Edge Energy Bounds of Graphs

P. Nageswari ¹ and P. B. Sarasija

Department of Mathematics
Noorul Islam Centre For Higher Education
Kumaracoil-629175, TamilNadu, India

Abstract

Let G be a graph with n vertices and m edges. In analog to the definition of Energy of a graph the Edge energy is defined as the sum of the absolute edge eigenvalues, the eigen values of the Edge Adjacency matrix. In this paper we investigate the bounds of the edge energy of graphs.

Mathematics Subject Classification: 05C50; 15A18

Keywords: Edge spectrum; Edge Energy of graph; Edge Energy Bounds

1 Introduction

In this paper we are concerned with finite, simple and undirected graphs. For notations and terminology see [1, 3, 4, 5]. Let G be such a graph possessing n vertices and m edges. [6, 7] The edge adjacency matrix denoted by $E(G)$ of a graph G is a real square symmetric matrix, determined by the adjacencies of edges (Two edges are said to be adjacent if they are incident to a common vertex) is defined as $[e]_{ij} = 1$ if edges i and j are adjacent and 0 otherwise.

The eigen values $\rho_1, \rho_2, \ldots, \rho_m$ of the edge adjacency matrix $E(G)$ are said to be edge eigenvalues of the graph G. A number of topological indices can be reformulated in terms of the edge degrees instead of the vertex degrees such as the total edge adjacency index and the edge connectivity index. The Energy of

¹Corresponding author
the graph $G$ was first defined by Ivan Gutman in 1978 [8]. During these days the energy of a graph is a much studied quantity in the mathematical literature (See, for example [8, 9, 10, 11, 12]). Motivated by the interesting results on energy of a graph, in this paper we define and analyse the Edge energy (Bond energy) as $EE(G) = \sum_{i=1}^{m} |\rho_i|$.

**Theorem 1.1.** If $G$ is a connected graph on $n$ vertices with $m$ edges and let $\rho_1, \rho_2, \ldots, \rho_m$ be its edge eigenvalues. Then $\sum_{i=1}^{m} \rho_i = 0$ and $\sum_{i=1}^{m} \rho_i^2 = 2 \sum_{1 \leq i < j \leq m} e_{ij}$.

**Proof.** We have $\sum_{i=1}^{m} \rho_i = \text{trace}[E(G)] = 0$.

Also $\sum_{i=1}^{m} \rho_i^2 = \text{trace}[E(G)]^2 = \sum_{i=1}^{m} \sum_{j=1}^{m} e_{ij} = 2 \sum_{1 \leq i < j \leq m} e_{ij} = 2 \sum_{1 \leq i < j \leq m} e_{ij}$, since $e_{ij} = 0$ or $1$.

**Theorem 1.2.** If $G$ is a connected graph on $n$ vertices with $m$ edges then the inequality $\sqrt{2} \sum_{1 \leq i < j \leq m} e_{ij} \leq EE(G) \leq \sqrt{2m} \sum_{1 \leq i < j \leq m} e_{ij}$ holds.

**Proof.** We have the Cauchy-Schwartz inequality $\left\{ \sum_{i=1}^{n} a_i b_i \right\}^2 \leq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2$. Considering $a_i = 1$ and $b_i = |\rho_i|$, we get $\left( \sum_{i=1}^{m} |\rho_i| \right)^2 \leq m \sum_{i=1}^{m} \rho_i^2 = 2 \sum_{1 \leq i < j \leq m} e_{ij}$, which gives $EE(G)^2 \leq 2m \sum_{1 \leq i < j \leq m} e_{ij}$, and leads to the upper bound.

And $EE(G)^2 = \left( \sum_{i=1}^{m} |\rho_i| \right)^2 \geq \sum_{i=1}^{m} |\rho_i|^2 = 2 \sum_{1 \leq i < j \leq m} e_{ij}$ which gives the lower bound. \qed

**Corollary 1.3.** The bounds of the path $P_n$, cycle $C_n$, complete graph $K_n$ are as follows

1) $\sqrt{2(n-2)} \leq EE(P_n) \leq \sqrt{2m(n-2)}$

2) $\sqrt{2m(n-2)} \leq EE(K_n) \leq m \sqrt{2(n-2)}$

3) $\sqrt{2n} \leq EE(C_n) \leq n \sqrt{2}$

Comparing the above bounds we get the path has the least edge energy and the complete graph the highest bound.

**Theorem 1.4.** Let $G$ be a connected graph on $n$ vertices with $m$ edges. Then $EE(G) \geq \left( \frac{1}{2} \sum_{1 \leq i < j \leq m} e_{ij} + m(m-1) \left( \text{det}E \right) ^{\frac{1}{n}} \right)$.
Edge energy bounds

1.6.  

(Corollary 1.6.)  
1) The upper bound for the cycle $C_n$ is $2 + \sqrt{2(m+1)(m-2)}$.  
2) The upper bound for the Path $P_n$ is $\frac{n-2}{m}[2 + (m-1)\sqrt{2}]$.  
3) The upper bound for the complete graph is $2(n-2)+\sqrt{2(m-1)(n-2)(m-2n+4)}$.  

Proof. The concept of the bounds of energy in [10] is used to investigate the edge energy bounds.

$$EE(G)^2 = \left(\sum_{i=1}^{m} |\rho_i| \right)^2 = \sum_{i=1}^{n} |\rho_i|^2 + 2 \sum_{i<j} |\rho_i| |\rho_j| = 2 \sum_{1\leq i<j \leq m} e_{ij} + m(m-1)AM \{|\rho_i| |\rho_j|\}$$

$$\geq 2 \sum_{1\leq i<j \leq m} e_{ij} + m(m-1)GM |\rho_i| |\rho_j| = 2 \sum_{1\leq i<j \leq m} e_{ij} + m(m-1)\left\{|\text{det}E|^2\right\}^{1/m}$$

Here $AM \{|\rho_i| |\rho_j|\}$ denotes the arithmetic mean of the $mC_2$ distinct terms and $GM$ the geometric mean which cannot exceed the arithmetic mean of the non-negative numbers. Hence the inequality follows.

**Theorem 1.5.** If $G$ is a connected graph on $n$ vertices with $m$ edges. Then

$$EE(G) \leq \rho_1 + \sqrt{(m-1)(2\sum_{1\leq i<j \leq m} e_{ij} \rho_1^2)}.$$

Proof. We follow the ideas of Koolen and Moulton [12, 13]. By using the Cauchy-Schwartz inequality to the vectors $(|\rho_2|, \ldots, |\rho_m|)$ and $(1, 1, \ldots, 1)$ with $m - 1$ entries we obtain the inequality $\left(\sum_{i=2}^{m} |\rho_i|\right)^2 \leq (m-1)(\sum_{i=2}^{m} \rho_i^2)$. Thus we have $EE(G) \leq \rho_1 + \sqrt{(m-1)(2\sum_{1\leq i<j \leq m} e_{ij} \rho_1^2)}$

Now consider the function $f(x) = x + \sqrt{(m-1)(2\sum_{1\leq i<j \leq m} e_{ij} x^2)}$. From $\sum_{i=1}^{m} \rho_i^2 = 2 \sum_{1\leq i<j \leq m} e_{ij}$, we get $x^2 = \rho_1^2 \leq 2 \sum_{1\leq i<j \leq m} e_{ij}$ then $x \leq \sqrt{2 \sum_{1\leq i<j \leq m} e_{ij}}$.

Also $f'(x) = 0$ implies $x = \sqrt{\frac{2}{m} \sum_{1\leq i<j \leq m} e_{ij}}$.

Here $f(x)$ decreases in the interval $\sqrt{\frac{2}{m} \sum_{1\leq i<j \leq m} e_{ij}} \leq x \leq \sqrt{\frac{2}{1\leq i<j \leq m} e_{ij}}$

and $\sqrt{\frac{2}{m} \sum_{1\leq i<j \leq m} e_{ij}} \leq \frac{2}{m} \sum_{1\leq i<j \leq m} e_{ij} \leq \rho_1$. Since the greatest edge eigenvalue is greater than the average edge degree. Hence $f(\rho_1) \leq f(\sqrt{\frac{2}{m} \sum_{1\leq i<j \leq m} e_{ij}})$ holds well. From this fact the inequality follows.

Corollary 1.6.  
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3) The upper bound for the complete graph is $2(n-2)+\sqrt{2(m-1)(n-2)(m-2n+4)}$.  


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Received: February 15, 2014