(\Omega, \text{div} 0, \mathbb{R}^3) + \text{grad} C^{2,\alpha}_0(\Omega)$$

where

$$C^{2,\alpha}_0(\Omega, \text{div} 0, \mathbb{R}^3) = \{ A \in C^{2,\alpha}(\Omega); \text{div} A = 0 \text{ in } \Omega, v \cdot A = 0 \text{ on } \partial \Omega \}$$

and

$$\| v_K \|_{C^{2,\alpha}(\Omega)} + \| \nabla \phi_K \|_{C^{1,\alpha}(\Omega)} \leq C$$

where the constant $C$ depends on $\Omega, \alpha, m_0, \| M \|_{C^{1,\alpha}(\Omega)}, K, \delta, \| D_T \|_{C^{2,\alpha}(\Omega)}$ and $\| b \|_{C^{1,\alpha}(\Omega)}$.

Corollary 4.3. In addition to the conditions of Proposition 4.2, assume that $S_K \in C^4([0, \infty)), f_K \in C^3([0, \infty))$ and $\partial \Omega$ is of class $C^{4,\alpha}$, and $M \in C^{2,\alpha}(\Omega, \mathbb{R}^3), b \in C^{2,\alpha}(\Omega, \mathbb{R}^3)$ and $v \cdot \text{curl} D_T \in C^{2,\alpha}(\Omega)$. Then

$$A_K = v_K + \nabla \phi_K \in C^{3,\alpha}(\Omega, \text{div} 0, \mathbb{R}^3) + \text{grad} C^{3,\alpha}(\Omega)$$

and

$$\| v_K \|_{C^{3,\alpha}(\Omega)} + \| \nabla \phi_K \|_{C^{2,\alpha}(\Omega)} \leq C$$

where $C$ depends on $\Omega, \alpha, m_0, \| M \|_{C^{2,\alpha}(\Omega)}, \| b \|_{C^{2,\alpha}(\Omega)}, K, \delta, \| D_T \|_{C^{2,\alpha}(\Omega)}$ and $\| v \cdot \text{curl} D_T \|_{C^{2,\alpha}(\Omega)}$.

The proof of Proposition 4.2 consists of several lemmas.

The following two lemma are due to [6].

Lemma 4.4. Let $\Omega$ be a bounded domain in $\mathbb{R}^3$ with a $C^2$ boundary.

(i) Let $w \in L^2(\Omega, \mathbb{R}^3)$. Then $w \in H^2(\Omega, \text{div})$ if and only if there exists a constant $C > 0$ such that

$$\int_{\Omega} \nabla \eta \cdot w \, dx \leq C \| \eta \|_{L^2(\Omega)}$$

for all $\eta \in C^\infty_0(\Omega)$.

(ii) Let $f \in L^2(\Omega), g \in H^{-1/2}(\partial \Omega)$ and $w \in H^2(\Omega, \text{div})$. Then $w$ satisfies the equation $\text{div} w = f$ in $\Omega$, $v \cdot w = g$ on $\partial \Omega$ if and only if

$$\int_{\Omega} \nabla \eta \cdot w \, dx = - \int_{\Omega} \eta f \, dx + \langle \eta, g \rangle_{H^{1/2}(\partial \Omega), H^{-1/2}(\partial \Omega)}$$

for all $\eta \in C^1(\Omega)$ where $(\cdot, \cdot)_{H^{1/2}(\partial \Omega), H^{-1/2}(\partial \Omega)}$ denotes the duality of $H^{1/2}(\partial \Omega)$ and $H^{-1/2}(\partial \Omega)$. In this case the above equality also holds for any $\eta \in H^1(\Omega)$. 

Lemma 4.5. Let $\Omega$ be a bounded domain in $\mathbb{R}^3$ without holes and with a $C^2$ boundary, and let $D_T \in H^{1/2}(\Omega)$. Let $D^e \in H^1(\Omega, \mathbb{R}^3)$ be a divergence-free extension of $D_T$. If $A_K \in \mathcal{H}^2(\Omega, \text{curl})$ is a minimizer of $\mathcal{H}_K$, then $M(x)A_K(x) + b(x) \in \mathcal{H}^2(\Omega, \text{div})$, $\nu \cdot (MA_K + b), \nu \cdot \text{curl} D^e \in H^{-1/2}(\partial \Omega)$, and

$$
\begin{aligned}
\left\{ \begin{array}{ll}
\text{div} (MA_K + b) = 0 & \text{in } \Omega, \\
\nu \cdot (MA_K + b) + \nu \cdot \text{curl} D^e = 0 & \text{on } \partial \Omega.
\end{array} \right.
\end{aligned}
$$

Proof. Since from (A2), $MA_K + b \in L^2(\Omega, \mathbb{R}^3)$ and from (4.2), we can see that $\text{div} (MA_K + b) = 0$ in $\Omega'$ and $(MA_K + b) \in \mathcal{H}^2(\Omega, \text{div} 0)$. Hence by Lemma 2.1, $\nu \cdot (MA_K + b) \in H^{-1/2}(\partial \Omega)$. Since $\text{curl} D^e \in L^2(\Omega, \mathbb{R}^3)$ and $\text{div} (\text{curl} D^e) = 0$ in $\Omega$, we also have $\nu \cdot \text{curl} D^e \in H^{-1/2}(\partial \Omega)$. From (4.2), we have

$$
\begin{aligned}
\nu \cdot (MA_K + b) &= -\nu \cdot \text{curl} (S_K'(|\text{curl} A_K|^2)\text{curl} A_K) \\
&= -\nu \cdot \text{curl} (S_K'(|\text{curl} A_K|^2)(\text{curl} A_K)_T) \\
&= -\nu \cdot \text{curl} D^e.
\end{aligned}
$$

We consider the following div-curl system.

$$
\left\{ \begin{array}{ll}
\text{curl} P = -M(x)A_K(x) - b(x) & \text{in } \Omega, \\
\text{div} P = 0 & \text{in } \Omega, \\
P_T = D_T & \text{on } \partial \Omega.
\end{array} \right.
$$

(4.5)

Lemma 4.6. The system (4.5) has a unique solution $P_K \in H^1(\Omega, \text{div} 0)$.

Proof. If we put $\tilde{P} = P - D^e$, then we see that (4.5) has a solution $P_K \in H^1(\Omega, \text{div} 0)$ if and only if the system

$$
\left\{ \begin{array}{ll}
\text{curl} P = -M(x)A_K(x) - b(x) - \text{curl} D^e & \text{in } \Omega, \\
\text{div} P = 0 & \text{in } \Omega, \\
P_T = 0 & \text{on } \partial \Omega.
\end{array} \right.
$$

(4.6)

has a solution $\tilde{P}_K \in H^1_{10}(\Omega, \text{div} 0)$. In fact, Since $\Omega$ is simply connected, it follows from [8, Proposition 4 and Remark 5] that

$$
\{ u \in L^2(\Omega, \mathbb{R}^3); \text{div} u = 0 \text{ in } \Omega, \nu \cdot u = 0 \text{ on } \partial \Omega \} = \text{curl} H^1_{10}(\Omega, \mathbb{R}^3) = \text{curl} H^1_{10}(\Omega, \text{div} 0).
$$

If we note that from Lemma 4.5,

$$
\text{div} (MA_K + b + \text{curl} D^e) = \text{div} (MA_K + b) = 0 \text{ in } \Omega,
$$

and $\nu \cdot (MA_K + b + \text{curl} D^e) = 0$ on $\partial \Omega$, then we see that (4.6) has a solution $\tilde{P}_K \in H^1_{10}(\Omega, \text{div} 0)$. Since $\Omega$ has no holes, the uniqueness of solution of (4.5) follows. 

$\square$
Let \( A_K \in \mathcal{H}^2(\Omega, \text{curl}) \) be a minimizer of \( \mathcal{H}^+_{K_0} \). Then we can decompose \( A_K \) so that \( A_K = v_K + \nabla \phi_K \) where \( v_K \in \mathcal{H}^2_{K_0}(\Omega, \text{curl}, \text{div} 0) := \{ v \in \mathcal{H}^2(\Omega, \text{curl}, \text{div}); \text{div} v = 0 \text{ in } \Omega, v \cdot \nu = 0 \text{ on } \partial \Omega \} \) and \( \phi_K \in H^1(\Omega) \). In fact, since \( \Omega \) is simply connected and has no holes, the following div-curl system

\[
\begin{aligned}
\text{curl} v &= \text{curl} A_K \quad \text{in } \Omega, \\
\text{div} v &= 0 \quad \text{in } \Omega, \\
v \cdot \nu &= 0 \quad \text{on } \partial \Omega
\end{aligned}
\]

has a unique solution \( v_K \in H^1(\Omega, \mathbb{R}^3) \) (cf. [15, Lemma 5.7] or [1, Theorem 3.3]). Since \( \text{curl}(A_K - v_K) = 0 \) in \( \Omega \) and \( \Omega \) is simply connected, there exists a function \( \phi_K \in H^1(\Omega) \) such that \( A_K = v_K + \nabla \phi_K \).

**Lemma 4.7.** If \( \nu \cdot \text{curl} D_T \in H^{1/2}(\partial \Omega) \), then \( \phi_K \in H^2(\Omega) \) and

\[
\| \nabla \phi_K \|_{H^2(\Omega)} \leq C(\| v_K \|_{L^2(\Omega)} + \| \nu \cdot \text{curl} D_T \|_{H^{1/2}(\partial \Omega)} + \| b \|_{H^1(\Omega)})
\]

where \( C \) depends on \( \Omega, m_0 \) and \( \| M \|_{C^1(\overline{\Omega})} \).

**Proof.** Taking (4.2) into consideration, we see that \( \phi_K \) satisfies the following Neumann equation

\[
\begin{aligned}
\text{div} (M(x)\nabla \phi) &= -\text{div} (M(x)v_K) - \text{div} b \quad \text{in } \Omega, \\
v \cdot (M(x)\nabla \phi) &= -\nu \cdot \text{curl} D_T - \nu \cdot (Mv_K) - \nu \cdot b \quad \text{on } \partial \Omega.
\end{aligned}
\]

We note that the compatibility condition holds. By [10, p. 160] or Murata and Kurata [14, Theorem 2.38], we see that \( \phi_K \in H^2(\Omega) \) and

\[
\begin{aligned}
\| \phi_K \|_{H^2(\Omega)} &\leq C(\| \phi_K \|_{L^2(\Omega)} + \| \text{div} (Mv_K) \|_{L^2(\Omega)} + \| \text{div} b \|_{L^2(\Omega)}) \\
&+ \| \nu \cdot \text{curl} D_T \|_{H^{1/2}(\partial \Omega)} + \| \nu \cdot (Mv_K) \|_{H^{1/2}(\partial \Omega)} \\
&+ \| \nu \cdot b \|_{H^{1/2}(\partial \Omega)}) \\
&\leq C_1(\| \phi_K \|_{L^2(\Omega)} + \| v_K \|_{H^1(\Omega)} + \| b \|_{H^1(\Omega)}) \\
&+ \| \nu \cdot \text{curl} D_T \|_{H^{1/2}(\partial \Omega)})
\end{aligned}
\]

where \( C_1 \) depends on \( \Omega, m_0 \) and \( \| M \|_{C^1(\overline{\Omega})} \). We may assume that \( \int_{\Omega} \phi_K dx = 0 \). Then we can remove the term \( \| \phi_K \|_{L^2(\Omega)} \) in the right hand side of the above inequality by the same reason as Remark 3.4. \( \square \)

We examine the regularity of \( P_K \).

**Lemma 4.8.** Assume that \( \Omega \) is simply connected bounded domain in \( \mathbb{R}^3 \) without holes, and with a \( C^3 \) boundary. Let \( P_K \in H^1(\Omega, \text{div} 0) \) be a unique solution of (4.5). Then \( P_K \in H^2(\Omega, \text{div} 0) \) and

\[
\| P_K \|_{H^2(\Omega)} \leq C(\| \text{curl} A_K \|_{L^2(\Omega)} + \| D_T \|_{H^{3/2}(\partial \Omega)} + \| \nu \cdot \text{curl} D_T \|_{H^{1/2}(\partial \Omega)})
\]

where \( C \) depends on \( \Omega, m_0 \) and \( \| M \|_{C^1(\overline{\Omega})} \).
Proof. Since $v_K \in H^1_{\text{div}}(\Omega, \text{div} 0)$ and $\Omega$ is simply connected, it follows from [8] that

$$\|v_K\|_{H^1(\Omega)} \leq C(\Omega) \|\text{curl } v_K\|_{L^2(\Omega)} = C(\Omega) \|\text{curl } A_K\|_{L^2(\Omega)}. \quad (4.9)$$

By Lemma 4.7 and (4.9), $M A_K + b = M(v_K + \nabla \phi_K) + b \in H^1(\Omega, \mathbb{R}^3)$, and

$$\|M A_K + b\|_{H^1(\Omega)} \leq C(\Omega, \|M\|_{C^1(\Omega)}) (\|v_K\|_{H^1(\Omega)} + \|
abla \phi_K\|_{H^1(\Omega)})$$

$$+ \|b\|_{H^1(\Omega)} \leq C(\Omega, \|M\|_{C^1(\Omega)}) (\|A_K\|_{H^1(\Omega)} + \|b\|_{H^1(\Omega)})$$

+ $\|D_T\|_{H^{3/2}(\partial \Omega)}$.

By the regularity of the div-curl system (4.5), we see that $P_K \in H^2(\Omega, \mathbb{R}^3)$ and the concluding estimate holds.

By the similar arguments as section 3, there exists a function $\psi_K \in H^1(\Omega)$ such that

$$S'_K(|\text{curl } v_K|^2) \text{curl } v_K = P_K + \nabla \psi_K.$$ 

Since

$$D_T = S'_K(|\text{curl } A_K|^2)(\text{curl } A_K) = (P_K + \nabla \psi_K)_T = D_T + (\nabla \psi)_T,$$

we have $(\nabla \psi_K)_T = 0$. Since $\partial \Omega$ is connected as $\Omega$ has no holes, we may assume that $\psi_K = 0$ on $\partial \Omega$. Thus from (2.11) $\psi_K$ is a weak solution of the following equation.

$$\begin{cases} \text{div } (f_K(|P_K + \nabla \psi|^2)(P_K + \nabla \psi)) = 0 & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial \Omega. \end{cases} \quad (4.10)$$

Similarly as (3.10), we have

$$\begin{cases} \text{curl } v_K = \text{curl } A_K = f_K(|P_K + \nabla \psi_K|^2)(P_K + \nabla \psi_K) & \text{in } \Omega, \\ \text{div } v_K = 0 & \text{in } \Omega, \\ \nu \cdot v_K = 0 & \text{on } \partial \Omega. \end{cases} \quad (4.11)$$

We examine $W^{1,q}$ regularity of $\psi_K$.

**Lemma 4.9.** Under the condition of Lemma 4.8, we have $\psi_K \in W^{1,q}(\Omega)$ for any $1 < q < \infty$ and

$$\|\psi_K\|_{W^{1,q}(\Omega)} \leq C(a_K + \|\text{curl } A_K\|_{L^2(\Omega)} + \|D_T\|_{H^{3/2}(\partial \Omega)})$$

$$+ \|\nu \cdot \text{curl } D_T\|_{H^{1/2}(\partial \Omega)} + \|b\|_{H^1(\Omega)}.$$

where $C$ depends on $\Omega, q, K, \delta, d_0, \|M\|_{C^1(\Omega)}$ and $m_0$. 
Since the proof is similar as that of Lemma 3.8, we omit it.

We note that by the Sobolev embedding theorem $W^{1,q}(\Omega) \hookrightarrow C^{0,\tau}(\Omega)$ for $\tau = 1 - 3/q > 0$, so we have $\psi_K \in C^{0,\tau}(\Omega)$ for any $\tau \in (0, 1)$ and $\|\psi_K\|_{C^{0,\tau}(\Omega)} \leq C(\Omega, \tau)\|\psi_K\|_{W^{1,q}(\Omega)}$.

We examine $W^{1,q}$ regularity of $v_K$.

**Lemma 4.10.** Under the condition of Lemma 4.8, we have $v_K \in W^{1,q}(\Omega, \mathbb{R}^3)$ for any $1 < q < \infty$ and

$$\|v_K\|_{W^{1,q}(\Omega)} \leq C(\Omega, q)\|\text{curl} A_K\|_{L^2(\Omega)} \leq C(\Omega, q)d_0\|P_K + \nabla \psi_K\|_{L^q(\Omega)}.$$  

In particular, $v_K \in C^{0,\tau}(\Omega)$ for any $\tau \in (0, 1)$ and

$$\|v_K\|_{C^{0,\tau}(\Omega)} \leq C(\Omega, \tau)\|v_K\|_{W^{1,q}(\Omega)}.$$  

**Proof.** By Lemma 4.8 and 4.9, we see that $P_K + \nabla \psi_K \in L^q(\Omega)$ for any $1 < q < \infty$. Thus

$$\text{curl} A_K = f_K(|P_K + \nabla \psi_K|^2)(P_K + \nabla \psi_K) \in L^q(\Omega, \mathbb{R}^3).$$

Since $v_K \in H^1(\Omega, \mathbb{R}^3)$ is a unique solution of (4.7) and $\Omega$ is simply connected, it follows from the regularity of div-curl system (4.7) (cf. [1]) that $v_K \in W^{1,q}(\Omega, \mathbb{R}^3)$ and

$$\|v_K\|_{W^{1,q}(\Omega)} \leq C(\Omega, q)\|\text{curl} A_K\|_{L^q(\Omega)} \leq C(\Omega, q)d_0\|P_K + \nabla \psi_K\|_{L^q(\Omega)}.$$  

\[\square\]

**Proof of Proposition 4.2.**

Step 1. $C^{1,\tau}$ estimate of $\phi_K$.

We know that $\phi_K \in H^2(\Omega)$ is a solution of (4.8). Since $v_K \in W^{1,q}(\Omega, \mathbb{R}^3)$ for any $1 < q < \infty$ from Lemma 4.10, we see that $\text{div}(Mv_K) + \text{div} b \in L^2(\Omega)$, and by the hypotheses and Lemma 4.10, we see that $v \cdot \text{curl} D_T - v \cdot (Mv_K) - \nu \cdot b \in W^{1-1/q, q}(\partial \Omega)$. Therefore we get $\phi_K \in W^{2,q}(\Omega) \hookrightarrow C^{1,1-3/q}(\Omega)$ for $1 - 3/q > 0$. Since $q$ is arbitrary, $\phi_K \in C^{1,\tau}(\Omega)$ for any $\tau \in (0, 1)$, and

$$\|\phi_K\|_{C^{1,\tau}(\Omega)} \leq C(\Omega, \tau, q)(\|\phi_K\|_{L^q(\Omega)} + \|v_K\|_{W^{1,q}(\Omega)} + \|b\|_{W^{1,q}(\Omega)} + \|v \cdot \text{curl} D_T\|_{W^{1-1/q, q}(\partial \Omega)}). \quad (4.12)$$

We can remove $\|\phi_K\|_{L^q(\Omega)}$ in the right hand side by using Lemma 4.7 and the Poincaré inequality.

Step 2. $C^{1,\tau}$ estimate of $P_K$. 

By Lemma 4.10 and Step 1, for any \(0 < \tau < 1\), \(MA_K = M(v_K + \nabla \phi_K) \in C^{0,\tau}(\Omega, \mathbb{R}^3)\). Since \(P_K\) is a solution of the div-curl system (4.5), it follows from \([15]\) that \(P_K \in C^{1,\tau}(\Omega, \mathbb{R}^3)\) and

\[
\|P_K\|_{C^{1,\tau}(\Omega)} \leq C(\Omega, \tau)\|MA_K + b\|_{C^{0,\tau}(\Omega)} + \|D_T\|_{C^{1,\tau}(\partial\Omega)}
\]

\[
\leq C(\|v_K\|_{C^{0,\tau}(\Omega)} + \|\nabla \phi_K\|_{C^{0,\tau}(\Omega)} + \|b\|_{C^{0,\tau}(\Omega)} + \|D_T\|_{C^{1,\tau}(\partial\Omega)})
\]

where \(C\) depends on \(\Omega, \tau\) and \(\|M\|_{C^{0,\tau}(\Omega)}\).

Step 3. \(C^{1,\theta}\) estimate of \(\psi_K\) for some \(\theta \in (0, 1)\).

We consider the equation (4.10). By Step 2, we see that

\[
A(x, z) = f_K(|P_K(x) + z|^2)(P_K(x) + z) \in C^{1,\tau}(\Omega \times \mathbb{R}^3, \mathbb{R}^3).
\]

By (2.14), we have

\[
\sum_{i,j=1}^3 \frac{\partial A_i}{\partial z_j}(x, z) \xi_i \xi_j \geq \lambda |\xi|^2
\]

for all \(\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3\) where \(\lambda = \lambda(K, \delta)\), and by the properties of \(f_K\), we have

\[
(1 + |z|^2) \left| \frac{\partial A_i}{\partial z_j}(x, z) \right| + (1 + |z|) \left( \left| \frac{\partial A_i}{\partial x_j}(x, z) \right| + |A_i(x, z)| \right) \leq \Lambda(1 + |z|^2)
\]

for some \(\Lambda > 0\). According to \([10, \text{Chapter 4, Theorem 6.5}]\), there exists \(\theta \in (0, 1)\) and \(C > 0\) depending on \(\Omega, \|P_K\|_{C^{1,\theta}(\Omega)}, \|\psi_K\|_{C^{0,\theta}(\Omega)}, \lambda\) and \(\Lambda\) such that \(\psi_K \in C^{1,\theta}(\Omega)\) and \(\|\psi_K\|_{C^{1,\theta}(\Omega)} \leq C\).

Step 4. \(C^{1,\theta}\) estimate of \(A_K\).

Since by Step 2 and 3, \(P_K + \nabla \psi_K \in C^{0,\theta}(\Omega, \mathbb{R}^3)\), it follows from the regularity of the div-curl system (4.11) that we see that \(v_K \in C^{1,\theta}(\Omega, \mathbb{R}^3)\) and

\[
\|v_K\|_{C^{1,\theta}(\Omega)} \leq C(\Omega, \theta)\|A_K\|_{C^{0,\theta}(\Omega)}
\]

\[
\leq C(\|P_K\|_{C^{0,\theta}(\Omega)} + \|\nabla \psi_K\|_{C^{0,\theta}(\Omega)})
\]

where \(C\) depends on \(\Omega, \theta, f_K\). Here we used the first equation of (4.11). If we return to the equation (4.8) and apply the Schauder theory, then we see \(\phi_K \in C^{2,\theta}(\Omega)\) and

\[
\|\phi_K\|_{C^{2,\theta}(\Omega)} \leq C(\|v_K\|_{C^{1,\theta}(\Omega)} + \|b\|_{C^{1,\theta}(\Omega)} + \|\nu \cdot \text{curl} D_T\|_{C^{1,\theta}(\partial\Omega)} + \|\phi_K\|_{C^{0}(\Omega)})
\]

Thus we have \(A_K = v_K + \nabla \phi_K \in C^{1,\theta}(\Omega, \mathbb{R}^3)\) and

\[
\|A_K\|_{C^{1,\theta}(\Omega)} \leq C(\|P_K\|_{C^{0,\theta}(\Omega)} + \|\nabla \psi_K\|_{C^{0,\theta}(\Omega)} + \|b\|_{C^{1,\theta}(\Omega)} + \|\nu \cdot \text{curl} D_T\|_{C^{1,\theta}(\partial\Omega)})
\]
where $C$ depends on $\Omega, \theta, f_K, m_0$ and $\|M\|_{C^{1,1}(\overline{\Omega})}$.

Step 5. $C^{2,\theta}$ estimate of $\psi_K$.
By Step 4 and the regularity of the div-curl system (4.5), we see that $P_K \in C^{2,\theta}(\overline{\Omega}, \mathbb{R}^3)$ and

$$
\|P_K\|_{C^{2,\theta}(\overline{\Omega})} \leq C(\|A_K\|_{C^{1,\theta}(\overline{\Omega})} + \|b\|_{C^{1,\theta}(\overline{\Omega})} + \|D_T\|_{C^{2,\theta}(\partial\Omega)})
$$

where $C$ depends on $\Omega, \theta$ and $\|M\|_{C^{1,1}(\overline{\Omega})}$. We rewrite the equation (4.10) into the form

$$
\begin{cases}
\sum_{i,j=1}^3 a_{ij}(x) \frac{\partial^2 \psi_K}{\partial x_i \partial x_j} = h(x) & \text{in } \Omega, \\
\psi_K = 0 & \text{on } \partial \Omega
\end{cases}
$$

(4.13)

where

$$
a_{ij}(x) = f_K(\|P_K + \nabla \psi_K\|^2) \delta_{ij} + 2f_K'(\|P_K + \nabla \psi_K\|^2)(P_{K,i} + \partial_i \psi_K)(P_{K,j} + \partial_j \psi_K),
$$

$$
h(x) = -2f_K'(\|P_K + \nabla \psi_K\|^2) \langle \nabla P_K(P_K + \nabla \psi_K), P_K + \nabla \psi_K \rangle - f_K(\|P_K + \nabla \psi_K\|^2) \text{div } P_K.
$$

Since $a_{ij}, h \in C^{0,\theta}(\overline{\Omega})$ and (4.13) is uniformly elliptic with lower bound $\lambda = \lambda(K, \delta) > 0$ from (2.14), we see that $\psi_K \in C^{2,\theta}(\overline{\Omega})$ and

$$
\|\psi_K\|_{C^{2,\theta}(\overline{\Omega})} \leq C\|h\|_{C^{0,\theta}(\overline{\Omega})} \leq C_1
$$

where $C_1$ depends on $\Omega, \theta$, $\|P_K\|_{C^{1,\theta}(\overline{\Omega})}$, $\|\nabla \psi_K\|_{C^{0,\theta}(\overline{\Omega})}$, $f_K$ and $f'_K$.

Step 6. $C^{2,\theta}$ estimate of $v_K$.
By Step 2 and 5, $P_K + \nabla \psi_K \in C^{1,\theta}(\overline{\Omega}, \mathbb{R}^3)$. By the regularity of div-curl system (4.11), we see that $v_K \in C^{2,\theta}(\overline{\Omega}, \mathbb{R}^3)$ and $\|v_K\|_{C^{2,\theta}(\overline{\Omega})} \leq C(\Omega, \theta)\|\text{curl } A_K\|_{C^{1,\theta}(\overline{\Omega})}$. In particular, since $v_K \in C^{1,\alpha}(\overline{\Omega}, \mathbb{R}^3)$, it follows form (4.8) that $\phi_K \in C^{2,\alpha}(\overline{\Omega})$ and

$$
\|\phi_K\|_{C^{2,\alpha}(\overline{\Omega})} \leq C(\|v_K\|_{C^{1,\alpha}(\overline{\Omega})} + \|b\|_{C^{1,\alpha}(\overline{\Omega})} + \|\nu \cdot \text{curl } D_T\|_{C^{1,\alpha}(\overline{\Omega})})
$$

where $C$ depends on $\Omega, \alpha$ and $\|M\|_{C^{1,\alpha}(\overline{\Omega})}$.

End of the proof of Proposition 4.2.
We have $A_K = v_K + \nabla \phi_K \in C^{2,\theta}(\overline{\Omega}, \mathbb{R}^3) + \text{grad} C^{2,\alpha}(\overline{\Omega}) \subset C^{1,\alpha}(\overline{\Omega}, \mathbb{R}^3)$.

When $\{\theta, \alpha\} = \alpha$, the proof is done.

When $\{\theta, \alpha\} = \theta$, since $A_K \in C^{1,\alpha}(\overline{\Omega}, \mathbb{R}^3)$ and $D_T \in C^{2,\alpha}(\partial \Omega, \mathbb{R}^3)$, taking (4.5) into consideration we see that $P_K \in C^{2,\alpha}(\overline{\Omega}, \mathbb{R}^3)$ and

$$
\|P_K\|_{C^{2,\alpha}(\overline{\Omega})} \leq C(\|v_K\|_{C^{1,\alpha}(\overline{\Omega})} + \|\nabla \phi_K\|_{C^{1,\alpha}(\overline{\Omega})} + \|D_T\|_{C^{2,\alpha}(\partial\Omega)})
$$

where $C$ depends on $\Omega, \alpha$ and $\|M\|_{C^{1,\alpha}(\overline{\Omega})}$. Therefore from (4.13), $\psi_K \in C^{2,\alpha}(\overline{\Omega})$ and $\|\psi_K\|_{C^{2,\alpha}(\overline{\Omega})} \leq C\|h\|_{C^{0,\alpha}(\overline{\Omega})} \leq C_1$ where $C_1$ depends $\Omega, \|P_K\|_{C^{1,\alpha}(\overline{\Omega})}, \|\nabla \psi_K\|_{C^{0,\alpha}(\overline{\Omega})}, \lambda$. 


and \( A \). Thus \( P_K + \nabla \psi_K \in C^{1,\alpha}(\Omega, \mathbb{R}^3) \). From (4.7), we see that \( \nu_K \in C^{2,\alpha}(\Omega, \mathbb{R}^3) \) and

\[
\| \nu_K \|_{C^{2,\alpha}(\Omega)} \leq C(\Omega, \alpha) \| \nabla A_K \|_{C^{1,\alpha}(\Omega)}.
\]

(4.14)

Hence we can write \( A_K = \nu_K + \nabla \phi_K \in C^{2,\alpha}(\Omega, \mathbb{R}^3) + \text{grad}C^{2,\alpha}(\Omega) \). This completes the proof of Proposition 4.2.

**Proof of Corollary 4.3.**

First we consider the equation (4.8). By the hypotheses and (4.14), \(-\text{div}(M\nu_K) - \text{div} b \in C^{1,\alpha}(\Omega)\) and \(-\nu \cdot D_T - \nu \cdot b \in C^{2,\alpha}(\Omega)\). Since \( \partial \Omega \) is of class \( C^{4,\alpha} \), we have \( \phi_K \in C^{3,\alpha}(\Omega) \) and

\[
\| \phi_K \|_{C^{3,\alpha}(\Omega)} \leq C(\| \nu_K \|_{C^{2,\alpha}(\Omega)} + \| b \|_{C^{2,\alpha}(\Omega)} + \| \nu \cdot \text{curl} D_T \|_{C^{2,\alpha}(\Omega)}).
\]

Next consider the equation (4.13). Since \( P_K \in C^{2,\alpha}(\Omega, \mathbb{R}^3) \) and \( \psi_K \in C^{2,\alpha}(\Omega, \mathbb{R}^3) \), we have \( h \in C^{1,\alpha}(\Omega) \). Therefore by the Schauder theory we have \( \psi_K \in C^{3,\alpha}(\Omega) \) and \( \| \psi_K \|_{C^{3,\alpha}(\Omega)} \leq C \| h \|_{C^{1,\alpha}(\Omega)} \leq C_1 \) where \( C_1 \) depends on \( \Omega, \alpha, \| P_K \|_{C^{2,\alpha}(\Omega)}, \| \psi_K \|_{C^{2,\alpha}(\Omega)}, f_K, f'_K \) and \( f''_K \).

Finally we consider the equation (4.11). Since \( f_K(\| P_K + \nabla \psi_K \|^2)(P_K + \nabla \psi_K) \in C^{2,\alpha}(\Omega, \mathbb{R}^3) \), it follows from the regularity estimate of the div-curl system (4.12) that we get \( \nu_K \in C^{3,\alpha}(\Omega, \mathbb{R}^3) \) and

\[
\| \nu_K \|_{C^{3,\alpha}(\Omega)} \leq C(\| P_K \|_{C^{2,\alpha}(\Omega)} + \| \nabla \psi_K \|_{C^{2,\alpha}(\Omega)}),
\]

where \( C \) depends on \( \Omega, \alpha \) and \( \| f_K \|_{C^{2,\alpha}} \). Hence we can write

\[
A_K = \nu_K + \nabla \phi_K \in C^{3,\alpha}(\Omega, \text{div} 0, \mathbb{R}^3) + \text{grad}C^{3,\alpha}(\Omega).
\]

This completes the proof of Corollary 4.3.

**Proposition 4.11.** Assume that \( \Omega, M(x), b(x) \) and \( D_T \) satisfy (A1), (A2) and (A4) with \( 0 < \alpha < 1 \), respectively. Let \( A_K = \nu_K + \nabla \phi_K \in C^{1,\alpha}(\Omega, \text{div} 0, \mathbb{R}^3) + \text{grad}C^{2,\alpha}(\Omega) \) be a solution of (4.2). Then for any \( 0 < \sigma < \alpha \), we have

\[
\lim_{\| D_T \|_{C^{2,\alpha}(\partial \Omega)} + \| b \|_{C^{1,\alpha}(\Omega)} \rightarrow 0} \| A_K \|_{C^{1,\sigma}(\Omega)} = 0.
\]

**Proof.** Assume that the conclusion is false. Then there exist \( 0 < \sigma < \alpha \), \( D_{n,T} \in C^{2,\alpha}(\partial \Omega, \mathbb{R}^3) \) and \( b_n \in C^{1,\alpha}(\Omega, \mathbb{R}^3) \) such that \( D_{n,T} \rightarrow 0 \) in \( C^{2,\alpha}(\partial \Omega, \mathbb{R}^3) \), \( b_n \rightarrow 0 \) in \( C^{1,\alpha}(\Omega, \mathbb{R}^3) \) and

\[
\liminf_{n \rightarrow \infty} \| A_{K,n} \|_{C^{1,\sigma}(\Omega)} > 0
\]

(4.15)

where \( A_{K,n} = \nu_{K,n} + \nabla \phi_{K,n} \) are corresponding solutions of (4.2) with \( b = b_n \) and \( D_T = D_{n,T} \), and \( \nu_{K,n} \in C^{1,\alpha}(\Omega, \text{div} 0, \mathbb{R}^3) \) and \( \phi_{K,n} \in C^{2,\alpha}(\Omega) \). Then from
the above estimates, we see that \( \{v_{K,n}\} \) and \( \{\phi_{K,n}\} \) are bounded in \( C^{2,\alpha}(\Omega, \mathbb{R}^3) \) and \( C^{2,\alpha}(\Omega) \), respectively. Since \( 0 < \sigma < \alpha \), passing to a subsequences, we may assume that \( A_{K,n} \to A^1 \) in \( C^{1,\sigma}(\Omega, \mathbb{R}^3) \). From (4.15), we have \( \|A^1\|_{C^{1,\sigma}(\Omega)} > 0 \).

If we take the inner product of the first equation of (4.2) with any \( H \in C^{1}(\Omega, \mathbb{R}^3) \), and then using the integration of parts, we have

\[
\int_{\Omega} S'_K(|\text{curl } A_{K,n}|^2) \text{curl } A_{K,n} \cdot \text{curl } H \, dx + \int_{\Omega} M A_{K,n} \cdot H \, dx \\
+ \int_{\Omega} b_n \cdot H \, dx + \int_{\partial\Omega} (D_{n,T} \times H_T) \cdot \nu \, dS = 0.
\]

Letting \( n \to \infty \), we get

\[
\int_{\Omega} S'_K(|\text{curl } A^1|^2) \text{curl } A^1 \cdot \text{curl } H \, dx + \int_{\Omega} M A^1 \cdot H \, dx = 0.
\]

Choosing \( H = A^1 \), it follows from the positivity of \( S'_K \) and positively definite-ness of the matrix \( M \) that \( A^1 = 0 \). This contradicts (4.15).

**Proof of Theorem 1.3.**

By Proposition 4.11, there exists a constant \( R_2 > 0 \) such that if \( \|D_T\|_{C^{2,\alpha}(\partial\Omega)} + \|b\|_{C^{1,\alpha}(\Omega)} \leq R_2 \), then \( \|\text{curl } A_K\|_{C^0(\Omega)} \leq \sqrt{K} \). Therefore \( A_K = v_K + \nabla \phi_K \in C^{2,\alpha}(\Omega, \text{div } 0, \mathbb{R}^3) + \text{grad} C^{2,\alpha}(\Omega) \) is a solution of (1.9). The uniqueness follows from the uniqueness of the weak solution of (4.2).

**Remark 4.12.** As similar as Remark 3.17, in our case where \( F(x, A) \) is of the form (1.7), in order to get a classical solution of (1.9), not only the boundary data but also the term \( b \) in (1.7) must be small.

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