Reconstruction of a Dynamically Distorted Signal with Respect to the Measuring Transducer Degradation

Aleksandr Shestakov
Department of Information-Measuring Technique
South-Ural State University
Chelyabinsk, Russian Federation

Georgy Sviridyuk
Department of Equations of Mathematical Physics
South-Ural State University
Chelyabinsk, Russian Federation

Minzilia Sagadeeva
Department of Information-Measuring Technique
South-Ural State University
Chelyabinsk, Russian Federation

Copyright © 2014 Aleksandr Shestakov, Georgy Sviridyuk and Minzilia Sagadeeva. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

The results of the Sobolev type equations theory are lately used extensively for the measurement of dynamically distorted signals. In this paper we consider the optimal measurement problem for the system with known multiplicative effect, which has the form of a monotonically decreasing scalar function that depends on time. Introduction of such factor to the measuring transducer model allows us to interpret it as a function describing the decrease in sensor sensitivity. The exact and approximate solutions of optimal measurement for the above system are
Aleksandr Shestakov, Georgy Sviridyuk and Minzilia Sagadeeva

constructed. The convergence of approximate solution to the exact one is proved.

Mathematics Subject Classification: MSC 47D06, 49J15, 93A30

Keywords: mathematical model of the measuring transducer, the optimal measurement, the approximate solution algorithm

1 Introduction

Previously [4] the first of the co-authors proposed to use one of the automatic control theory models as a mathematical model of the measuring transducer (MT). However, the proposed model was mathematically incorrect, therefore technically reasonable hypothesis (such as ”method of sliding modes”) that allowed to embody a mathematical model ”in metal” [5] have been proposed. Later the second co-author proposed to use the ideas and methods of the optimal control theory [9], [10] to solve such problem. The resulting mathematical model become the basis for optimal measurement theory [7], [8]. The results of this theory in some cases were ”brought to number” [6].

However, a careful study of the MT mathematical model has revealed some of its shortcomings. Particularly the model does not take into account its degradation during the measurement. This article seeks to eliminate this annoying drawback. To achieve the goal we use the results of [2], [3]. The article besides the introduction and bibliography contains two parts. In the first one we consider an exact solution and in the second approximate solutions proposed in [7], [8] for the MT model are presented. References do not claim to be complete but only represent the tastes and preferences of the authors.

2 Exact Solution

Let $L$ and $M$ be square matrices of order $n$. Following [7], [8], we call the sets $\rho^L(M) = \{\mu \in \mathbb{C} : \det(\mu L - M) \neq 0\}$ and $\sigma^L(M) = \mathbb{C} \setminus \rho^L(M)$ $L$-resolvent set and $L$-spectrum of $M$ respectively. It is easy to show that either $\rho^L(M) = \emptyset$ or $\sigma^L(M)$ for $L$-spectrum of $M$ consists of a finite set of points. Furthermore we note that the set $\rho^L(M)$ and $\sigma^L(M)$ do not change at transition to another basis.

Define the matrix-valued functions $(\mu L - M)^{-1}, R^L_{\mu}(M) = (\mu L - M)^{-1}L$ and $L^L_{\mu}(M) = L(\mu L - M)^{-1}$ with the domain $\rho^L(M)$ which are called the $L$-resolvent, the right and the left $L$-resolvent of $M$ respectively.

Definition 1 The matrix $M$ is said to be $(L,p)$-regular, $p \in \{0\} \cup \mathbb{N}$, if $\rho^L(M) \neq \emptyset$ and $\infty$ is a removable singularity ($p = 0$) or a pole of order $p \in \mathbb{N}$ for $L$-resolvent of $M$. 
Let the matrix \( M \) be a \((L,p)\)-regular one, \( p \in \{0\} \cup \mathbb{N} \). Consider the Leontiev type system

\[
\begin{align*}
L\dot{x}(t) &= a(t)Mx(t) + Du(t), \\
y(t) &= Nx(t).
\end{align*}
\]

(1)

(2)

The system (1) was named so, because of its apparent similarity to the balance model of V. Leontiev. The condition \( \ker L \neq \{0\} \) corresponds to the presence of resonances in the sensor circuits \[1\]. Here \( x = (x_1, x_2, \ldots, x_n) \) and \( \dot{x} = (\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n) \) are the vector-functions of a state and a rate of state change for the MT respectively; \( u = (u_1, u_2, \ldots, u_n) \) and \( y = (y_1, y_2, \ldots, y_n) \) are the vector-functions of measurements and observations for the MT correspondingly; \( L \) and \( M \) are matrices representing the mutual velocity of state changes and the actual state of the MT. The scalar function \( a : (0, \tau) \to \mathbb{R}_+ \) describes the time variation of the parameters of the MT; \( D \) and \( N \) are square matrices of order \( n \), characterizing the interference of the measurement parameters and the relationship between system state and observation correspondingly.

Equations (1) and (2) are supplemented by the Showalter – Sidorov condition

\[
[R_\alpha^L(M)]^p+1(x(0) - x_0) = 0,
\]

(3)

which is in this situation more applicable than the traditional Cauchy condition \[6\], \[11\], \[12\]. Consider the penalty functional in the form

\[
J(u) = \beta \sum_{q=0}^{\tau} \left\| y^{(q)}(t) - y_0^{(q)}(t) \right\|^2_y dt + (1 - \beta) \sum_{q=0}^{\theta} \left\langle F_q u^{(q)}(t), u^{(q)}(t) \right\rangle_U dt,
\]

(4)

where \( \beta \in (0, 1] \), spaces \( \mathcal{U} \) and \( \mathcal{Y} \) are Hilbert ones, matrices \( F_q \) of order \( n \) are positively defined, \( \theta = 0, 1, \ldots, p + 1, \tau \in \mathbb{R}_+ \), and \( y_0 = (y_{01}, y_{02}, \ldots, y_{0n}) \) are the observations obtained in the field experiment results. Let \( \mathcal{X} = \{x \in L_2((0, \tau); \mathbb{R}^n) : \dot{x} \in L_2((0, \tau); \mathbb{R}^n)\} \) be a state space, \( \mathcal{U} = \{u \in L_2((0, \tau); \mathbb{R}^n) : u^{p+1} \in L_2((0, \tau); \mathbb{R}^n)\} \) be a measuring space and \( \mathcal{Y} = N[\mathcal{X}] \) be a space of observations for some fixed \( \tau \in \mathbb{R}_+ \). We distinguish a convex and closed subset \( \mathcal{U}_0 \) in the space \( \mathcal{U} \) which we call a set of admissible measurements. Our aim is to find the optimal measurement \( v \in \mathcal{U}_0 \) such that

\[
J(v) = \min_{u \in \mathcal{U}_0} J(u).
\]

(5)

Problem (1)–(5) is called the optimal measurement problem. A pair \((x, v) \in \mathcal{X} \times \mathcal{U}\) is named the exact solution of this problem if it satisfies the system (1), (2) almost everywhere in \((0, \tau)\) (where \( u = v \)), the conditions (3) (for some vector \( x_0 \in \mathbb{R}^n \)) and (4), (5).

To find the exact solution we use the results \[3\].
Theorem 2 Let $M$ be $(L, p)$-regular, $p \in \{0\} \cup \mathbb{N}$ and $\det M \neq 0$. Then for any vectors $x_0 \in \mathbb{R}^n$ and a separated from zero function $a \in C^{p+1}([0, \tau]; \mathbb{R}_+)$ there exists a unique exact solution $(x, v)$ of problem (1)–(5), whereas

$$x(t) = \lim_{k \to \infty} \left[ \left( L - \frac{1}{k} M \int_0^t a(\zeta) d\zeta \right)^{-1} L \right]^{k} x_0 +$$

$$+ \int_0^t \left( L - \frac{1}{k} M \int_s^t a(\zeta) d\zeta \right)^{-1} L \left( L - \frac{1}{k} M \right)^{-1} (kL_k^L(M))^p Dv(s) ds +$$

$$+ \sum_{q=0}^p \left( M^{-1} \left( (kL_k^L(M))^{p+1} - I_n \right) L \right)^q M^{-1} \left( I_n - (kL_k^L(M))^{p+1} \right) \left( \frac{1}{a(t)} \frac{d}{dt} \right)^q Dv(t) \frac{a(t)}{a(t)}.$$

(6)

3 Approximate Solutions

We present an algorithm for construction of an approximate solution for the problem (1)–(5). Firstly we construct a finite-dimensional space $U^\ell$ of trigonometric polynomial vectors of the form

$$u^\ell(t) = \text{col} \left( \sum_{j=0}^{\ell} \tilde{c}_{1j} \cos jt + \tilde{b}_{1j} \sin jt, \sum_{j=0}^{\ell} \tilde{c}_{2j} \cos jt + \tilde{b}_{2j} \sin jt, \ldots, \sum_{j=0}^{\ell} \tilde{c}_{nj} \cos jt + \tilde{b}_{nj} \sin jt \right),$$

(7)

then substitute $u^\ell$ into (6) instead of $v$ and obtain

$$\tilde{x}_k^\ell(t) = \left( L - \frac{A(t)}{k} M \right)^{-1} L x_0 +$$

$$+ \int_0^t \left( L - \frac{A(t)}{k} M \int_s^t a(\zeta) d\zeta \right)^{-1} L \left( L - \frac{1}{k} M \right)^{-1} (kL_k^L(M))^p Du^\ell(s) ds +$$

$$+ \sum_{q=0}^p \left( M^{-1} \left( (kL_k^L(M))^{p+1} - I_n \right) L \right)^q M^{-1} \left( I_n - (kL_k^L(M))^{p+1} \right) \left( \frac{1}{a(t)} \frac{d}{dt} \right)^q Du^\ell(t) \frac{a(t)}{a(t)},$$

(8)

where $A(t) = \int_0^t a(s) ds$. After taking the integral in (8) and derivatives, the result $\tilde{x}_k^\ell = \tilde{x}_k^\ell(t)$ is substituted in (4). We get a continuous and strongly convex
Reconstruction of a dynamically distorted signal

functional on the space $\mathbf{R}^{2\ell n}$ in the form $J_k^\ell = J_k^\ell(\tilde{b}_{10}, \tilde{b}_{11}, \ldots, \tilde{b}_{1\ell}, \tilde{b}_{20}, \tilde{b}_{21}, \ldots, \tilde{b}_{2\ell}, \ldots, \tilde{b}_{n0}, \tilde{b}_{n1}, \ldots, \tilde{b}_{n\ell}, \tilde{c}_{10}, \ldots, \tilde{c}_{n\ell}, \ldots, \tilde{c}_{\alpha_1}, \ldots, \tilde{c}_{\alpha_\ell})$. Denote the projection of $\mathcal{U}_\theta$ on $\mathcal{U}^\ell$ by $\mathcal{U}_\theta^{\ell}$. This set is obviously also convex and closed, and it corresponds to a convex closed set $\mathbf{R}^{2\ell n}_\theta \subset \mathbf{R}^{2\ell n}$. By the Mazur theorem the functional $J_k^\ell$ has a unique minimum in $\mathbf{R}^{2\ell n}_\theta$. Substituting the coordinates $(b_{10}, \ldots, c_{n\ell})$ of the minimum point of the functional $J_k^\ell$ into (7) we get an approximate optimal measurement $v_k^\ell$. Replacing the function $u^\ell$ by $v_k^\ell$ in (8) we get a pair $(x_k^\ell, v_k^\ell)$, which is called an approximate solution of the problem (1)–(5). Thus, we have the

**Theorem 3** Let the conditions of Theorem 2 be fulfilled. Then for any vector $x_0 \in \mathbf{R}^n$ and a separated from zero function $a \in C^{p+1}([0, \tau]; \mathbf{R}_+)$ there exists a unique approximate solution $(x_k^\ell, v_k^\ell)$ of problem (1)–(5).

Let us discuss the convergence of the approximate solutions $(x_k^\ell, v_k^\ell)$ of the problem (1)–(5) to the exact solution $(x, v)$ for $k, \ell \to \infty$. First of all, we fix $\ell \in \mathbb{N}$ and consider the problem (1)–(5) in case when $\mathcal{U} = \mathcal{U}^\ell$, $\mathcal{U}_\theta = \mathcal{U}_\theta^\ell$. By Theorem 2, there exists a unique exact solution (1)–(5), which is denoted by $(x^\ell, v^\ell)$. Arguing as in [8], due to the strong convexity of the constructed functional $J$ on $\mathcal{U}_\theta^\ell$, closeness and convexity of the set $\mathcal{U}_\theta^\ell$ we obtain the existence of $\lim_{k \to \infty} (x_k^\ell, v_k^\ell) = (x^\ell, v^\ell)$. Then noting that $\mathcal{U} = \bigcup_{\ell=1}^{\infty} \mathcal{U}^\ell$ and $\mathcal{U}_\theta = \bigcup_{\ell=1}^{\infty} \mathcal{U}_\theta^\ell$ we conclude that $\{(x^\ell, v^\ell)\}$ is a minimizing sequence for the functional (4). Thus we have proved

**Theorem 4** Let the conditions of Theorem 2 be fulfilled, then for an arbitrary vector $x_0 \in \mathbf{R}^n$ and a separated from zero function $a \in C^{p+1}([0, \tau]; \mathbf{R}_+)$ $\lim_{l \to \infty} \lim_{k \to \infty} (x_k^\ell, v_k^\ell) = (x, v)$.

**Acknowledgements.** The authors thank Professor A.V. Keller for valuable comments and clarifications, which have improved the results.

**References**


Received: December 15, 2013