On $p$-$(\alpha, \beta)$-Normal Operators

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Abstract

An operator $T \in B(H)$ is said to be $p$-$(\alpha, \beta)$ normal operators for $0 < p \leq 1$ if $\alpha^2(T^*T)^p \leq (TT^*)^p \leq \beta^2(T^*T)^p$, $0 \leq \alpha \leq 1 \leq \beta$. In this paper, we prove that if $T$ is $p$-$(\alpha, \beta)$ - normal operator then $T^n$ is $\frac{p}{n}$-$(\alpha, \beta)$ - normal operator for all positive integer $n$. Moreover, we prove that if $T = U|T|$ is $p - (\alpha, \beta)$ - normal operator for $0 < p < 1$, the Aluthge transform $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ is $(p + \frac{1}{2}) - (\alpha, \beta)$ - normal operator. In this paper, some of the properties and structure of $p - (\alpha, \beta)$ normal operators are discussed.

Mathematics Subject Classification: Primary 47B33; Secondary 47B37

Keywords: $p$- $(\alpha, \beta)$ - normal operators, $(\alpha, \beta)$ - normal operators, Aluthge Transformation

1. Introduction and Preliminaries

Let $H$ be a separable complex Hilbert space and $B(H)$ denote the algebra of bounded linear operators on an infinite dimensional separable Hilbert space

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An operator $T$ is said to be normal if $TT^* = T^*T$ and hyponormal if $TT^* \leq T^*T$. An operator $T$ is said to be dominant if $\text{ran } (T - \lambda I) \subseteq \text{ran } (T - \lambda I)^*$ for all $\lambda \in \mathbb{C}$ or equivalently there exists a real number $M_{\lambda}$ for each $\lambda \in \mathbb{C}$ such that $\| (T - \lambda I)x \| \leq M_{\lambda} \| (T - \lambda I)x \|$ for each $x \in H$. If there exists a constant $M$ such that $M_{\lambda} \leq M$ for all $\lambda \in \mathbb{C}$, then $T$ is called $M$-hyponormal and if $M = 1$, $T$ is hyponormal. The class of Hyponormal operators has been studied by many authors. In recent years this class has been generalized, in some sense, to the larger sets of so-called $p$-hyponormal, log hyponormal, Posinormal, etc [7], [8], [5], [1] and [2]. An operator $T \in B(H)$ is said to be $p$-hyponormal for $0 < p < 1$ if $(TT^*)^p \leq (T^*T)^p$, $p$-posinormal if $(TT^*)^p \leq c^2(T^*T)^p$, $(\alpha, \beta)$-normal operators if $\alpha^2T^*T \leq TT^* \leq \beta^2T^*T$, $0 \leq \alpha \leq 1 \leq \beta$ [6].

The example of an $M$-hyponormal operator given by Wadhwa [12], the weighted shift operator defined by $T e_1 = e_2$, $T e_2 = 2e_3$ and $T e_i = e_{i+1}$ for $i \geq 0$, is not an $p - (\alpha, \beta)$ normal, which is neither normal nor hyponormal. So it is clear that the class of $p - (\alpha, \beta)$ normal lies between hyponormal and $M$-hyponormal operators. Now the inclusion relation becomes

\[
\text{Normal} \subseteq \text{Hyponormal} \subseteq (\alpha, \beta) \text{ normal} \subseteq p - (\alpha, \beta) \text{ normal} \subseteq M \text{- hyponormal} \subseteq \text{Dominant}
\]

S.S Dragomir and M.S. Moslehian [3] and [6] has given various inequalities between the operator norm and numerical radius of $(\alpha, \beta)$-normal operators. Weyl type theorems and composition operators of $(\alpha, \beta)$ have been studied by D. Senthil Kumar and Sherin Joy S.M [10, 11]. As a generalisation of $(\alpha, \beta)$-normal operators, we introduce $p/(\alpha, \beta)$-normal operators. When $p = 1$, this coincide with $(\alpha, \beta)$-normal operators.

2. on $p - (\alpha, \beta)$ normal operators

**Proposition 2.1.** If $T \in B(H)$, then the following statements are equivalent:

(i) $T \in p - (\alpha, \beta)$ normal operators.

(ii) $\text{Range } |T|^p = \text{Range } |T^*|^p$, $\text{ker } T^p = \text{ker } T^*^p$.

(iii) There exist $S_1, S_2 \in B(H)$ such that $|T^*|^p = |T|^p S_1$, $|T|^p = |T^*|^p S_2$.

(iv) There exist positive operators $P_1, P_2$ such that $|T^*|^2p = |T|^p P_1 |T|^p$, $|T|^2p = |T|^*^p P_2 |T|^*^p$.

**Theorem 2.2.** Let $T$ be an $p - (\alpha, \beta)$ normal operator then

(i) $\lambda T$ is $p - (\alpha, \beta)$ normal operator

(ii) the translate $T - \lambda$ need not be $p - (\alpha, \beta)$ normal operator.
Therefore $\lambda T$ is $p-(\alpha, \beta)$ normal operator.

(ii) The operator $T = U^* - 2$, where $U$ is the unilateral shift since $2 \notin \sigma(U^*)$ [5], $T$ is $p-(\alpha, \beta)$ normal operator but $U^*$ is not $p-(\alpha, \beta)$ normal, $U$ is $p-(\alpha, \beta)$ normal.

**Theorem 2.3.** Let $T = U|T| \in B(H)$ be the polar decomposition of $T$. Then $T \in p-(\alpha, \beta)$ normal operator iff there exist a positive number $\alpha, \beta$ such that $\alpha \|T^p x\| \|T^p y\| \leq \| (U|T|^2 p x, y) \| \leq \beta \|T^p x\| \|T^p y\|$

**Proof.** By assumption $T$ is $p-(\alpha, \beta)$ normal operator

$$|(U|T|^2 p x, y)| = \| (U|T|^2 p x, U^* y) \| = \|T^p x\| \|T^p U^* y\|$$

Similarly

$$|(U|T|^2 p x, y)| = \| (|T|^2 p x, U^* y) \| = \|T^p x\| \|T^p U^* y\|$$

Therefore

$$\alpha \|T^p x\| \|T^p y\| \leq |(U|T|^2 p x, y)| \leq \beta \|T^p x\| \|T^p y\|.

**Theorem 2.4** (Mc Carthy [2]). Let $A \geq 0$. Then

(i) $(Ax, x)^r \leq \|x\|^{2(r-1)}(A^r x, x)$ if $r \geq 1$

(ii) $(Ax, x)^r \geq \|x\|^{2(r-1)}(A^r x, x)$ if $0 \leq r \leq 1$.

**Theorem 2.5.** If $T$ is $p-(\alpha, \beta)$ normal operator then $T$ is $M$-paranormal.

**Proof.** Let $T = U|T|$ be the polar decomposition of $T$. Since $T$ is $p-(\alpha, \beta)$ normal operator,

$$\alpha^2 |T|^{2p} \leq U|T|^{2p} U^* \leq \beta^2 |T|^{2p}$$

$$\Rightarrow U|T|^{2p} U^* \leq \beta^2 |T|^{2p}$$

$$\Rightarrow |T|^{2p} \leq \beta^2 U^* |T|^{2p} U.$$
\[ \beta^2 \left( \| T \|^{2p} \left( \frac{T_x}{\| T_x \|},\frac{T_x}{\| T_x \|} \right) \right) \]

\[ 0 \leq p \leq 1 \]

\[ \geq \beta^2 \left( \| T \|^{2p} \| T_x \|^{1/p} \| T_x \|^{2} \right) \]

\[ \geq \beta^2 (U^* |T|^{2p} |T|x, |T|x)^{1/p} \| T_x \|^{2} \]

\[ \geq (|T|^{2p+2} x, x)^{1/p} \| T_x \|^{2} \]

\[ \| T_x \|^4 \| T_x \|^4 / p \]

\[ \geq \| T_x \|^{4} \]

\[ \| T_x \|^4 \leq \beta^2 \| T^2 x \|^2 \]

Therefore, \( \| T_x \|^2 \leq \beta \| T^2 x \| \) T is M-paranormal.

**Corollary 2.6.** If \( T \) is \( p - (\alpha, \beta) \) normal operator and \( V \) is isometry, then \( VTV^* \) is also \( p - (\alpha, \beta) \) normal operator.

**Proof.** Let \( T \in B(H) \), there exists some positive operators \( K_1 \) and \( K_2 \) such that

\[ |T^*|^{2p} = |T|^p K_1 |T|^p \]

\[ |T|^{2p} = |T^*|^p K_2 |T^*|^p \]

Let

\[ |T_0^*|^{2p} = V |T^*|^{2p} V^* \]

\[ = V |T|^p K_1 |T|^p V^* \]

\[ = |T_0|^p V K_1 V^* |T_0|^p ( \text{ since } V \text{ is isometry } ). \]

Let

\[ |T_0|^{2p} = V |T|^{2p} V^* \]

\[ = V |T^*|^p K_2 |T^*|^p V^* \]

\[ = |T_0|^p V K_2 V^* |T_0|^p ( \text{ since } V \text{ is isometry } ). \]

\( VTV^* \) is \( p - (\alpha, \beta) \) normal operator.
Theorem 2.7. Let $T$ be $p-(\alpha, \beta)$ normal operator, then there exist $M > 0$ such that
\[ \| T^k x \|^2 \leq M_p \| T^{k-1} x \| \| T^{k+1} x \| \] for all unit vectors $x \in H$.

Proof. $T$ is $p-(\alpha, \beta)$ normal operator

\[
\begin{align*}
\| T^{k+1} x \|^2 &= (T^{k+1} x, T^{k+1} x) \\
&= (T^* T T^k x, T^k x) \\
&= ( (|T|^{2p})^{1/p} T^k x, T^k x) \\
&\geq (|T|^{2p} T^k x, T^k x)^{1/p} \| T^k x \|^2 (1 - 1/p) \\
&\geq \beta^{-2/p}(|T|^{2p} T^k x, T^k x)^{1/p} \| T^k x \|^2 (1 - 1/p) \\
&\geq \beta^{-2/p}(T^2 T^{k-1} x, T^{k-1} x)^{1/p} \| T^k x \|^2 (1 - 1/p) \\
&\geq \beta^{-2/p}(T^2 T^{k-1} x, T^{k-1} x)^{p+1/p} \| T^k x \|^2 (1 - 1/p) \| T^{k-1} x \|^2 (1 - 1/p) \\
&\geq \beta^{-2/p} \| T^k x \|^4 \| T^{k-1} x \|^2 \\
\| T^k x \|^2 &\leq \beta^{1/p} \| T^{k-1} x \| \| T^{k+1} x \|
\end{align*}
\]

\[ \Box \]

Theorem 2.8. Let $T$ be $p-(\alpha, \beta)$ normal operator and $S$ is a self-adjoint operator on $H$. If $T S$ is a contraction, then $\alpha \leq \| T S \|^p \leq \beta$.

Proof. $T$ is $p-(\alpha, \beta)$ normal operator, we have

\[
\begin{align*}
\alpha \| T S x \|^p &\leq \| T^* S^* x \|^p \leq \beta \| T S x \|^p \\
\alpha \| x \|^p &\leq \| (S T)^* x \|^p \leq \beta \| x \|^p \\
\alpha &\leq \| S T \|^p \leq \beta \\
\alpha &\leq \| T S \|^p \leq \beta.
\end{align*}
\]

\[ \Box \]

Lemma 2.9. If $T$ is $p-(\alpha, \beta)$ normal operator such that $\alpha \beta = 1$, then $T^*$ is also $p-(\alpha, \beta)$ normal operator.

Proof. From the definition of $p-(\alpha, \beta)$ normal operator

\[
\begin{align*}
\alpha^2 (T^* T)^p &\leq (T T^*)^p \leq \beta^2 (T^* T)^p \\
\alpha^4 (T^* T)^p &\leq \alpha^2 (T T^*)^p \leq \alpha^2 \beta^2 (T^* T)^p
\end{align*}
\]

and

\[
\begin{align*}
\alpha^2 \beta^2 (T^* T)^p &\leq \beta^2 (T T^*)^p \leq \beta^4 (T^* T)^p
\end{align*}
\]

(2.1)
From above two equations, 
\[ \alpha^2(T^*T)^p \leq \alpha^2\beta^2(T^*T)^p \leq \beta^2(TT^*)^p. \]

Therefore, 
\[ \alpha^2(TT^*)^p \leq (T^*T)^p \leq \beta^2(TT^*)^p \]

\( T^* \) is also \( p - (\alpha, \beta) \) normal operator if \( \alpha\beta = 1 \). \( \square \)

**Theorem 2.10.** If \( T \) is \( p - (\alpha, \beta) \) normal operator and \( S \) is an unitary operator such that \( TS = ST \) then \( C = TS \) is also \( p - (\alpha, \beta) \) normal operator.

**Proof.**
\[
\alpha^2(C^*C)^p \leq CC^*p \leq \beta^2(C^*C)^p \\
\alpha^2((TS)^*TS)^p \leq TS(TS)^*p \leq \beta^2((TS)^*TS)^p \\
\alpha^2(S^*T^*TS)^p \leq TSS^*T^*p \leq \beta^2(S^*T^*TS)^p \\
\alpha^2(T^*S^*ST)^p \leq (TT^*)^p \leq \beta^2(T^*S^*ST)^p \\
\alpha^2(T^*T)^p \leq (TT^*)^p \leq \beta^2(T^*T)^p (\text{since } S \text{ is unitary}).
\]

Hence, \( TS \) is \( p - (\alpha, \beta) \) normal operator. \( \square \)

3. **Aluthge Transformation on powers of \( p - (\alpha, \beta) \) - normal operators**

An operator \( T \) can be decomposed into \( T = U|T| \) where \( U \) is partial isometry and \( |T| \) is square root of \( T^*T \) with \( N(U) = N(|T|) \) and this kernal condition \( N(U) = N(|T|) \) uniquely determines \( U \) and \( |T| \) in the polar decomposition of \( T \). In this section \( T = U|T| \) denotes the polar decomposition satisfying the kernal condition \( N(U) = N(|T|) \).

In this section, we consider new properties as an extension of \( p \) - hyponormal operators using the generalized Aluthge transform. In this section, we prove that if \( T \) is \( p - (\alpha, \beta) \) - normal operator then \( T^n \) is \( \frac{n}{p} - (\alpha, \beta) \) - normal operator for all positive integer \( n \). Moreover, we prove that if \( T = U|T| \) is \( p - (\alpha, \beta) \) - normal operator for \( 0 < p < 1 \), the Aluthge transform \( \tilde{T} = |T|^{\frac{1}{p}}U|T|^{\frac{1}{p}} \) is \( (p + \frac{1}{2}) - (\alpha, \beta) \) - normal operator.

For an operator \( T = U|T| \) defines \( \tilde{T} \) as follows:
\[ \tilde{T}_{s,t} = |T|^sU|T|^t \]
for \( s, t > 0 \) which is called the generalized Aluthge transform of \( T \). In this section, we will study \( p - (\alpha, \beta) \) - normal operators using their generalized Aluthge transform.

**Theorem 3.1.** **Furuta Inequality** [4] Let \( A \geq B \geq 0 \). Then for all \( r > 0 \),
\[
(1) \ (B^{\frac{1}{p}}A^pB^{\frac{1}{p}})^{\frac{1}{q}} \geq (B^{\frac{1}{p}}B^pB^{\frac{1}{p}})^{\frac{1}{q}} \\
(2) \ (A^{\frac{1}{p}}A^pA^{\frac{1}{p}})^{\frac{1}{q}} \geq (A^{\frac{1}{p}}B^pA^{\frac{1}{p}})^{\frac{1}{q}}
\]
for \( p \geq 0, q \geq 1 \) with \( (1 + r)q \geq p + r \).
Theorem 3.2. Let $T = U |T|$ be the polar decomposition of $p - (\alpha, \beta)$ - normal operator for $0 < p \leq 1$ then

1) $\tilde{T}_{s,t} = |T|^s U |T|^t$ is $\frac{p + \min(s, t)}{s + t}$ - normal for $s, t > 0$ such that $\max(s, t) \geq p$.

2) $\tilde{T}_{s,t}$ is $(\alpha, \beta)$ - normal for $0 < s, t \leq p$.

Proof. Let $T$ be $p - (\alpha, \beta)$ - normal operator then

$$\alpha^2 (T^* T)^p \leq (TT^*)^p \leq \beta^2 (T^* T)^p$$

$$\alpha^2 |T|^{2p} \leq |T^*|^{2p} \leq \beta^2 |T|^{2p}$$

(1) Assume $A = \alpha^2 |T|^{2p}$, $B = |T^*|^{2p}$ and $C = \beta^2 |T|^{2p}$. Then,

$$\frac{(\tilde{T}_{s,t}^* \tilde{T}_{s,t})^{p + \min(s, t)}}{s + t} = (|T|^s U^* |T|^{2s} U |T|^t) \frac{p + \min(s, t)}{s + t}$$

$$= U^*(|T|^s |T|^{2s} |T|^t) \frac{p + \min(s, t)}{s + t} U$$

$$= U^*(\beta^{-2s/p} B^t/2p C^s/p B^{t/2p}) \frac{p + \min(s, t)}{s + t} U$$

$$\geq \beta^{-2s/p} (p + \min(s, t)) \frac{p + \min(s, t)}{s + t} \geq \beta^{-2s/p} (p + \min(s, t)) |T|^{2(p + \min(s, t))}$$

(2) Assume $\tilde{T}_{s,t}^* \tilde{T}_{s,t}$ then

$$\frac{(\tilde{T}_{s,t}^* \tilde{T}_{s,t})^{p + \min(s, t)}}{s + t} = (|T|^s U^* |T|^{2s} U |T|^t) \frac{p + \min(s, t)}{s + t}$$

$$= U^*(|T|^s |T|^{2s} |T|^t) \frac{p + \min(s, t)}{s + t} U$$

$$= U^*(\alpha^{-2s/p} B^t/2p A^s/p B^{t/2p}) \frac{p + \min(s, t)}{s + t} U$$

$$\leq \alpha^{-2s/p} (p + \min(s, t)) \frac{p + \min(s, t)}{s + t} \leq \alpha^{-2s/p} (p + \min(s, t)) |T|^{2(p + \min(s, t))}$$

So, we have,

$$|T|^{2(p + \min(s, t))} \geq \alpha^{2s/p} (\tilde{T}_{s,t}^* \tilde{T}_{s,t}) \frac{p + \min(s, t)}{s + t} \quad (3.1)$$

and

$$|T|^{2(p + \min(s, t))} \leq \beta^{2s/p} (\tilde{T}_{s,t}^* \tilde{T}_{s,t}) \frac{p + \min(s, t)}{s + t} \quad (3.2)$$
From (3.1), (3.2), (3.3) and (3.4), we have

\[
(\widetilde{T}_{s,t}\widetilde{T}_{s,t})^{\frac{p+\min(s,t)}{s+t}} = (|T|^sU|T|^{2t}U^*|T|^s)^{\frac{p+\min(s,t)}{s+t}}
\]
\[
= (\frac{C}{\beta^2})^{s/2p}T^{\frac{2t}{s/2p}(\frac{C}{\beta^2})^{s/2p}}^{\frac{p+\min(s,t)}{s+t}}
\]
\[
= (\frac{C}{\beta^2})^{s/2p}B^\frac{1}{p}(\frac{C}{\beta^2})^{s/2p}T^{\frac{2t}{s/2p}(\frac{C}{\beta^2})^{s/2p}}^{\frac{p+\min(s,t)}{s+t}}
\]
\[
\leq \beta^{-2s/p(p+\min(s,t))}T^{\frac{2t}{s/2p}(\frac{C}{\beta^2})^{s/2p}}^{\frac{p+\min(s,t)}{s+t}}
\]
\[
\leq (\beta)\frac{p+\min(s,t)}{p}\left(\widetilde{T}_{s,t}\widetilde{T}_{s,t}\right)^{\frac{p+\min(s,t)}{s+t}}
\]

Applying Lowner - Heinz Inequality, we have
\[
\min\left(s,t\right)\left(\frac{p+\min(s,t)}{p}\right)^{\frac{p+\min(s,t)}{s+t}}
\]
\[
(\min\left(s,t\right)+\frac{1}{p}(p+\min(s,t)))^{\frac{p+\min(s,t)}{s+t}}
\]
\[
2(p+\min(s,t))\left(\beta\right)^{\frac{p+\min(s,t)}{p}}
\]
\[
\leq \left(\alpha\right)\frac{p+\min(s,t)}{p}\left(\widetilde{T}_{s,t}\widetilde{T}_{s,t}\right)^{\frac{p+\min(s,t)}{s+t}}
\]

So, we have
\[
(\widetilde{T}_{s,t}\widetilde{T}_{s,t})^{\frac{p+\min(s,t)}{s+t}} \leq \left(\beta\right)^{\frac{p+\min(s,t)}{p}}\left(\widetilde{T}_{s,t}\widetilde{T}_{s,t}\right)^{\frac{p+\min(s,t)}{s+t}}
\]

(3.3)

and

(3.4)

From (3.1), (3.2), (3.3) and (3.4), we have
\[
(\alpha)^{\frac{p+\min(s,t)}{p}}(\widetilde{T}_{s,t}\widetilde{T}_{s,t})^{\frac{p+\min(s,t)}{s+t}} \leq (\widetilde{T}_{s,t}\widetilde{T}_{s,t})^{\frac{p+\min(s,t)}{s+t}} \leq (\beta)\frac{p+\min(s,t)}{p}\left(\widetilde{T}_{s,t}\widetilde{T}_{s,t}\right)^{\frac{p+\min(s,t)}{s+t}}
\]

(3.4)

Since \[
1 \geq \left(\frac{1}{p}(s/p) + (t/p)\right) \geq (s/p) + (t/p)
\]

Therefore \(\widetilde{T}_{s,t}\) is \(\frac{p+\min(s,t)}{s+t}\) - normal.

(2) Applying Lowner - Heinz Inequality, we have
\[
\alpha^{2s/p}T^{2s} \leq |T|^s \leq \beta^{2s/p}T^{2s}
\]
If $0 < s, t \leq p$, we have
\[
\tilde{T}_{s,t}^* \tilde{T}_{s,t} = |T|^t |U^*| T^{2s} U |T|^t \\
\geq \frac{|T|^t |U^*| T^{2s} U |T|^t}{\beta^{\frac{2s}{p}}} \\
\geq \beta^{\frac{2s}{p}} |T|^t |U^*| T^{2s} U |T|^t \\
\geq \beta^{\frac{2s}{p}} |T|^{2(s+t)}
\]

Similarly, we have
\[
\tilde{T}_{s,t}^* \tilde{T}_{s,t} \geq \beta^{\frac{2s}{p}} |T|^{2(s+t)}. \quad (3.5)
\]

Since $|T^*| \geq |T|^{\frac{1}{\alpha}}, |T^*| \leq \beta^{\frac{1}{\beta}} |T|$,
\[
\tilde{T}_{s,t}^* \tilde{T}_{s,t} \leq \beta^{\frac{2s}{p}} |T|^{2(s+t)}. \quad (3.6)
\]

\[
\tilde{T}_{s,t}^* \tilde{T}_{s,t} = |T|^t |U| T^{2t} U |T|^t \\
= |T|^s |T^*|^{2t} |T|^s
\]

Since $|T^*| \geq |T|^{\frac{1}{\alpha}}, |T^*| \leq \beta^{\frac{1}{\beta}} |T|$, we have
\[
\tilde{T}_{s,t}^* \tilde{T}_{s,t} \leq \beta^{\frac{2s}{p}} |T|^{2(s+t)}. \quad (3.7)
\]

\[
\tilde{T}_{s,t}^* \tilde{T}_{s,t} \geq \alpha^{\frac{2s}{p}} |T|^{2(s+t)}. \quad (3.8)
\]

From (3.5), (3.6), (3.7) and (3.8), we have
\[
\alpha^{\frac{2(s+t)}{p}} \tilde{T}_{s,t} \tilde{T}_{s,t} \leq \tilde{T}_{s,t} \tilde{T}_{s,t} \leq \beta^{\frac{2(s+t)}{p}} \tilde{T}_{s,t} \tilde{T}_{s,t}. \quad \text{Therefore } \tilde{T}_{s,t} \text{ is } (\alpha, \beta) \text{- normal}. \quad \Box
\]

**Corollary 3.3.** Let $T = U |T|$ be $p - (\alpha, \beta)$ - normal operator for $0 < p < 1$, then
(1) $\tilde{T} = |T|^t U |T|^t$ is $(p + \frac{1}{2}) - (\alpha, \beta)$ - normal for $0 < p < 1/2$.
(2) $\tilde{T}$ is $(\alpha, \beta)$ - normal operator for $1/2 \leq p < 1$.

**Theorem 3.4.** Let $T = U |T|$ be $p - (\alpha, \beta)$ - normal operator for $0 < p < 1$, then
(1) $\beta^{-(p+1)(n-1)} (T^* T)^{p+1} \leq (T^* T^n)^{\frac{p+1}{n}} \leq \alpha^{-(p+1)(n-1)} (T^* T)^{p+1}$
(2) $\beta^{-(p+1)(n-1)} (T^n T^*)^{\frac{p+1}{n}} \leq (T T^*)^{p+1} \leq \alpha^{-(p+1)(n-1)} (T^n T^*)^{\frac{p+1}{n}}$ holds for all positive integer $n$.

**Proof.** (1) Let $A_n = (T^* T^n)^{\frac{p}{n}} = |T^n|^{\frac{2p}{n}}$ and $B_n = (T^n T^*)^{\frac{p}{n}} = |T^n|^{\frac{2p}{n}}$ for all positive integer $n$.

By induction $\beta^{-(p+1)(n-1)} (T^* T)^{p+1} \leq (T^* T^n)^{\frac{p+1}{n}} \leq \alpha^{-(p+1)(n-1)} (T^* T)^{p+1}$ holds for $n = k$. Since $A_k = (T^k T^k)^{p/k} \geq \beta^{-(k-1)} (T^* T)^{p} \geq \beta^{-(k+1)} |T|^2$. 


\[ A_k = (T^*T)^{p/k} \leq \alpha^{-(k-1)} (T^*T)^p \leq \alpha^{-(k+1)} B_1. \]

It follows that
\[
(T^{(k+1)*}T^{k+1}) \frac{p+1}{k+1} = (U^* | T^* | T^k | T^* | U) \frac{p+1}{k+1} \\
= U^* (| T^* | T^k | T^* |) \frac{p+1}{k+1} U \\
= U^* (B_1^{\frac{1}{2p}} A_k^{\frac{1}{2p}} B_1^{\frac{1}{2p}}) \frac{p+1}{k+1} U \\
\geq \beta \frac{k(p+1)}{p} (T^*T)^{p+1}. 
\]

Similarly
\[
(T^{(k+1)*}T^{k+1}) \frac{p+1}{k+1} \leq \alpha \frac{k(p+1)}{p} (T^*T)^{p+1}. 
\]
Hence \( \beta \frac{-(p+1)}{p} (n-1) (T^*T)^{p+1} \leq (T^n T^n^*) \frac{p+1}{n} \leq \alpha \frac{-(p+1)}{p} (n-1) (T^*T)^{p+1} \).

(2) Assume that \( \beta \frac{-(p+1)}{p} (n-1) (T^n T^n^*) \frac{p+1}{n} \leq (T^*T)^{p+1} \leq \alpha \frac{-(p+1)}{p} (n-1) (T^n T^n^*) \frac{p+1}{n} \) holds for \( n = k \). Then
\[ A_1 = (T^*T)^p \geq \beta^{-2} (TT^*)^p \geq \beta^{-2(k+1)} (T^{k} T^{k*})^{p/k} \geq \beta^{-(k+1)} B_k. \]
Similarly \( A_1 \leq \alpha^{-(k+1)} B_k \). Hence we have
\[
(T^{(k+1)*}T^{k+1}) \frac{p+1}{k+1} = (U^* | T^* | T^k | T^* | U) \frac{p+1}{k+1} \\
= U^* (| T^* | T^k | T^* |) \frac{p+1}{k+1} U \\
= U^* (A_1^{\frac{1}{2p}} B_1^{\frac{1}{2p}} A_1^{\frac{1}{2p}}) \frac{p+1}{k+1} U \\
\leq \beta \frac{k(p+1)}{p} (U^* A_1^{\frac{1}{2p}} A_1^{\frac{1}{2p}} A_1^{\frac{1}{2p}}) \frac{p+1}{k+1} U \text{ by Furuta Inequality} \\
\leq \beta \frac{k(p+1)}{p} |T^*|^{2(p+1)}. 
\]
Similarly, \( (T^{(k+1)*}T^{k+1}) \frac{p+1}{k+1} \geq \alpha \frac{k(p+1)}{p} |T^*|^{2(p+1)}. \)

Therefore, \( \beta \frac{-(p+1)}{p} (n-1) (T^n T^n^*) \frac{p+1}{n} \leq (T^*T)^{p+1} \leq \alpha \frac{-(p+1)}{p} (n-1) (T^n T^n^*) \frac{p+1}{n} \) holds for all positive integer \( n \).

\[ \square \]

**Corollary 3.5.** If \( T \) is \( p - (\alpha, \beta) \) normal operator then \( T^n \) is \( \frac{p}{n} - (\alpha, \beta) \) normal operator for all positive integer \( n \).

**Proof.** Let \( T \) be \( p - (\alpha, \beta) \) normal operator, then by Theorem 3.4, we have

So, \( T^n \) is \( \frac{p}{n} - (\alpha, \beta) \) normal operator.

\[ \square \]
4. Properties of \( p-(\alpha, \beta) \) - normal operators

**Definition 4.1.** An operator \( T \) is called totally \( p-(\alpha, \beta) \) - normal, if the translate \( T - \lambda \) is \( p-(\alpha, \beta) \) - normal for all \( \lambda \in C \).

**Definition 4.2.** An operator \( T \) is called algebraically \( p-(\alpha, \beta) \) - normal, if there exists a non constant polynomial \( p \) such that \( p(T) \) is \( p-(\alpha, \beta) \) - normal.

**Definition 4.3.** Let \( T \in B(H) \). The reducing point spectrum of \( T \), \( \sigma_{pr}(T) \) is the set of scalars \( \lambda \) for which there exists \( x \neq 0 \) such that \( Tx = \lambda x \) and \( T^* x = \overline{\lambda} x \).

**Definition 4.4.** An operator \( T \in B(H) \) is called finite if \( \|TX - XT - I\| \geq 1 \) for every \( X \in B(H) \).

Equivalently if \( \sigma_{pr}(T) \neq 0 \) then \( T \) is a finite operator.

**Theorem 4.5.** Every totally \( p-(\alpha, \beta) \) - normal operator is finite.

*Proof.* Since \( \ker(T - \lambda)^p = \ker((T - \lambda)^*)^p \) and since \( T - \lambda \) is \( p-(\alpha, \beta) \) - normal for every \( \lambda \), we get \( \sigma_p(T) = \sigma_{pr}(T) \). And since \( \partial \sigma(T) \subset \sigma_p(T) \), we get \( \sigma_{pr}(T) \neq 0 \). \( \square \)

**Definition 4.6.** Let \( \mathcal{M}^2 = \{AB : A, B \in \mathcal{M}\} \) where \( \mathcal{M} \) denote the class of all self adjoint operators. This class is defined and its various properties are studied in [9]

**Theorem 4.7.** An \( p-(\alpha, \beta) \) - normal operator \( T \in \mathcal{M}^2 \) is necessarily normal.

*Proof.* Let \( T = AB \) with \( A = A^* \) and \( B = B^* \). Consider the following two cases:

Case 1: \( A \) is injective, Then \( AT^* = TA \) and hence \( T \) is normal [9] Theorem 3.a.

Case 2: \( A \) has a non-trivial null space \( N \).

Let \( A = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} \) and \( B = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} \)

with respect to \( N^* \oplus N \) where \( A_1 \) is an injective self adjoint operator and \( B_1 \) is self adjoint. Then \( T = \begin{pmatrix} A_1B_1 & A_1B_2 \\ 0 & 0 \end{pmatrix} \)

Since \( A_1B_1 \) is the restriction of \( T \) to an invariant subspace, \( A_1B_1 \) is \( p-(\alpha, \beta) \) - normal and hence by case 1 it is normal. By [9] (Theorem 4) \( N \) is a reducing invariant subspace of \( T \) and hence \( A_1B_2 = 0 \). This implies \( T \) is normal. \( \square \)

**Theorem 4.8.** Every \( p-(\alpha, \beta) \) - normal operator can be written as \( T = A \oplus S \) where \( A \) is normal and \( S \) is \( p-(\alpha, \beta) \) - normal with \( \sigma_w(S) = \sigma(S) \).

*Proof.* By Weyl’s theorem \( \sigma(T) \setminus \sigma_w(T) = \pi_{00}(T) \). Let \( N \) be a closed linear subspace of \( H \) generated by \( \lambda_i \in \pi_{00}(T) \bigcup N(T - \lambda_i) \) Then \( N \) is reduced by \( T \). The decomposition \( H = N \oplus N^* \) gives \( T = A \oplus S \) where \( A \) is normal and \( S \) is \( p-(\alpha, \beta) \) - normal and \( \sigma_w(S) = \sigma(S) \). \( \square \)
References

1. A. Aluthge, *on p-hyponormal operators for 0 < p < 1*, Integral Equations operator theory.

Received: February 21, 2014