Double Domination in Graphs Under Some Binary Operations

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1 Introduction

Let $G = (V(G), E(G))$ be a graph. For any vertex $x \in V(G)$, the open neighborhood of $x$ is the set $N_G(x) = \{y \in V(G) : xy \in E(G)\}$ and the closed neighborhood of $x$ in $G$ is the set $N_G[x] = N_G(x) \cup \{x\}$. If $X \subseteq V(G)$, the open neighborhood of $X$ in $G$ is the set $N_G(X) = \bigcup_{x \in X} N_G(x)$. The closed neighborhood of $X$ in $G$ is the set $N_G[X] = N_G(X) \cup X$.

A subset $S$ of $V(G)$ is a dominating set in $G$ if $N_G[S] = S \cup N_G(S) = V(G)$ where $N_G(S) = \{v \in V(G) : xv \in E(G) \text{ for some } x \in S\}$. Equivalently, a subset $S$ of $V(G)$ is a dominating set in $G$ if for every $v \in V(G) \setminus S$, there exists $x \in S$ such that $xv \in E(G)$. Furthermore, the minimum cardinality of a dominating set in $G$, denoted by $\gamma(G)$, is the domination number of $G$.

Let $G = (V(G), E(G))$ be a graph with no isolated vertices. A subset $S$ of $V(G)$ is a total dominating set in $G$ if $N_G(S) = V(G)$. That is, every vertex in $V(G)$ is adjacent to some vertex in $S$. The minimum cardinality of a total dominating set of $G$, denoted by $\gamma_t(G)$, is the total domination number of $G$. Moreover, a subset $S$ of $V(G)$ is a double dominating set of $G$ if for each $x \in V(G)$, $|N_G[x] \cap S| \geq 2$. The double domination number of $G$, denoted by $\gamma_{dd}(G)$, is the minimum cardinality of a double dominating set of $G$. A double dominating set of $G$ with cardinality $\gamma_{dd}(G)$ is called a $\gamma_{dd}$-set.

Double dominating set and double domination number were first defined and introduced by F. Harary and T. W. Haynes in [3] as cited in [4]. They also established the Nordhaus-Gaddum inequalities for double domination. Blidia, Chellali and Haynes [2] characterized the trees having equal paired and double domination number. Atapour, Khodkar and Sheikholeslami [1] established upper bounds on the double domination subdivision number for arbitrary graphs in terms of vertex degree. Khelifi et al. [5] studied the concept in relation to $\gamma_{dd}$-critical graphs. Khelifi and Chellali [6] further studied the effects of removal of any vertex on the double domination number of graph. They also investigated various properties of a $\gamma_{dd}$-vertex critical graphs.

2 Some Preliminary Results

The first two theorems are according to F. Harary and T. W. Haynes [4].

**Theorem 2.1** For any graph $G$ without isolated vertices of order $n \geq 2$, $2 \leq \gamma_{dd}(G) \leq n$.

**Theorem 2.2** Let $G$ be a graph of order $n \geq 2$ and suppose $G$ has no isolated vertices. Then $\gamma_{dd}(G) = 2$ if and only if $G$ has two full nodes (degree $n - 1$).

**Theorem 2.3** Let $G$ be a graph of order $n \geq 2$ and suppose $G$ has no isolated vertices. Then $\gamma_{dd}(G) = n$ if and only if for each $v \in V(G)$, $v$ is either a leaf or a support vertex.
Proof: Suppose \( \gamma_{dd}(G) = n \). Suppose there exists \( v \in V(G) \) that is neither a leaf nor a support. Then \( |N_G(v)| \geq 2 \) and \( |N_G(x)| \geq 2 \) for every \( x \in N_G(v) \). Since \( G \) has no isolated vertices, it follows that \( S = V(G) \setminus \{v\} \) is a double dominating set of \( G \). Hence, \( \gamma_{dd}(G) \leq |S| = n - 1 \), contrary to our assumption. Therefore every vertex \( v \) of \( G \) is either a leaf or a support. The converse is clear. \( \square \)

Theorem 2.4 Let \( G \) be a graph without isolated vertices. Then

\[
\gamma_{dd}(G) \geq \gamma(G) + 1
\]

Proof: Let \( S \) be a \( \gamma_{dd} \)-set of \( G \). If \( S = V(G) \), then \( S \) is not a \( \gamma \)-set of \( G \) (since \( \gamma(G) \leq n - 1 \)). Hence \( \gamma_{dd}(G) \geq \gamma(G) + 1 \). So suppose \( V(G) \setminus S \neq \emptyset \). Let \( x \in V(G) \setminus S \). Then there exists \( y, z \in S \) (\( y \neq z \)) such that \( xy, xz \in E(G) \). Let \( S^* = S \setminus \{y\} \) and let \( w \in V(G) \setminus S^* \). If \( w = y \), then \( w \in S \). Since \( S \) is a total dominating set, there exists \( q \in S \) such that \( wq \in E(G) \). Hence \( q \in S^* \) such that \( wq \in E(G) \). Suppose \( w \neq y \). Then \( w \in V(G) \setminus S \). Hence there exists \( a, b \in S \) (\( a \) or \( b \) may be \( y \)) (\( a \neq b \)) such that \( aw, bw \in E(G) \). Thus, \( S^* \) is a dominating set of \( G \). Therefore, \( \gamma(G) \leq |S^*| = |S| - 1 \), that is, \( \gamma_{dd}(G) \geq \gamma(G) + 1 \). \( \square \)

3 Join of Graphs

The join of two graphs \( G \) and \( H \), denoted by \( G + H \), is the graph with vertex-set \( V(G + H) = V(G) \cup V(H) \) and edge-set \( E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\} \).

Theorem 3.1 Let \( G \) and \( H \) be any non-trivial graphs. Then a subset \( S \) of \( V(G + H) \) is a double dominating set if and only if one of the following holds:

(i) \( S \subseteq V(G) \) and \( S \) is a double dominating set in \( G \).

(ii) \( S \subseteq V(H) \) and \( S \) is a double dominating set in \( H \).

(iii) \( |S \cap V(G)| = 1 \) and \( |S \cap V(H)| = 1 \) where \( S \cap V(G) \) and \( S \cap V(H) \) are dominating sets of \( G \) and \( H \), respectively.

(iv) \( |S \cap V(G)| = 1 \) and \( S \cap V(H) \) is a dominating set in \( H \) with \( |S \cap V(H)| \geq 2 \).

(v) \( |S \cap V(H)| = 1 \) and \( S \cap V(G) \) is a dominating set in \( G \) with \( |S \cap V(G)| \geq 2 \).

(vi) \( |S \cap V(G)| \geq 2 \) and \( |S \cap V(H)| \geq 2 \).
Proof: Let $S \subseteq V(G + H)$ be a double dominating set of $G + H$. Consider the following cases:

Case 1: $S \cap V(G) = \phi$ or $S \cap V(H) = \phi$

Suppose $S \cap V(G) = \phi$. Then $S \subseteq V(H)$. Since $S$ is a double dominating set of $G + H$, it follows that $S$ is a double dominating set of $H$. Similarly if $S \cap V(H) = \phi$. Then $S \subseteq V(G)$. Thus $S$ is a double dominating set of $G$.

Case 2: $S_G = S \cap V(G) \neq \phi$ and $S_H = S \cap V(H) \neq \phi$

If $|S_G| \geq 2$ and $|S_H| \geq 2$, then (vi) holds. So suppose (vi) does not hold. Consider the following subcases:

Subcase 1: $|S_G| = 1$ and $|S_H| = 1$

Let $x \in V(G) \setminus S_G$. Since $S$ is a double dominating set and $|S_H| = 1$, there exists $y \in S_G$ such that $xy \in E(G)$. Thus $S_G$ is a dominating set of $G$. Similarly, $S_H$ is a dominating set of $H$.

Subcase 2: $|S_G| = 1$ and $|S_H| \geq 2$ or $|S_H| = 1$ and $|S_G| \geq 2$

Assume that $|S_G| = 1$ and $|S_H| \geq 2$. Let $a \in V(H) \setminus S_H$. Since $|S_G| = 1$ and $S$ is a double dominating set of $G + H$, there exists $b \in S_H$ such that $ab \in E(H)$. This implies that $S_H$ is a dominating set of $H$. Similarly, if $|S_H| = 1$ and $|S_G| \geq 2$, then $S_G$ is a dominating set of $G$.

The converse is easy. \hfill \Box

The next result is an immediate consequence of Theorem 3.1.

Corollary 3.2 Let $G$ and $H$ be any non-trivial graphs. Then

$$2 \leq \gamma_{dd}(G + H) \leq 4$$

Theorem 3.3 Let $G$ and $H$ be any non-trivial graphs. Then $\gamma_{dd}(G + H) = 2$ if and only if one of the following holds:

(i) $\gamma_{dd}(G) = 2$

(ii) $\gamma_{dd}(H) = 2$

(iii) $\gamma(G) = 1$ and $\gamma(H) = 1$

Proof: Suppose $\gamma_{dd}(G + H) = 2$. Let $S \subseteq V(G + H)$ be a minimum double dominating set of $G + H$. By Theorem 3.1, $S$ is a double dominating set of $G$ or $S$ is a dominating set of $H$ or $|S \cap V(G)| = 1$ and $|S \cap V(H)| = 1$, where $S \cap V(G)$ and $S \cap V(H)$ are dominating sets of $G$ and $H$, respectively. Hence, $\gamma_{dd}(G) = 2$ or $\gamma_{dd}(H) = 2$ or $\gamma(G) = 1$ and $\gamma(H) = 1$.

For the converse, suppose $\gamma_{dd}(G) = 2$. Let $S^*$ be a $\gamma_{dd}$-set of $G$. By Theorem 3.1, $S^*$ is a $\gamma_{dd}$-set of $G + H$. Thus $\gamma_{dd}(G + H) = |S| = 2$. Similarly, if $\gamma_{dd}(H) = 2$, then $\gamma_{dd}(G + H) = 2$. Moreover if $\gamma(G) = 1$ and $\gamma(H) = 1$, then by Theorem 3.1, it follows that $\gamma_{dd}(G + H) = 2$. \hfill \Box
Theorem 3.4 Let $G$ and $H$ be any non-trivial graphs such that $\gamma_{dd}(G + H) \neq 2$. Then $\gamma_{dd}(G + H) = 3$ if and only if one of the following holds:

(i) $\gamma_{dd}(G) = 3$

(ii) $\gamma_{dd}(H) = 3$

(iii) $\gamma(H) = 2$

(iv) $\gamma(G) = 2$.

Proof: Suppose $\gamma_{dd}(G + H) = 3$. Let $S \subseteq V(G + H)$ be a $\gamma_{dd}$-set of $G + H$. Then by Theorem 3.1, $S$ is a double dominating set of $G$ or $S$ is a double dominating set of $H$ or $|S \cap V(G)| = 1$ and $S \cap V(H)$ is a dominating set of $H$, or $|S \cap V(G)| = 1$ and $S \cap V(G)$ is a dominating set in $G$. Since $\gamma_{dd}(G + H) \neq 2$, it follows that $\gamma_{dd}(G) = 3$ or $\gamma_{dd}(H) = 3$ or $\gamma(H) = 2$ or $\gamma(G) = 2$.

For the converse, Suppose (i) holds. Let $S$ be a $\gamma_{dd}$-set of $G$. Then by Theorem 3.1, $S$ is a double dominating set of $G + H$. Hence, $\gamma_{dd}(G + H) \leq |S| = 3$. Since $\gamma_{dd}(G + H) \neq 2$, it follows that $\gamma_{dd}(G + H) = 3$. Similarly, $\gamma_{dd}(G + H) = 3$ if (ii) holds.

Next, suppose that (iii) holds, i.e., $\gamma(H) = 2$. Let $S_1$ be a $\gamma$-set of $H$. Pick any $v \in V(H)$ and let $S_2 = S_1 \cup \{v\}$. Then by Theorem 3.1(iv), $S_2$ is a double dominating set of $G + H$. Hence, $\gamma_{dd}(G + H) \leq |S| = 3$. Again, since $\gamma_{dd}(G + H) \neq 2$, it follows that $\gamma_{dd}(G + H) = 3$. A similar argument is used to show that the $\gamma_{dd}(G + H) = 3$ if (iv) holds.

\[\square\]

Theorem 3.5 Let $G$ be any graph and let $K_1 = \langle v \rangle$. Then $S \subseteq V(K_1 + G)$ is a double dominating set of $K_1 + G$ if and only if either:

(i) $v \in S$ and $S \cap V(G)$ is a dominating set of $G$, or

(ii) $v \notin S$ and $S \cap V(G)$ is a double dominating set of $G$.

Proof: Let $S$ be a double dominating set of $K_1 + G$ and put $S_G = V(G) \cap S$. Consider the following cases:

Case 1: $v \in S$

Let $x \in V(G) \setminus S_G$. Since $S$ is a double dominating set, there exists $y \in S_G$ such that $xy \in E(G)$. Hence $S_G$ is a dominating set of $G$.

Case 2: $v \notin S$

Let $x \in V(G)$. Since $S$ is a double dominating set of $K_1 + G$ and $v \notin S$, it follows that $|N_{K_1 + G}[x] \cap S| = |N_G[x] \cap S_G| \geq 2$. Hence $S$ is a double dominating set of $G$.

The converse is clear. \[\square\]

The next result follows from Theorem 2.4 and Theorem 3.5.
Corollary 3.6 Let $G$ be a non-complete graph. Then,

$$\gamma_{dd}(G + K_1) = \gamma(G) + 1$$

4 Corona of Graphs

Let $G$ and $H$ be graphs of order $n$ and $m$, respectively. The corona $G \circ H$ of $G$ and $H$ is the graph obtained by taking one copy of $G$ and $n$ copies of $H$, and then joining the $i$th vertex of $G$ to every vertex of the $i$th copy of $H$. For every $v \in V(G)$, denote by $H^v$ the copy of $H$ whose vertices are attached one by one to the vertex $v$. Denote by $v + H^v$ the subgraph of the corona $G \circ H$ corresponding to the join $\langle \{v\}\rangle + H^v$.

Theorem 4.1 Let $G$ and $H$ be any graphs without isolated vertices of orders $n$ and $m$ respectively. Then $S \subseteq V(G \circ H)$ is a double dominating set of $G \circ H$ if and only if $S = S_1 \cup S_2 \cup S_3$ such that $S_1 \subseteq V(G)$, $S_2 = \cup_{v \in S_1}D_v$, where $D_v$ is a dominating set in $H^v$ for each $v \in S_1$, and $S_3 = \cup_{v \notin S_1}D_v^*$, where $D_v^*$ is a double dominating set in $H^v$ for each $v \notin S_1$.

Proof: Suppose $S \subseteq V(G \circ H)$ is a double dominating set of $G \circ H$. Consider the following cases:

Case 1: $S \cap V(G) = \phi$

Let $S_1 = S \cap V(G)$. Then $S_2 = \cup_{v \in S_1}D_v = \phi$. Let $v \in V(G)$ and set $D_v^* = S \cap V(H^v)$. Since $S$ is a double dominating set of $G \circ H$ and $v \notin S$, it follows that $D_v^*$ is a double dominating set of $H^v$. Let $S_3 = \cup_{v \in V(G)}D_v^*$, then $S = S_3$.

Case 2: $S \cap V(G) \neq \phi$

Let $S_1 = S \cap V(G)$ and let $v \in S_1$. Set $D_v = S \cap V(H^v)$. Let $x \in V(H^v) \setminus D_v$. Since $S$ is a double dominating set of $G \circ H$, there exists $y \in D_v$ such that $xy \in E(G \circ H)$ (hence $xy \in E(H^v)$). Thus $D_v$ is a dominating set of $V(H^v)$. Let $S_2 = \cup_{v \in S_1}D_v$. Now, let $w \notin S_1$ and let $D_w^* = S \cap V(H^w)$. Since $S$ is a double dominating set of $G \circ H$, it follows that $D_w^*$ is a double dominating set of $H^w$. Let $S_3 = \cup_{w \notin S_1}D_w^*$. Then $S = S_1 \cup S_2 \cup S_3$.

For the converse, suppose $S = S_1 \cup S_2 \cup S_3$, where $S_1$, $S_2$ and $S_3$ satisfy the given properties. Let $x \in V(G \circ H)$. Consider the following cases:

Case 1: $x \in S$

If $x \in V(G)$, then $x \in S_1$. Hence by assumption, $D_x$ is a dominating set of $H^x$. It follows that there exists $y \in D_x \subseteq S$ such that $xy \in (G \circ H)$. Suppose $x \notin V(G)$. Let $v \in V(G)$ such that $x \in V(H^v)$. If $v \in S$, then $xv \in E(G \circ H)$. If $v \notin S$, then, by assumption, $D_v^*$ is a double dominating set of $H^v$. This implies that there exists $z \in D_v^*$ such that $xz \in E(G \circ H)$.
Case 2: \( x \notin S \)

If \( x \in V(G) \), then \( D^*_x \) is a double dominating set of \( H^x \). Hence \( |D^*_x| \geq 2 \). This implies that there exists \( a, b \in D^*_x \) (\( a \neq b \)) such that \( xa, xb \in E(G \circ H) \). Suppose \( x \notin V(G) \). Again, let \( v \in V(G) \) such that \( x \in V(H^v) \). If \( v \in S \), then \( xv \in E(G \circ H) \) and \( D_v \) is a dominating set of \( H^v \). Since \( x \in V(H^v) \setminus D_v \), there exists \( q \in D_v \subseteq S \) such that \( qx \in E(G \circ H) \). If \( v \notin S \), then \( v \notin S_1 \). Hence, by assumption, \( D^*_v \) is a double dominating set of \( H^v \). Since \( x \in V(H^v) \setminus D^*_v \), there exists \( c, d \in D^*_v \) (\( c \neq d \)) such that \( xc, xd \in E(G \circ H) \). Accordingly, \( S \) is a double dominating set of \( G \circ H \).

**Corollary 4.2** Let \( G \) and \( H \) be any graphs without isolated vertices of orders \( n \) and \( m \) respectively. Then, \( \gamma_{dd}(G \circ H) = n (1 + \gamma(H)) \)

**Proof:** Let \( D \) be a \( \gamma \)-set of \( H \). For each \( v \in V(G) \), let \( D_v \subseteq V(H^v) \) such that \( D_v \approx D \). Then \( S = V(G) \cup (\bigcup_{v \in V(G)} D_v) \) is a double dominating set of \( G \circ H \) by Theorem 4.1. Hence,

\[
\gamma_{dd}(G \circ H) \leq |S| = |V(G)| + \sum_{v \in V(G)} |D_v| = n + n\gamma(H) = n (1 + \gamma(H)).
\]

Next, let \( S \) be a \( \gamma_{dd} \)-set of \( G \circ H \). By Theorem 4.1, \( S = S_1 \cup S_2 \cup S_3 \), where \( S_1, S_2, S_3 \) satisfy the conditions in the theorem. Hence, by Theorem 2.4,

\[
\gamma_{dd}(G \circ H) = |S_1| + |S_2| + |S_3|
\]
\[
= |S_1| + \sum_{v \in S_1} |D_v| + \sum_{v \notin S_1} |D_v^*|
\]
\[
\geq |S_1| + |S_1| \gamma(H) + (n - |S_1|) \gamma_{dd}(H)
\]
\[
\geq |S_1| + |S_1| \gamma(H) + (n - |S_1|)(\gamma(H) + 1)
\]
\[
= n + n\gamma(H)
\]
\[
= n (1 + \gamma(H))
\]

Therefore \( \gamma_{dd}(G \circ H) = n (1 + \gamma(H)) \).

5 Lexicographic Product of Graphs

The lexicographic product \( G[H] \) of two graphs \( G \) and \( H \) is the graph with vertex-set \( V(G[H]) = V(G) \times V(H) \) and edge-set \( E(G[H]) \) satisfying the following conditions: \((x, u)(y, v) \in E(G[H])\) if and only if either \( xy \in E(G) \) or \( x = y \) and \( uv \in E(H) \).

Observe that a subset \( C \) of \( V(G[H]) = V(G) \times V(H) \) can be written as \( C = \bigcup_{x \in S} (x \times T_x) \), where \( S \subseteq V(G) \) and \( T_x \subseteq V(H) \) for every \( x \in S \). Henceforth, we shall use this form to denote any subset \( C \) of \( V(G[H]) = V(G) \times V(H) \).
Theorem 5.1 Let $G$ and $H$ be any connected non-trivial graphs. A non-empty subset $C = \bigcup_{x \in S} \{x \times T_x\}$ of $V(G[H])$ is a double dominating set of $G[H]$ if and only if $S$ is a dominating set of $G$ and satisfies each of the following:

(a) For each $x \in V(G) \setminus S$ such that $|N_G(x) \cap S| = 1$, $|T_y| \geq 2$ for $y \in N_G(x) \cap S$.

(b) $T_x$ is a double dominating set of $H$ for each $x \in S \setminus N(S)$.

(c) For each $z \in S$ with $|N_G(z) \cap S| = 1$, either $T_z$ is a dominating set of $H$ or $|T_w| \geq 2$ for $w \in N_G(z) \cap S$.

Proof: Suppose $C = \bigcup_{x \in S} \{x \times T_x\}$ is a double dominating set of $G[H]$. Then $S$ is a dominating set of $G$. Let $x \in V(G) \setminus S$ with $|N_G(x) \cap S| = 1$, say $N_G(x) \cap S = \{y\}$. Pick any $a \in V(H)$. Then $(x,a) \notin C$. Since $C$ is a double dominating set, there exists $(y,b),(y,c) \in C \cap N_{G[H]}((x,a))$. Thus, $|T_y| \geq 2$. Next, let $z \in S$ with $|N_G(z) \cap S| = 1$. Let $\{w\} = N_G(z) \cap S$. If $T_z$ is a dominating set, then we are done. Suppose $T_z$ is not a dominating set. Then there exists $a \in V(H) \setminus T_z$ such that $ab \notin E(H)$ for all $b \in T_z$. Since $(z,a) \notin C$ and $C$ is a double dominating set of $G[H]$, there exists $p,q \in T_w$ ($p \neq q$) such that $(w,p),(w,q) \in N_{G[H]}((z,a))$. This implies that $|T_w| \geq 2$. Finally, let $x \in S \setminus N(S)$ and let $c \in V(H) \setminus T_x$. Since $C$ is a double dominating set, there exists $a,b \in T_x$ ($a \neq b$) such that $(x,a),(x,b) \in N_G((x,c))$. This implies that $a,b \in N_G(c)$. Moreover, since $C$ is a total dominating set of $G[H]$, $T_x$ is a total dominating set of $H$. Thus, $T_x$ is a double dominating set of $H$.

For the converse, suppose $S$ is a dominating set of $G$ satisfying (a), (b), and (c). Let $(x,a) \in V(G[H])$. Consider the following cases:

Case 1: $x \in S$

If $x \in S \setminus N(S)$, then $T_x$ is a double dominating set of $H$ by (b). If $a \in T_x$, then there exists $b \in T_x \setminus \{a\}$ such that $ab \in E(H)$. Hence, there exists $(x,b) \in C$ such that $(x,a) \in C \cap N_{G[H]}((x,b))$. If $a \notin T_x$, then there exists $(x,b) \in C \cap N_{G[H]}((x,c))$ such that $(x,a) \in N_{G[H]}((x,c)) \cap N_{G[H]}((x,d))$. Suppose $N_G(x) \cap S = \{w\}$. Pick any $p \in T_w$. Then $(x,a) \in C \cap N_{G[H]}((w,p))$. Suppose $a \notin T_x$. If $T_x$ is a dominating set, then there exists $b \in T_x$ such that $ab \in E(H)$. Hence $(w,p),(x,b) \in C \cap N_{G[H]}((x,a))$. If $T_x$ is not a dominating set, then $|T_w| \geq 2$ by (c). This implies that there exists $q \in T_w \setminus \{p\}$. Therefore, $(w,p),(w,q) \in C \cap N_{G[H]}((x,a))$.

Case 2: $x \notin S$

Suppose $x \notin S$. Then $(x,a) \notin C$. From our assumption that (a) holds, it follows that $|T_y| \geq 2$ if $N_G(x) \cap S = \{y\}$. Pick $b,c \in T_y$, where $b \neq c$. Then $(y,b),(y,c) \in C \cap N_{G[H]}((x,a))$. Suppose $|N_G(x) \cap S| \geq 2$. Choose any $y,z \in N_G(x) \cap S$, where $y \neq z$. Pick any $s \in T_y$ and $t \in T_z$. Then $(y,s),(z,t) \in C \cap N_{G[H]}((x,a))$.

Accordingly, $C$ is a double dominating set of $G$. \qed
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Proof: Let $S_1$ and $S_2$ be $\gamma$- and $\gamma_t$-sets of $G$ and let $D_1$ be a $\gamma_{dd}$-set of $H$ and $D_2 = \{a, b\}$, where $a, b \in V(H)$. Set $C_1 = \cup_{x \in S_1} \{x\} \times D_1 = S_1 \times D_1$ and $C_2 = \cup_{x \in S_2} \{x\} \times D_2 = S_2 \times D_2$. By Theorem 5.1, $C_1$ and $C_2$ are double dominating sets of $G[H]$. Hence, $\gamma_{dd}(G[H]) \leq |C_1| = \gamma(G)\gamma_{dd}(H)$ and $\gamma_{dd}(G[H]) \leq |C_2| = 2\gamma_t(G)$. Thus, $\gamma_{dd}(G[H]) \leq \min\{2\gamma_t(G), \gamma(G)\gamma_{dd}(H)\}$.

Consider the graphs $P_4[P_3]$ and $S_4[P_3]$ in Figure 1.

![Graphs P_4[P_3] and S_4[P_3]](image)

Figure 1: Lexicographic product of different graphs resulting to different $\gamma_{dd}$. In (a), $\gamma_{dd}(P_4[P_3]) = 2\gamma_t(P_4) = 2(2) = 4 < \gamma(P_4)\gamma_{dd}(P_3) = 2(4) = 8$, while in (b), $\gamma_{dd}(S_4[P_3]) = \gamma(S_4)\gamma_{dd}(P_3) = 1(3) = 3 < 2\gamma_t(S_4) = 2(2) = 4$. These graphs in Figure 1 show that the bounds given in Corollary 5.2 are sharp.

References


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