Abstract

Given two vertices $u$ and $v$ of a connected graph $G$, the closed interval $I_G[u,v]$ is that set of all vertices lying in some $u$-$v$ geodesic in $G$. If $S \subseteq V(G)$, then $I_G[S] = \cup\{I_G[u,v] : u, v \in S\}$. Let $v_i \in V(G)$ for $i = 1, 2, ..., n$. We select vertices of $G$ as follows: Select $v_1$ and let $S_1 = \{v_1\}$. Select another vertex $v_2 \neq v_1$ and let $S_2 = \{v_1, v_2\}$. Then successively select vertex $v_k \in S_k' - 1$ and let

$$S_k = S_{k-1}' \cup \{v_k\} \cup \{u \in V(G) : u \in I_G[v_k, w] \text{ for some } w \in S_{k-1}'\}.$$ 

The sequential geodetic number of $G$, denoted by $sgn(G)$, is the smallest $k$ such that there is a sequence $\langle v_1, v_2, ..., v_k \rangle$ for which $S_k = V(G)$. The set $S = S_k' = \{v_1, v_2, ..., v_k\}$ with $v_1, v_2, ..., v_k \in S_k'$ for which $S_k = V(G)$ is a sequential geodetic cover of $G$. The sequential geodetic number is again inspired by the achievement and avoidance games. In this paper, some connected graphs $G$ with $sgn(G)$ equals $|V(G)| - 1$ and those equal to $|V(G)|$ are characterized. It is shown that the geodetic number ($gn$), closed geodetic number ($cgn$) and the sequential geodetic number ($sgn$) coincide for some particular graphs. Further, for the complete bipartite graph $K_{m,n}$ these three graph invariants are determined.
1 Introduction

Let $G$ be a connected graph. A $u$-$v$ geodesic, for vertices $u$ and $v$ in $G$, is any shortest path in $G$ joining $u$ and $v$. The length of a $u$-$v$ geodesic is called the distance $d_G(u, v)$ between $u$ and $v$. If $S \subseteq V(G)$, we define the closure of $S$, to be the set $I_G[S]$ given by $I_G[S] = \bigcup \{I_G[u, v] : u, v \in S\}$. By a geodetic cover of $G$ we mean a subset $S$ of $V(G)$ such that $I_G[S] = V(G)$. The number $gn(G)$ given by $gn(G) = \min\{|S| : I_G[S] = V(G)\}$ is called the geodetic number of $G$. A geodetic cover $S$ of $G$ with $|S| = gn(G)$ is called the geodetic basis of $G$. A subset $S$ of $V(G)$ is said to be a closed geodetic subset if there is a positive integer $k$ and a sequence of sets $S_1 = \{v_1\}$, $S_2 = \{v_1, v_2\}$, ..., $S_k = \{v_1, v_2, \ldots, v_k\}$ such that $S_k = S$ and $v_i \notin I_G[S_{i-1}]$ for all $i = 3, 4, \ldots, k$. A geodetic cover $S$ of $G$ is called a closed geodetic cover of $G$ if $S$ is a closed geodetic subset of $V(G)$. In this case, we refer to the set $S_k$ being a canonical representation of $S$. We denote by $C^*(G)$ the set of all closed geodetic covers of $G$. The closed geodetic number $cgn(G)$ of $G$ is given by

$$cgn(G) = \min\{|S| : S \in C^*(G)\}.$$

Some results concerning closed geodetic numbers of graphs are found in [?]

2 Sequential Geodetic Number of Some Graphs

For the purpose of this study, the following definitions of a geodetic sequence, sequential geodetic cover and sequential geodetic basis are introduced.

**Definition 2.1** Let $G$ be a connected graph. A sequence $\langle v_1, v_2, \ldots, v_k \rangle$ of the vertices in $G$ is a geodetic sequence if it generates a sequence $S_1, S_2, \ldots, S_k$ of subsets of $V(G)$ satisfying the following: (1) $v_1 \neq v_2$ for which $S_1 = \{v_1\}$ and $S_2 = \{v_1, v_2\}$; (2) $v_i \notin S_{i-1}'$ for $3 \leq i \leq k$, with $S_{i-1}' = \{v_1, v_2, \ldots, v_{i-1}\}$ that determines $\langle v_1, v_2, \ldots, v_k \rangle$ for which

$$S_i = S_{i-1}' \cup \{v_i\} \cup \{u \in V(G) : u \in I_G[v_i, w] \text{ for some } w \in S_{i-1}'\}.$$

The sequence $S_1, S_2, \ldots, S_k$ of subsets of $V(G)$ satisfies the set inclusion

$$S_1 \subseteq S_2 \subseteq S_3 \subseteq \ldots \subseteq S_k.$$

Since $V(G)$ is finite, there exists an integer $k$ such that $S_k = V(G)$.

**Definition 2.2** Let $G$ be a connected graph of order $n$. A subset $S = \{v_1, v_2, \ldots, v_k\}$ of $V(G)$ is a sequential geodetic cover of $G$ if $\langle v_1, v_2, \ldots, v_k \rangle$ is a geodetic sequence in $G$ and $S_k = V(G)$. We denote by $S^*(G)$ the collection of all
sequential geodetic covers of $G$. A sequential geodetic cover of minimum cardinality is called sequential geodetic basis of $G$. The sequential geodetic number $sgn(G)$ of a graph $G$ is the cardinality of a sequential basis, that is,

$$sgn(G) = \min\{|S| : S \in S^*(G)\}.$$  

Equivalently, $sgn(G) = \min\{k \in \mathbb{Z}^+ : S_k = V(G)\}$. If $S \in S^*(G)$, then there exists a $k \in \mathbb{Z}^+$ such that $\langle v_1, v_2, \ldots, v_k \rangle$ is a geodetic sequence and $S_k = V(G)$.

**Remark 2.3** In view of Definition 2.1, $S_k = S_{k-1}' \cup \{v_k\} \cup (I_G[v_k, w])$ where $w \in S_{k-1}'$, $3 \leq k \leq n$ and $S_2 = S_2'$.

**Remark 2.4** For a connected graph $G$ of order $n \geq 3$, $sgn(G) \geq 3$.

**Example 2.5** Consider the octahedron in Figure ??.

\[ G : \]

![Octahedron Graph](image)

Figure 1: The octahedron graph $G$

Let $S_1 = \{v_1\}$ and $S_2 = \{v_1, v_2\}$. The vertex $v_3 \notin S_2$ so that

$$S_3 = \{v_1, v_2\} \cup \{v_3\} \cup \{u \in V(G) : u \in I_G[v_3, w] \exists w \in S_2\}.$$  

Thus, $S_3 = \{v_1, v_2, v_3\} \cup I_G[v_3, v_1] \cup I_G[v_3, v_2]$. That is, $S_3 = \{v_1, v_2, v_3\} \cup \{v_3, u_1, u_2, u_3, v_2, v_1\} \cup \{v_3, v_2\}$. Hence, $S_3 = \{v_1, v_2, v_3, u_1, u_2, u_3\} = V(G)$. That is, $\langle v_1, v_2, \ldots, v_k \rangle$ is a geodetic sequence of $G$ and $S = \{v_1, v_2, v_3\}$ is in $S^*(G)$. It follows that $sgn(G) \leq 3$. By Remark ??, $sgn(G) \geq 3$. Therefore, $sgn(G) = 3$.

We now consider some relationships between $sgn(G)$ and $cgn(G)$ and between $sgn(G)$ and $gn(G)$ of a connected graph $G$.

**Theorem 2.6** Let $G$ be a connected graph of order $n \geq 3$. If $cgn(G) = 2$, then $sgn(G) = 3$.

**Proof.** Suppose $cgn(G) = 2$ and let $S = \{u, v\}$ be a closed geodetic basis of $G$. Then $I_G[u, v] = V(G)$. Since $|V(G)| \geq 3$, $sgn(G) \geq 3$ by Remark ??.
there exists a vertex \( w \) different from \( u \) and \( v \) which lies in some \( u-v \) geodesic. Consider the sequence \( \langle v_1, v_2, v_3 \rangle \) where \( v_1 = u \), \( v_2 = w \), and \( v_3 = v \). Then

\[
S_3 = \{v_1, v_2, v_3\} \cup \{u' \in V(G) : u' \in I_G[v_3, w] \text{ for some } w \in S_2\}
\]

\[
= \{v_1, v_2, v_3\} \cup \{u' \in V(G) : u' \in I_G[v_3, v_1] \text{ or } u' \in I_G[v_3, v_2]\}
\]

\[
S_3 = V(G).
\]

Thus, \( \langle v_1, v_2, v_3 \rangle \) is a geodetic sequence and \( S = \{v_1, v_2, v_3\} \) is a sequential geodetic cover. It follows that \( sgn(G) \leq 3 \). However, by Remark ??, \( sgn(G) = 3 \).

The converse of Theorem ?? is not true. Consider, for example the cycle \( C_5 \). It can be verified that \( sgn(C_5) = 3 \) but \( cgn(C_5) = 3 \).

**Theorem 2.7** [?] Let \( G \) be a connected graph. Then, \( gn(G) = 2 \) if and only if \( cgn(G) = 2 \) and \( gn(G) = 3 \) if and only if \( cgn(G) = 3 \).

The following are consequences of Theorem ??.

**Corollary 2.8** Let \( G \) be a connected graph of order \( n \geq 3 \). If \( gn(G) = 2 \), then \( sgn(G) = 3 \).

**Proof.** If \( gn(G) = 2 \), then by Theorem ??, \( cgn(G) = 2 \). Thus by Theorem ??, \( sgn(G) = 3 \). \[\square\]

**Corollary 2.9** For a path \( P_n \) of order \( n \), \( sgn(P_n) = 3 \) for all integer \( n \geq 3 \).

**Proof.** Let \( P_n \) be the path of order \( n \geq 3 \). Then, \( cgn(P_n) = 2 \). By Theorem ??, \( sgn(P_n) = 3 \). \[\square\]

**Theorem 2.10** For a cycle \( C_n \) of order \( n \geq 3 \), \( sgn(C_n) = 3 \).

**Proof.** Let \( C_n = [x_1, x_2, x_3, \ldots, x_n, x_1] \) be a cycle of order \( n \geq 3 \). Then

\[
\begin{align*}
gn(C_n) = cgn(C_n) &= \begin{cases} 
2, & \text{if } n \text{ is even} \\
3, & \text{if } n \text{ is odd}
\end{cases}
\end{align*}
\]

If \( n \) is even, then by Theorem ??, \( sgn(C_n) = 3 \). If \( n \) is odd, then consider the sequence \( \langle x_1, x_2, x_{\lceil \frac{n}{2} \rceil+1} \rangle \). That is, \( S_1 = \{x_1\} \) and \( S_2 = \{x_1, x_2\} \). Clearly, \( x_{\lceil \frac{n}{2} \rceil+1} \notin S_2 \) and

\[
S_3 = \{x_1, x_2, x_{\lceil \frac{n}{2} \rceil+1}\} \cup \{u \in V(C_n) : u \in I_{C_n}[x_{\lceil \frac{n}{2} \rceil+1}, w] \text{ } \exists \text{ } w \in S_2\}
\]

\[
= V(C_n) = V(G).
\]
Hence, \( \langle x_1, x_2, x_{\lceil \frac{n}{2} \rceil + 1} \rangle \) is a geodetic sequence and \( S = \{x_1, x_2, x_{\lceil \frac{n}{2} \rceil + 1}\} \) is a sequential geodetic cover. This, together with Remark 2 imply that \( sgn(C_n) = 3 \).

In [1], Albia showed that
\[
sgn(P_n) = \begin{cases} 
2, & \text{if } n = 4 \\
3, & \text{if } n \geq 5
\end{cases}
\]
He also added that for \( n \geq 5 \), \( sgn(C_n) = 3 \). The following results give the sequential geodetic numbers of these graphs.

**Theorem 2.11** For \( n \geq 4 \), \( sgn(P_n) = 3 \).

**Proof.** Let \( P_n = [x_1, x_2, x_3, \ldots, x_n] \) be a path of order \( n \geq 4 \). If \( n = 4 \), then \( P_4 = P_4 \). Hence, \( sgn(P_4) = sgn(P_4) = 3 \). Suppose \( n > 4 \). Then \( P_n \) is connected and by Remark 2, \( sgn(P_n) \geq 3 \). Let \( \langle x_1, x_3, x_2 \rangle \) be a sequence of vertices in \( P_n \). Then \( S_1 = \{x_1\} \) and \( S_2 = \{x_1, x_3\} \). Clearly, \( x_2 \notin S_2 \) and so \( \langle x_1, x_3, x_2 \rangle \) is a geodetic sequence. That is, set \( S = \{v_1, v_2, v_3\} \) with \( v_1 = x_1, v_2 = x_3, \) and \( v_3 = x_2 \), is a sequential geodetic basis of \( G \).

**Theorem 2.12** For \( n \geq 5 \), \( sgn(C_n) = 3 \).

**Proof.** Let \( C_n = [x_1, x_2, x_3, \ldots, x_n, x_1] \) be the cycle of order \( n \geq 5 \). Then \( C_n \) is connected and by Remark 2, \( sgn(C_n) \geq 3 \). Consider the sequence \( \langle x_1, x_3, x_2 \rangle \). Then \( S_1 = \{x_1\} \) and \( S_2 = \{x_1, x_3\} \). Clearly, \( x_2 \notin S_2 \) and so if \( u \in V(C_n) \setminus S \), then \( u = x_i \) where \( 4 \leq i \leq n \), and \( u \in I_{C_n}[w, x_2] \). This means that \( S_3 = V(C_n) \). That is \( S = \{v_1, v_2, v_3\} \) with \( v_1 = x_1, v_2 = x_3 \) and \( v_3 = x_2 \), is a sequential geodetic basis and \( sgn(C_n) = 3 \).

The next result shows that a sequential geodetic cover may not be a closed geodetic cover.

**Theorem 2.13** Let \( G = K_{m,n} \), where \( m, n > 2 \), let \( U \) and \( W \) be the partite sets of \( V(G) \). Let \( S \subseteq V(G) \). Then \( S \in S^*(G) \) if and only if \( S \) is any of the following:

(i) \( S = U \);
(ii) \( S = W \);
(iii) \( S = U \cup \{w\} \) for some \( w \in W \);
(iv) \( S = W \cup \{u\} \) for some \( u \in U \);
(v) \( U \cup \{w, w'\} \) for some \( w, w' \in W \);
(vi) \( S = W \cup \{u, u'\} \) for some \( u, u' \in U \).
Proof. Suppose $S = \{v_1, v_2, \ldots, v_k\}$ is a sequential geodetic cover of $G = K_{m,n}$. Let $U' = S \cap U$ and $W' = S \cap W$. Since $m,n > 2$, we have $|U'| > 2$ or $|W'| > 2$. Suppose that $|U'| > 2$. Let $i = \min\{n \in \mathbb{Z}^+ : v_n \in U'\}$. Then $i < j$. If $i > 3$, then $v_1, v_2, v_3 \in W$ so that $v_i \in I_G[v_1,v_3]$ or $v_i \in I_G[v_2,v_3]$. In other words, $v_i \in S_3$, a contradiction. Thus, $i = 1,2,3$.

Case 1. Suppose $i = 1$. If $j > 3$, then $v_2, v_3 \in W$ so that $v_j \in I_G[v_2,v_3] \subset S_3$, a contradiction. Consequently, $j = 2$ or $j = 3$. If $j = 2$, then either $v_3 \in W$ and $v_n \in U$ for all $n = 1,2,3, \ldots, k$ or $v_n \in U$ for all $n = 1,2,3, \ldots, k$. This implies that $S \setminus \{v_3\} \subset U$ or $S \subset U$. If $j = 3$, then $v_2 \in W$ but $S \setminus \{v_2\} \subset U$.

Case 2. Suppose that $i = 2$. If $j > 3$, then $v_1, v_3 \in W$ so that $v_j \in I_G[v_1,v_3] \subset S_3$, a contradiction. This means that $j = 3$. Consequently, $S \setminus \{v_1\} \subset U$.

Case 3. Suppose that $i = 3$. If $j > 4$, then $v_1, v_2, v_4 \in W$ so that $v_j \in I_G[v_2,v_4]$, a contradiction. This means that $j = 4$. Consequently, $S \setminus \{v_1,v_2\} \subset U$.

Now, let $T \subset U$. Then $T_1 = T \cup W$ where $l = k,k-1$, or $k-2$. This implies that $T_1 = V(G)$ if and only if $T = U$. Therefore, either $S = U \cup \{w\}$ or $S = U \cup \{w,w'\}$ for some $w,w' \in W$. Similarly, if $|W| > 2$, then either $S = W$ or $S = W \cup \{u\}$ or $S = W \cup \{u,u'\}$ for some $u,u' \in U$.

Conversely, suppose that $U$, and $W$ are the partite sets of $V(G) = K_{m,n}$ with $|U| = m$ and $|W| = n$ where $m,n > 2$. Let $S = U$. For every distinct vertices $u,u' \in U$ and for every $w \in W$, $w \in I_G[u,u']$. Moreover, for every distinct vertices $u,u' \in U$ a $u-u'$ geodesic in $G$ is of the form $[u,w,u']$ where $w \in W$. Thus,

$$u'' \notin I_G[u,u'] \text{ for all } u'' \in U \setminus \{u,u'\}. \quad (*)$$

Hence, any ordering of the elements of $U$ forms a geodetic sequence in $G$. Since $W \subset I_G[u,u'] \subset S_m$ and $U \subset S_m$, we have $V(G) \subset S_m$. Consequently, since $S_m \subset V(G)$, $V(G) = S_m$. Therefore, $S \in S^*(G)$.

Parallel arguments will show that $W \in S^*(G)$. Let $w \in W$ and consider the set $S = \{w\} \cup U = \{v_1, v_2, \ldots, v_{m+1}\}$ where $v_1 = w$ and $v_i = u_1 \in U$ for $i = 2,3,\ldots,m+1$. Clearly, $v_1 \neq v_2$ and by $(*)$, $v_i \notin S_{i-1}$ for all $i = 3,4,\ldots,m+1$. Note that $S_m = S_{m+1}$ and by the result above, $S_{m+1} = V(G)$. Therefore, $S \in S^*(G)$.

Similarly, if $S = \{u\} \cup W$ for some $u \in U$, then $S \in S^*(G)$.

Now, let $w,w' \in W$. Consider the set $S = \{w,w'\} \cup U = \{v_1, v_2, \ldots, v_{m+2}\}$ where $v_1 = w$, $v_2 = w'$ and $v_i = u_{i-2}$ for $i = 3,4,\ldots,m+2$. Clearly, $v_1 \neq v_2$ and by $(*)$, $v_i \notin S_{i-1}$ for all $i = 3,4,\ldots,m+2$. Since $w,w' \in I_G[v_i,v_j]$ for all $3 \leq i \neq j \leq m+2$, $S_{m} = S_{m+2}$. By the result above, $S_{m+2} = V(G)$.

The remaining case is proved similarly.

The next corollary is an immediate consequence of Theorem ??.

Corollary 2.14 For $m,n > 2$, $\operatorname{sgn}(K_{m,n}) = \min\{m,n\}$.

Proof. By Theorem ??,

$$\operatorname{sgn}(K_{m,n}) = \min\{m,n,m+1,n+1,m+2,n+2\} = \min\{m,n\}.$$
Cagaanan [4] showed that $gn(K_{m,n}) = \min\{m, n, 4\}$, for all $m, n > 2$. The next corollary is a consequence of this result and Corollary ??.

**Corollary 2.15** Let $m, n > 2$. Then $sgn(K_{m,n}) = gn(K_{m,n})$ if and only if $\min\{m, n\} = 3$ or 4.

Aniversario [2] showed that $cgn(K_{m,n}) = gn(K_{m,n})$ if and only if $\min\{m, n\} \leq 4$ for all $m, n \geq 2$. This result together with Corollary ?? give the next corollary.

**Corollary 2.16** $sgn(K_{m,n}) = cgn(K_{m,n})$ if and only if $m, n \geq 2$.

**Theorem 2.17** Every closed geodetic basis of a connected graph $G$ can be extended to a sequential geodetic cover.

**Proof.** Let $G$ be a connected graph and $S = \{u_1, u_2, \ldots, u_k\} \in C^*(G)$ in a canonical form with $k = cgn(G)$. Then $u_i \notin I_G[u_m, u_n]$ for all integers $i, m, n$ with $m < n < i$ and $1 \leq i \leq k$. Consider the following cases:

Case 1. Suppose $I_G[u_i, u_j] = \{u_i, u_j\}$ for some integers $i, j$ where $1 \leq i, j \leq k$ and $i \neq j$. Then $u_i$ and $u_j$ are adjacent. Consider the sequence $\langle v_1, v_2, \ldots, v_k \rangle$ where $v_1 = u_i$, $v_2 = u_j$, and $\{v_3, v_4, \ldots, v_k\} = \{u_1, u_2, \ldots, u_k\} \setminus \{u_i, u_j\}$. Then $S_1 = \{u_i\}$ and $S_2 = \{u_i, u_j\}$. Clearly $v_3 \notin S_2$. By assumption, $u_r \notin I_G[S_{r-1}]$ for all $r$, $3 \leq r \leq k$. Since $S_r \subseteq I_G[S_r]$ for all $r$, it follows that $u_r \notin S'_{r-1}$ for all $r$, where $3 \leq r \leq k$. This means that $\langle v_1, v_2, \ldots, v_k \rangle$ is a geodetic sequence and $S'_k = \{v_1, v_2, \ldots, v_k\} = V(G)$. Therefore, $S'_k = \{u_1, u_2, \ldots, u_k\} = S$ is a sequential geodetic cover.

Case 2. Suppose $I_G[u_i, u_j] \neq \{u_i, u_j\}$ for all $i, j$ where $1 \leq i \leq j \leq k$. In particular, $I_G[u_1, u_2] \neq \{u_1, u_2\}$. Thus, there exists a vertex $w$ such that $w \in I_G[u_1, u_2]$.

Suppose $|V(G)| = 3$ and consider the sequence $\langle u_1, u_2, w \rangle$. Clearly, $w \notin S_2 = \{u_1, u_2\}$ and so $\langle u_1, u_2, w \rangle$ is a geodetic sequence in $G$. If $I_G[u_1, u_2] = V(G)$, then $S_3 = S_2 \cup \{w\} \cup \{u \in V(G) : u \in I_G[w, u_i]\}$ for some $u_i \in S_2 = V(G)$.

This implies that $S = \{u_1, u_2, w\}$ is a sequential geodetic basis of $G$. If $I_G[u_1, u_2] \neq V(G)$, then consider the sequence $\langle v_1, v_2, \ldots, v_k \rangle$ where $v_1 = u_1$, $v_2 = u_3, v_3 = u_2$ and $v_r = u_r$ for $r = 4, 5, \ldots, k$. Thus, $v_r \notin S'_{r-1}$ for all $r$, $4 \leq r \leq k$. Also, $S'_k = S \cup \{w\} = V(G)$. Therefore, $S'_k = \{v_1, v_2, \ldots, v_k\} = \{u_1, u_2, \ldots, u_k\} = S$ is a sequential geodetic cover of $G$. 

**Theorem 2.18** Let $G$ be a connected graph of order $n \geq 4$. Then $sgn(G) = n$ if and only if $G = K_n$. 
Proof. Let \( \langle v_1, v_2, \ldots, v_k \rangle \) be a sequence of vertices in \( K_n \) and \( i, j \) be integers such that \( 1 \leq i, j \leq k \). Then \( d(v_i, v_j) = 1 \) for all \( i, j \) with \( i \neq j \). Consequently, \( S_i = \{v_1, v_2, \ldots, v_i\} \) for all \( i \). This implies that \( v_i \notin S'_{i-1} \) for all \( i \) and \( \langle v_1, v_2, \ldots, v_k \rangle \) is a geodetic sequence. Furthermore, \( S_k = V(K_n) \) if and only if \( k = n \). Hence, \( S_i = \{v_1, v_2, \ldots, v_n\} \) is a sequential geodetic cover of \( K_n \) and is the minimum such cover. Therefore, \( sgn(K_n) = n \).

Suppose \( sgn(G) = n \) and suppose further that \( G \neq K_n \). Then there exist \( u \) and \( v \) in \( V(G) \) such that \( d_G(u, v) = 2 \). Let \( v_2 = u \) and \( v_3 = v \). If \( I_G[u, v] = V(G) \), then let \( v_1 = w \), where \( w \in I_G(u, v) \). Then \( \langle v_1, v_2, v_3 \rangle \) is a geodetic sequence and \( S = \{v_1, v_2, v_3\} \) is a sequential geodetic cover. Hence, \( sgn(G) = 3 < n \). If \( I_G[u, v] \neq V(G) \), pick \( v_1 = v \notin V(G) \), pick \( v_1 = w \notin I_G[u, v] \). Then \( \langle v_1, v_2, v_3 \rangle \) is a geodetic sequence. If \( S_3 = V(G) \), then \( S = \{v_1, v_2, v_3\} \) is a sequential geodetic cover. This implies that \( sgn(G) = 3 < n \). If \( S_3 \notin V(G) \), then continue until \( S_k = V(G) \). Since \( \{v_1, v_2, v_3\} \subset S_3 \), it follows that \( S = \{v_1, v_2, \ldots, v_k\} \subset S_k \). That is, \( |S| = k < n \). Hence, \( sgn(G) \leq k < n \) and the conclusion follows.

\[\text{Theorem 2.19} \; \text{Let} \; G \; \text{be a connected graph of order} \; n > 4. \text{Then} \; sgn(G) = n - 1 \; \text{if and only if} \; G = K_1 + \cup m_j K_j, \; \text{where} \; 2 \leq \Sigma m_j = n - 1. \]

\[\text{Proof.} \; \text{Suppose} \; G = K_1 + \cup m_j K_j, \; \text{where} \; 2 \leq \Sigma m_j \; \text{and} \; |V(G)| > 4. \; \text{Then} \; G - w \; \text{has at least two components. Note that each component of} \; G \; \text{is complete. Let} \; u, v \in V(G) \setminus \{w\}. \; \text{If} \; u \; \text{and} \; v \; \text{do not belong to the same component, then} \; d_G(u, v) = 2 \; \text{and a} \; u-v \; \text{geodesic is of the form} \; \{u, w, v\}. \; \text{If} \; u \; \text{and} \; v \; \text{are in the same component, then} \; I_G[u, v] = \{u, v\}. \; \text{Claim first that} \; S = (V(G) \setminus \{w\}) \; \text{is a sequential geodetic cover of} \; G. \]

\[\text{Let} \; \langle v_1, v_2, \ldots, v_{n-1} \rangle \; \text{be a sequence of vertices in} \; G - w. \; \text{Then} \; S_1 = \{v_1\}, \; S_2 = \{v_1, v_2\} \; \text{and either} \; S_i = \{v_1, v_2, \ldots, v_i\} \; \text{or} \; S_i = \{v_1, v_2, \ldots, v_i\} \cup \{w\} \; \text{for all} \; i, \; \text{where} \; 3 \leq i \leq n - 1. \; \text{Clearly,} \; v_i \notin S'_{i-1} \; \text{for all} \; i. \; \text{If} \; d_G(v_1, v_2) = 2, \; \text{then by assumption there exists a vertex} \; v_i \; \text{for some} \; i \; \text{such that either} \; w \in I_G[v_1, v_i] \; \text{or} \; w \in I_G[v_i, v_2]. \; \text{Therefore, any ordering of the elements in the sequence forms a geodesic sequence} \; G - w. \; \text{Moreover,} \; S_{n-1} = \{v_1, v_2, \ldots, v_{n-1}\} \cup \{w\} = V(G). \; \text{Therefore,} \; S = \{v_1, v_2, \ldots, v_{n-1}\} \; \text{is a sequential geodetic cover. It remains to show that} \; S \; \text{is of minimum cardinality.} \]

\[\text{Let} \; T \subseteq S \; \text{and} \; |T| = k < n - 1. \; \text{Then using the same argument as above, either} \; S'_k = T \; \text{or} \; S'_k = T \cup \{w\}. \; \text{Consequently, if} \; |T| \leq n - 2, \; \text{then} \; |S'_k| \leq n - 1 \; \text{and so} \; S_k \neq V(G). \; \text{Therefore,} \; S \; \text{is a sequential geodetic basis.} \]

\[\text{Conversely, suppose that} \; sgn(G) = n - 1. \; \text{Claim first that} \; d_G(u, v) \leq 2 \; \text{for all} \; u, v \in V(G). \; \text{On the contrary, suppose that there exist} \; u, v \in V(G) \; \text{with} \; d_G(u, v) > 3. \; \text{If} \; I_G[u, v] = V(G), \; \text{then pick} \; v_1 = u, \; v_2 = w, \; \text{and} \; v_3 = v, \; \text{where} \; w \in I_G(u, v) \; \text{and} \; d_G(u, v) = 1. \; \text{Then} \; \langle v_1, v_2, v_3 \rangle \; \text{is a geodesic sequence and} \; S = \{v_1, v_2, v_3\} \; \text{is a sequential geodetic cover.} \; \text{By assumption,} \; |V(G)| > 4. \; \text{Hence,} \; sgn(G) = 3 < |V(G)| - 1. \; \text{This is a contradiction. If} \; I_G[u, v] \neq V(G), \]


then let $v_1 = u$, $v_2 = w'$, and $v_3 = v$ where $w' \notin I_G[u,v]$. Clearly, $\langle v_1, v_2, v_3 \rangle$ is a geodetic sequence. Since $d_G(u,v) = 3$, it follows that $|I_G[u,v]| \geq 4$. Hence, $|S_3| \geq 5$. This implies that if $S = \{v_1, v_2, \ldots, v_k\}$ is a sequential geodetic cover of $G$, then $|S_k| \geq k + 2$, that is, $k \leq n - 2$. It follows that $sgn(G) \leq k \leq n - 2$, a contradiction. Therefore, $d_G(u,v) = 1$ or $d_G(u,v) = 2$ for distinct vertices $u$ and $v$ of $G$.

Since $G \neq K_n$, there exist $u, v \in V(G)$ such that $d_G(u,v) = 2$. Let $w \in I_G(u,v)$.

Claim 1. If $I_G[u,v] = \{u, w, v\}$.

Assume that there exists $w' \neq w$ such that $w' \in I_G[u,v]$. If $I_G[u,v] = V(G)$, then pick $v_1 = u$, $v_2 = w'$, and $v_3 = v$. Then $v_3 \notin S_2 = \{u, w'\}$ and $\langle v_1, v_2, v_3 \rangle$ is a geodetic sequence. Hence, $S = \{v_1, v_2, v_3\}$ is a sequential geodetic cover. Since $n \geq 5$, $|S| \leq n - 2$. Thus, $sgn(G) \leq n - 2$ which is a contradiction. If $I_G[u,v] \neq V(G)$, then pick $v_1 = u$, $v_2 = w'$ and $v_3 = v$ where $w' \notin I_G[u,v]$. Since $|I_G[u,v]| \geq 4$, it follows that $|S_3| \geq 5$. Consequently, if $S = \{v_1, v_2, \ldots, v_k\}$ is a sequential geodetic cover, then $|S_k| \geq k + 2$. That is, $k \leq n - 2$. This is a contradiction. Thus, $[u, w, v]$ is the only $u - v$ geodesic in $G$.

Claim 2. $d_G(w, x) = 1$ for all $x \in V(G) \setminus \{w\}$.

Suppose on the contrary, that there exists $x \in V(G) \setminus \{w\}$ such that $d_G(w, x) = 2$. Clearly, $x \neq u, v$. Let $z \in I_G(w, x)$. Now, $d_G(u, x) \leq 2$ and $d_G(x, v) \leq 2$. Consequently, either $z = u$ or $z = v$. Suppose $z = v$.

Then, $d_G(u, x) = 1$ or $d_G(u, x) = 2$.

Case 1. If $d_G(u, x) = 1$, then $w, x \in I_G(u, v)$. If $I_G[u,v] = V(G)$, then pick $v_1 = u$, $v_2 = x$, $v_3 = v$ and consider the sequence $\langle v_1, v_2, v_3 \rangle$. Then $S_1 = \{u\}$ and $S_2 = \{u, x\}$. Clearly, $v = v_3 \notin S_2$. Thus, $\langle v_1, v_2, v_3 \rangle$ is a geodetic sequence and $S_3 = V(G)$. Hence, $S = \{v_1, v_2, v_3\}$ is a sequential geodetic cover. Moreover, $sgn(G) = 3 \leq n - 2$, a contradiction. If $I_G[u,v] \neq V(G)$, then pick $v_1 = u$, $v_2 = w'$ and $v_3 = v$ where $w' \notin I_G[u,v]$. Clearly, the sequence $\langle v_1, v_2, v_3 \rangle$ is a geodesic sequence. Since $|I_G[v_1, v_3]| \geq 4$, it follows that $|S_3| \geq 5$. This means that if $S = \{v_1, v_2, \ldots, v_k\} \in S^*(G)$, then $sgn(G) \leq k \leq n - 2$, a contradiction.

Case 2. If $d_G(u, x) = 2$, then let $v_1 = x$, $v_2 = v$ and $v_3 = u$. Using the same argument as above, this case leads to a contradiction also.

Similarly, it is impossible to have $z = u$. Therefore the claim holds.

Claim 3. There does not exist $w' \in V(G)$, $w' \neq w$ such that $d_G(w', x) = 1$ for all $x \in V(G) \setminus \{w\}$.

Suppose $w' \in V(G)$, $w' \neq w$ and $d_G(w', x) = 1$ for all $x \in V(G) \setminus \{w'\}$. By definition of $u$ and $v$ above, $w' \neq u, v$. Since $d_G(u,w') = d_G(v,w') = 1$, $w' \in I_G(u,v)$. This contradicts Claim 1 above. Therefore, the claim holds.

The above claims imply that $G = \langle \{w\} \rangle + H$, where $H$ is a subgraph of $G$.

Claim 4. $H$ is not connected.
Consider the vertices $u$ and $v$ above. Clearly, $u, v \in V(G)$, by Claims 1 and 2, $[u, w, v]$ is the only $u$-$v$ geodesic in $G$. Therefore, there is no path joining $u$ and $v$ in $H$. This means that $H$ is not connected.

Claim 5. Every component of $H$ is complete.

Let $C$ be a component of $H$. If $|V(C)| = 1$, then we are done. Suppose that $|V(C)| \geq 2$. Let $x, y \in V(C)$. If $d_G(x, y) = 2$, then by Claims 3 and 4, $[x, w, y]$ is the only $x$-$y$ geodesic in $G$. Hence, $x$ and $y$ cannot be in one component of $H$, a contradiction. Therefore, $d_G(x, y) = 1$. Since $x$ and $y$ are arbitrary, $C$ is complete. Thus, $H = \bigcup m_j K_j$. This completes the proof of the theorem.

**Example 2.20** The graph $G = K_1 + (2K_2 \cup K_3)$ where $K_1 = \{x\}$ shown in Figure ?? has $\text{sgn}(G) = 7$. The sequential geodetic basis of $G$ is the set $V(G) \setminus \{x\}$.

![Figure 2: The graph $K_1 + (2K_2 \cup K_3)$](image)

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**References**


On sequential geodetic numbers of some connected graphs


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