Coupled Fixed Point for Map Satisfying the Mixed Monotone Property in Partially Ordered Complete NIFMS

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Abstract

In this paper, we define non-Archimedean intuitionistic fuzzy metric space (Shortly, NIFMS), and prove a coupled fixed point theorems for map satisfying the mixed monotone property in partially ordered complete NIFMS.

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1. Introduction

Fuzzy set theory was first introduced by Zadeh[13] in 1965. Kramosil and Michalek[5] introduced the concept of fuzzy metric space in 1975, which opened an avenue for further development of analysis in such spaces. Later on, it is modified that a few concepts of mathematical analysis have been generalized by George and Veeramani[1], and also, Vasuki and Veeramani[12] generalized the fixed point theorem in fuzzy metric space. Afterwards, many articles have been published on fixed point theorems under different contractive condition in fuzzy metric spaces. Also, Luong et.al.[6] proved coupled fixed points in partially ordered metric spaces, and Samanta and Mohinta[10] studied coupled fixed point theorems for mapping having the mixed monotone property. Also,
Hu[4] studied common coupled fixed point theorems for contractive mappings in fuzzy metric space, and Park et.al.[7] defined an IFMS and proved a fixed point theorem in IFMS, Park[9] studied common coupled fixed point theorems for compatible mappings in an IFMS.

In this paper, we define non-Archimedean intuitionistic fuzzy metric space (Shortly, NIFMS), and prove a coupled fixed point theorems for map satisfying the mixed monotone property in partially ordered complete NIFMS.

2. Preliminaries

In this section, we recall some definitions, properties and known results in NIFMS as following:

Let us recall (see [11]) that a continuous \( t \)-norm is a operation \( * : [0, 1] \times [0, 1] \rightarrow [0, 1] \) which satisfies the following conditions: (a) \( * \) is commutative and associative, (b) \( * \) is continuous, (c) \( a * 1 = a \) for all \( a \in [0, 1] \), (d) \( a * b \leq c * d \) whenever \( a \leq c \) and \( b \leq d \) \( (a, b, c, d \in [0, 1]) \). Also, a continuous \( t \)-conorm is a operation \( \diamond : [0, 1] \times [0, 1] \rightarrow [0, 1] \) which satisfies the following conditions: (a) \( \diamond \) is commutative and associative, (b) \( \diamond \) is continuous, (c) \( a \diamond 0 = a \) for all \( a \in [0, 1] \), (d) \( a \diamond b \geq c \diamond d \) whenever \( a \leq c \) and \( b \leq d \) \( (a, b, c, d \in [0, 1]) \).

**Definition 2.1.** ([7]) The 5-tuple \( (X, M, N, *, \diamond) \) is said to be a non-Archimedean intuitionistic fuzzy metric space (shortly, NIFMS) if \( X \) is an arbitrary set, \( * \) is a continuous \( t \)-norm, \( \diamond \) is a continuous \( t \)-conorm and \( M, N \) are fuzzy sets on \( X \times X \times (0, \infty) \) satisfying the following conditions; for all \( x, y \in X \) and \( t > 0 \), such that
\( (a) M(x, y, t) > 0, \)
\( (b) M(x, y, t) = 1 \) if and only if \( x = y, \)
\( (c) M(x, y, t) = M(y, x, t), \)
\( (d) M(x, y, s) * M(y, z, t) \leq M(x, z, \max\{s, t\}), \)
\( (e) M(x, y, \cdot) : (0, \infty) \rightarrow (0, 1] \) is continuous,
\( (f) N(x, y, t) > 0, \)
\( (g) N(x, y, t) = 0 \) if and only if \( x = y, \)
\( (h) N(x, y, t) = N(y, x, t), \)
\( (i) N(x, y, s) \diamond N(y, z, t) \geq N(x, z, \max\{s, t\}), \)
\( (j) N(x, y, \cdot) : (0, \infty) \rightarrow (0, 1] \) is continuous.

Note that \( (M, N) \) is called a NIFM on \( X \). The functions \( M(x, y, t) \) and \( N(x, y, t) \) denote the degree of nearness and the degree of non-nearness between \( x \) and \( y \) with respect to \( t \), respectively.

**Remark 2.2.** Every non-Archimedean intuitionistic fuzzy metric space is an intuitionistic fuzzy metric space.

**Definition 2.3.** ([10]) A partially ordered set is a set \( P \) and a binary relation \( \preceq \), denoted by \( (X, \preceq) \) such that for all \( a, b, c \in P, \)
\( (a) a \preceq a \) (reflexivity),
\( (b) a \preceq b \) and \( b \preceq c \) implies \( a \preceq c \) (transitivity),
\( (c) a \preceq b \) and \( b \preceq a \) implies \( a = b \) (anti-symmetry).
Lemma 2.5. ([8]) Let \( x \in X \) be a NIFMS.

(a) \( \{x_n\} \) is said to be convergent to \( x \in X \) by \( \lim_{n \to \infty} x_n = x \) if for all \( t > 0 \),
\[
\lim_{n \to \infty} M(x_n, x, t) = 1, \quad \lim_{n \to \infty} N(x_n, x, t) = 0.
\]
(b) \( \{x_n\} \subset X \) is called a Cauchy sequence if for all \( t > 0 \) and \( p = 1, 2, 3, \ldots \)
\[
\lim_{n \to \infty} M(x_n, x_{n+p}, t) = 1, \quad \lim_{n \to \infty} N(x_n, x_{n+p}, t) = 0.
\]
(c) \( X \) is complete if and only if every Cauchy sequence converges in \( X \).

Lemma 2.5. ([8]) Let \( \{x_n\} \) be a sequence in an NIFMS \( X \) with \( t \star t \geq t \) and \( t \circ t \leq t \). If there exist a number \( k \in (0, 1) \) such that for all \( x, y \in X \), \( t > 0 \) and \( n = 1, 2, \ldots \),
\[
M(x_{n+2}, x_{n+1}, t) \geq M(x_{n+1}, x_n, t), \quad N(x_{n+2}, x_{n+1}, t) \leq N(x_{n+1}, x_n, t),
\]

then \( \{x_n\} \) is a Cauchy sequence in \( X \).

Definition 2.6. ([9]) Let \( X \) be an NIFMS. Also, let \( F : X \times X \to X \). An element \( (x, y) \in X \times X \) is called a coupled fixed point of the mapping \( F \) if \( F(x, y) = x \), \( F(y, x) = y \).

Definition 2.7. ([10]) Let \( (X, \preceq) \) be a partially ordered set and \( F : X \times X \to X \). The mapping \( F \) is said to satisfy the mixed monotone property if \( F(x, y) \) is monotone non-decreasing in \( x \) and is monotone non-increasing in \( y \), that is, for any \( x, y \in X \),
\[
\begin{align*}
x_1, x_2 \in X, & \quad x_1 \preceq x_2 \Rightarrow F(x_1, y) \preceq F(x_2, y), \\
y_1, y_2 \in X, & \quad y_1 \preceq y_2 \Rightarrow F(x, y_1) \succeq F(x, y_2).
\end{align*}
\]

3. Main result and example

Theorem 3.1. Let by \( (X, \preceq) \) be a partially ordered set and \( X \) is a complete NIFMS. Let \( F : X \times X \to X \) be a continuous mapping satisfying the mixed monotone property on \( X \) such that there exist two elements \( x_0, y_0 \in X \) with \( x_0 \preceq F(x_0, y_0) \) and \( y_0 \preceq F(y_0, x_0) \) and
\[
\begin{align*}
M(F(x, y), F(u, v), kt) & \geq \max\{M(F(x, y), x, t), M(F(u, v), x, t)\} \\
& \quad \star \max\{M(F(x, y), u, t), M(F(u, v), u, t)\} \\
N(F(x, y), F(u, v), kt) & \leq \min\{N(F(x, y), x, t), N(F(u, v), x, t)\} \\
& \quad \diamond \min\{N(F(x, y), u, t), N(F(u, v), u, t)\} \\
& \quad \diamond \min\{N(F(x, y), u, t), N(F(u, v), x, t)\},
\end{align*}
\]
for all \( x, y, u, v \in X \) with \( x \succeq u, y \preceq v \) and \( k \in (0, 1) \). then \( F \) has a coupled fixed point in \( X \).
Proof. Let \( x_0, y_0 \in X \) with \( x_0 \preceq F(x_0, y_0) \) and \( y_0 \succeq F(y_0, x_0) \). Define the sequences \( \{x_n\} \) and \( \{y_n\} \) in \( X \) such that for all \( n = 0, 1, 2, \ldots \),

\[
x_{n+1} = F(x_n, y_n) \quad \text{and} \quad y_{n+1} = F(y_n, x_n).
\]

We prove that \( x_n \preceq y_n \) for all \( n = 0, 1, 2, \ldots \). Since \( x_0 \preceq F(x_0, y_0) \), \( y_0 \succeq F(y_0, x_0) \) and \( x_1 = F(x_0, y_0) \), \( y_1 = F(y_0, x_0) \), we have \( x_0 \preceq x_1 \) and \( y_0 \succeq y_1 \). Now, suppose that for some \( n, x_n \preceq x_{n+1} \) and \( y_n \succeq y_{n+1} \). Then by Definition 2.7,

\[
x_{n+2} = F(x_{n+1}, y_{n+1}) \preceq F(x_n, y_{n+1}) \preceq F(x_n, y_n) = x_{n+1},
\]

\[
y_{n+2} = F(y_{n+1}, x_{n+1}) \preceq F(y_n, x_{n+1}) \preceq F(y_n, x_n) = y_{n+1}.
\]

Thus by the mathematical induction, for all \( n \in \mathbb{N} \),

\[
x_0 \preceq x_1 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots,
\]

\[
y_0 \succeq y_1 \succeq \cdots \succeq y_n \succeq y_{n+1} \succeq \cdots.
\]

From (3.1), we have

\[
M(F(x_n, y_n), F(x_{n-1}, y_{n-1}), kt)
\]

\[
\geq \max\{M(F(x_n, y_n), x_n, t), M(F(x_{n-1}, y_{n-1}), x_n, t)\}
\]

\[
\quad \times \max\{M(F(x_n, y_n), x_{n-1}, t), M(F(x_{n-1}, y_{n-1}), x_{n-1}, t)\}
\]

\[
\quad \times \max\{M(F(x_n, y_n), x_{n-1}, t), M(F(x_{n-1}, y_{n-1}), x_{n-1}, t)\},
\]

\[
N(F(x_n, y_n), F(x_{n-1}, y_{n-1}), kt)
\]

\[
\leq \min\{N(F(x_n, y_n), x_n, t), N(F(x_{n-1}, y_{n-1}), x_n, t)\}
\]

\[
\quad \diamond \min\{N(F(x_n, y_n), x_{n-1}, t), N(F(x_{n-1}, y_{n-1}), x_{n-1}, t)\}
\]

\[
\quad \diamond \min\{M(F(x_n, y_n), x_{n-1}, t), M(F(x_{n-1}, y_{n-1}), x_{n-1}, t)\}.
\]

Hence

\[
M(x_{n+1}, x_n, kt)
\]

\[
\geq \max\{M(x_{n+1}, x_n, t), M(x_n, x_n, t)\}
\]

\[
\quad \times \max\{M(x_{n+1}, x_{n-1}, t), M(x_n, x_{n-1}, t)\}
\]

\[
\quad \times \max\{M(x_{n+1}, x_{n-1}, t), M(x_n, x_{n-1}, t)\},
\]

\[
N(x_{n+1}, x_n, kt)
\]

\[
\leq \min\{N(x_{n+1}, x_n, t), N(x_n, x_n, t)\}
\]

\[
\quad \diamond \min\{N(x_{n+1}, x_{n-1}, t), N(x_n, x_{n-1}, t)\}
\]

\[
\quad \diamond \min\{N(x_{n+1}, x_{n-1}, t), N(x_n, x_{n-1}, t)\}
\]

\[
\leq \min\{N(x_{n+1}, x_{n-1}, t), N(x_n, x_{n-1}, t)\}
\]

\[
\geq \min\{N(x_{n+1}, x_n, t) \diamond N(x_n, x_{n-1}, t), N(x_n, x_{n-1}, t)\}.
\]
Since
\[ M(x_{n+1}, x_n, t) \geq M(x_n, x_{n-1}, \frac{t}{k^n}) \geq \cdots \geq M(x_1, x_0, \frac{t}{k^n}), \]
\[ N(x_{n+1}, x_n, t) \leq N(x_n, x_{n-1}, \frac{t}{k^n}) \leq \cdots \leq N(x_1, x_0, \frac{t}{k^n}). \]

Therefore
\[ \lim_{n \to \infty} M(x_{n+1}, x_n, t) \geq 1, \quad \lim_{n \to \infty} N(x_{n+1}, x_n, t) \leq 0. \]

Also, since
\[ M(x_n, x_{n+p}, t) \geq M(x_n, x_{n+1}, t) \ast \cdots \ast M(x_{n+p-1}, x_{n+p}, t), \]
\[ N(x_n, x_{n+p}, t) \leq N(x_n, x_{n+1}, t) \odot \cdots \odot N(x_{n+p-1}, x_{n+p}, t), \]
\[ \lim_{n \to \infty} M(x_n, x_{n+p}, t) \geq 1 \ast \cdots \ast 1 = 1, \quad \lim_{n \to \infty} N(x_n, x_{n+p}, t) \leq 0 \odot \cdots \odot 0 = 0. \]

Hence \( \{ x_n \} \) is a Cauchy sequence in \( X \).

Similarly, by same method, \( \{ y_n \} \) is a Cauchy sequence in \( X \). Since \( X \) is a complete NIFMS, there exist \( x, y \in X \) such that
\[ \lim_{n \to \infty} x_n = x, \quad \lim_{n \to \infty} y_n = y, \]
we have
\[ x = \lim_{n \to \infty} x_n = \lim_{n \to \infty} F(x_{n-1}, y_{n-1}) \]
\[ = F(\lim_{n \to \infty} x_{n-1}, \lim_{n \to \infty} y_{n-1}) = F(x, y), \]
\[ y = \lim_{n \to \infty} y_n = \lim_{n \to \infty} F(y_{n-1}, x_{n-1}) \]
\[ = F(\lim_{n \to \infty} y_{n-1}, \lim_{n \to \infty} x_{n-1}) = F(y, x). \]

Hence \( x = F(x, y), \ y = F(y, x) \). Therefore \( F \) has a coupled fixed point in \( X \). This completes the proof of the theorem. \( \Box \)

**Theorem 3.2.** Let by \((X, \preceq)\) be a partially ordered set and \( X \) is a complete NIFMS. Let \( F : X \times X \to X \) be a continuous mapping satisfying the mixed monotone property on \( X \) such that there exist two elements \( x_0, y_0 \in X \) with \( x_0 \preceq F(x_0, y_0) \) and \( y_0 \succeq F(y_0, x_0) \). Suppose that for all \( x, y, u, v \in X \) with \( x \succeq u, \ y \preceq v \) and \( k \in (0, 1) \).

\[
(3.2) \quad M(F(x, y), F(u, v), kt) \geq \max\{M(x, u, t), M(y, v, t)\},
\quad N(F(x, y), F(u, v), kt) \leq \min\{N(x, u, t), N(y, v, t)\}.
\]

Also, suppose either
(a) \( F \) is a continuous or
(b) \( X \) has the following property;
   (i) if a non-decreasing sequence \( \{ x_n \} \to x \), then \( x_n \preceq x \) for all \( n \),
   (ii) if a non-increasing sequence \( \{ y_n \} \to y \), then \( y_n \succeq y \) for all \( n \).

Then there exist \( x, y \in X \) such that \( x = F(x, y) \) and \( y = F(y, x) \), that is, \( F \) has a coupled fixed point in \( X \).
Proof. Let $x_0, y_0 \in X$ with $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$. Define the sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that for all $n = 0, 1, 2, \ldots$,

$$x_{n+1} = F(x_n, y_n) \quad \text{and} \quad y_{n+1} = F(y_n, x_n).$$

Hence by same method of Theorem 3.1, $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in $X$. Since $X$ is a complete NIFMS, there exist $x, y \in X$ such that

$$\lim_{n \to \infty} x_n = x, \quad \lim_{n \to \infty} y_n = y.$$

Suppose that the hypothesis (a) holds, we get as $n \to \infty$,

$$x = \lim_{n \to \infty} x_n = \lim_{n \to \infty} F(x_{n-1}, y_{n-1}) = F(\lim_{n \to \infty} x_{n-1}, \lim_{n \to \infty} y_{n-1}) = F(x, y),$$

$$y = \lim_{n \to \infty} y_n = \lim_{n \to \infty} F(y_{n-1}, x_{n-1}) = F(\lim_{n \to \infty} y_{n-1}, \lim_{n \to \infty} x_{n-1}) = F(y, x).$$

Hence $x = F(x, y), y = F(y, x)$. Therefore $F$ has a coupled fixed point in $X$.

Finally, suppose that the hypothesis (b) holds. Then, since $\{x_n\}$ is non-decreasing and $\{x_n\} \to x$ and $\{y_n\}$ is non-increasing and $\{y_n\} \to y$, we have $x_n \preceq x$ and $y_n \succeq y$ for all $n$. From (3.2),

$$M(F(x_n, y_n), F(x, y), kt) \geq \max\{M(x_n, x, t), M(y_n, y, t)\},$$

$$N(F(x_n, y_n), F(x, y), kt) \leq \min\{N(x_n, x, t), N(y_n, y, t)\}.$$

Taking $n \to \infty$, $M(x, F(x, y), kt) = 1$ and $N(x, F(x, y), kt) = 0$. Thus $x = F(x, y)$. Similarly, we can get $y = F(y, x)$. Hence $F$ has a coupled fixed point in $X$.

**Example 3.3.** Let $X = [0, 1]$ with NIFM $M, N$ defined by

$$M(x, y, kt) = \frac{kt}{kt + |x - y|}, \quad N(x, y, kt) = \frac{|x - y|}{kt + |x - y|}$$

for all $x, y \in X$ and $k \in [\frac{1}{2}, 1)$.

We consider the following conditions: For $x, y \in X$, $x \preceq y$ if and only if $x, y \in [0, 1]$ and $x \leq y$, where $\preceq$ is the usual ordering of real numbers, then $(X, \preceq)$ is partially ordered set.

Also, $X$ is a complete NIFMS. Moreover, $X$ has the following property:

(i) if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all $n$,

(ii) if a non-increasing sequence $\{y_n\} \to y$, then $y_n \succeq y$ for all $n$.

Let $F : X \times X \to X$ defined by

$$F(x, y) = \begin{cases} 
\frac{x - y}{2} & \text{if } x, y \in [0, 1], \ x \geq y, \\
0 & \text{if } x < y
\end{cases}$$

Then $F$ is continuous and satisfy the mixed monotone property. Also, there exist $x_0 = 0, y_0 = 0$ in $X$ such that

$$x_0 = 0 \preceq F(0, 0) = F(x_0, y_0), \ y_0 = 0 \succeq F(0, 0) = F(y_0, x_0).$$
Then \((0, 0)\) is a coupled fixed point of \(F\) in \(X\).

**References**


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