Correlation Theorem for Two-Sided Quaternion Fourier Transform

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Abstract

In this paper we establish correlation theorem for the two-sided quaternion Fourier transform (QFT). A consequence of the theorem is also investigated.

Keywords: quaternion Fourier transform, correlation

I. Introduction

The quaternion Fourier transform (QFT) is a nontrivial generalization of the classical Fourier transform (FT) using quaternion algebra. A number of already known and useful properties of this extended transform are generalizations of the corresponding properties of the FT with some modifications (see, for example, [1, 2, 3, 4, 5]). One of the most powerful properties of the QFT is the convolution theorem. Recently, in [6] authors proposed the convolution theorem for the two-sided QFT, which describes the relationship between the two-sided QFT and convolution of two quaternion function.

Therefore, the main objective of the present paper is to establish the correlation for the two-sided QFT, which is a generalization of correlation theorem of the classical FT. We find that the correlation theorem does not work well for the right-sided quaternion Fourier transform and left-sided quaternion Fourier transform.
The quaternion algebra over $\mathbb{R}$, denoted by
\[ \mathbb{H} = \{ q = q_0 + iq_1 + jq_2 + kq_3 \}, \quad q_0, q_1, q_2, q_3 \in \mathbb{R}, \] (1)
is an associative non-commutative four-dimensional algebra, which the quaternion units $i, j, k$ obey the following multiplication rules:
\[ i^2 = j^2 = k^2 = -1, \quad ij = -ji, \quad ik = -ki, \quad jk = -kj, \text{ and } ijk = -1. \] (2)
The quaternion conjugate is defined by
\[ \bar{q} = q_0 - iq_1 - jq_2 - kq_3, \] (3)
which is an anti-involution, that is,
\[ \bar{\bar{p}} = p, \quad p + \bar{q} = \bar{\bar{p}} + \bar{q}, \quad \bar{pq} = \bar{q}\bar{p}. \] (4)
The norm of a quaternion is defined as
\[ |q| = \sqrt{q\bar{q}} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}. \] (5)
It is not difficult to check that
\[ |qp| = |p||q|, \quad \forall p, q \in \mathbb{H}. \] (6)
We further get the inverse
\[ q^{-1} = \frac{\bar{q}}{|q|^2}. \]
This fact shows that $\mathbb{H}$ is a skew field, that means, every nonzero element has a multiplicative inverse.

For the sake of further simplicity, we will use the real vector notations
\[ x = (x_1, x_2) \in \mathbb{R}^2, \quad f(x) = f(x_1, x_2), \quad f(\omega) = f(\omega_1, \omega_2), \]
and so on when there is confusion.

2. Main Results

In this section we introduce the definition of the two-sided QFT and establish the correlation of two quaternion-valued functions associated with the two-sided QFT.

**Definition 2.1 (Two-sided QFT)** Let $f$ be in $L^2(\mathbb{R}^2; \mathbb{H})$. The two-sided QFT of the quaternion function $f$ is the transform given by the integral
\[ \mathcal{F}_q[f](\omega) = \int_{\mathbb{R}^2} e^{-i\omega_1 x_1} f(x) e^{-i\omega_2 x_2} d^2x. \] (7)

**Theorem 2.1 (Inverse two-sided QFT)** Suppose that $f \in L^2(\mathbb{R}^2; \mathbb{H})$ and $\mathcal{F}_q[f] \in L^1(\mathbb{R}^2; \mathbb{H})$. Then the two-sided QFT is an invertible transform and its inverse is given by
\[ f(x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{i\omega_1 x_1} \mathcal{F}_q[f](\omega) e^{i\omega_2 x_2} d^2\omega. \] (8)
Correlation theorem

**Definition 2.2 (Quaternion Correlation)** Suppose \( f \) and \( g \) are quaternion functions in \( L^2(\mathbb{R}^2; \mathbb{H}) \). The quaternion correlation of the functions is given by

\[
(f \otimes g)(x) = \int_{\mathbb{R}^2} f(x + y)g(y) \, d^2y.
\]

The following theorem is the main result of this paper, which describes how the two-sided QFT behaves under correlations.

**Theorem 2.2** Let \( f, g \in L^2(\mathbb{R}^2; \mathbb{H}) \) be two quaternion-valued functions. The two-sided QFT of correlation of \( f \) and \( g \) takes the form \( \mathcal{F}_q(f \otimes g)(\omega) \)

\[
\mathcal{F}_q(f \otimes g)(\omega) = \left( \mathcal{F}_q(f_0)(\omega) + i\mathcal{F}_q(f_1)(\omega) \right) \left( \mathcal{F}_q(g_0)(\omega) - j\mathcal{F}_q(g_2)(\omega_1, \omega_2) \right)
\]

\[
+ \left( \mathcal{F}_q(f_0)(\omega_1, -\omega_2) + i\mathcal{F}_q(f_1)(\omega_1, -\omega_2) \right) \left( -i\mathcal{F}_q(g_1)(\omega) - k\mathcal{F}_q(g_3)(\omega_1, -\omega_2) \right)
\]

\[
+ \left( j\mathcal{F}_q(f_2)(\omega_1, -\omega_2) + k\mathcal{F}_q(f_3)(\omega_1, -\omega_2) \right) \left( \mathcal{F}_q(g_0)(\omega_1, -\omega_2) - j\mathcal{F}_q(g_2)(\omega) \right)
\]

\[
+ \left( j\mathcal{F}_q(f_2)(\omega) + k\mathcal{F}_q(f_3)(\omega) \right) \left( -i\mathcal{F}_q(g_1)(\omega_1, -\omega_2) - k\mathcal{F}_q(g_3)(\omega) \right).
\]

**Proof.** Applying the two-sided QFT definition gives

\[
\mathcal{F}_q(f \otimes g)(\omega)
\]

\[
= \int_{\mathbb{R}^2} e^{-i\omega_1 x_1} \left( \int_{\mathbb{R}^2} f(x + y) \, d^2y \right) e^{-j\omega_2 x_2} \, d^2x.
\]

By inserting the change of variables \( z = x + y \) to the above expression we immediately obtain

\[
\mathcal{F}_q(f \otimes g)(\omega)
\]

\[
= \int_{\mathbb{R}^2} e^{-i\omega_1 (x_1 - y_1)} \left( \int_{\mathbb{R}^2} f(z) \, d^2y \right) e^{-j\omega_2 (x_2 - y_2)} \, d^2z
\]

\[
+ \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-i\omega_1 (x_1 - y_1)} \left( f_0(z) + i f_1(z) + j f_2(z) + k f_3(z) \right)
\]

\[
\left( g_0(y) - j g_2(y) \right) + \left( -i g_1(y) - k g_3(y) \right) e^{-j\omega_2 (x_2 - y_2)} \, d^2y \, d^2z
\]

\[
+ \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-i\omega_1 (x_1 - y_1)} \left( f_0(z) + i f_1(z) \right) \left( g_0(y) - j g_2(y) \right) e^{-j\omega_2 (x_2 - y_2)} \, d^2y \, d^2z
\]

\[
+ \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-i\omega_1 (x_1 - y_1)} \left( j f_2(z) + k f_3(z) \right) \left( g_0(y) - j g_2(y) \right) e^{-j\omega_2 (x_2 - y_2)} \, d^2y \, d^2z
\]
\[ + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-i \omega_1 (x_1 - y_1)} \left( (j f_2(z) + k f_3(z))(-i g_1(y) - k g_3(y)) \right) e^{-j \omega_2 (x_2 - y_2)} \ d^2y \ d^2z \]

\[ = \int_{\mathbb{R}^2} e^{-i \omega_1 z_1} \left( f_0(z) + i f_1(z) \right) \]

\[ \left( \int_{\mathbb{R}^2} e^{i \omega_1 y_1} g_0(y) e^{i \omega_2 y_2} \ d^2y - j \int_{\mathbb{R}^2} e^{-i \omega_1 y_1} g_2(y) e^{i \omega_2 y_2} \ d^2y \right) e^{-j \omega_2 z_2} \ d^2z \]

\[ -i \int_{\mathbb{R}^2} e^{i \omega_1 y_1} g_1(y) e^{i \omega_2 y_2} \ d^2y - k \int_{\mathbb{R}^2} e^{-i \omega_1 y_1} g_3(y) e^{i \omega_2 y_2} \ d^2y \right) e^{-j \omega_2 z_2} \ d^2z \]

\[ + \int_{\mathbb{R}^2} e^{-i \omega_1 z_1} \left( j f_2(z) + k f_3(z) \right) \]

\[ \left( \int_{\mathbb{R}^2} e^{i \omega_1 y_1} g_0(y) e^{i \omega_2 y_2} \ d^2y - j \int_{\mathbb{R}^2} e^{i \omega_1 y_1} g_2(y) e^{i \omega_2 y_2} \ d^2y \right) e^{-j \omega_2 z_2} \ d^2z \]

\[ + \int_{\mathbb{R}^2} e^{i \omega_1 y_1} g_1(y) e^{i \omega_2 y_2} \ d^2y - k \int_{\mathbb{R}^2} e^{i \omega_1 y_1} g_3(y) e^{i \omega_2 y_2} \ d^2y \right) e^{-j \omega_2 z_2} \ d^2z \]

\[ = \int_{\mathbb{R}^2} e^{-i \omega_1 z_1} \left( f_0(z) + i f_1(z) \right) \left( F_q\{g_0\}(-\omega) - j F_q\{g_2\}(\omega_1, -\omega_2) \right) e^{-j \omega_2 z_2} \ d^2z \]

\[ + \int_{\mathbb{R}^2} e^{-i \omega_1 z_1} \left( f_0(z) + i f_1(z) \right) \left( -i F_q\{g_3\}(\omega_1, -\omega_2) - k F_q\{g_3\}(\omega_1, -\omega_2) \right) e^{-j \omega_2 z_2} \ d^2z \]

\[ + \int_{\mathbb{R}^2} e^{-i \omega_1 z_1} \left( j f_2(z) + k f_3(z) \right) \left( F_q\{g_0\}(\omega_1, -\omega_2) - j F_q\{g_2\}(-\omega) \right) e^{-j \omega_2 z_2} \ d^2z \]

\[ + \int_{\mathbb{R}^2} e^{-i \omega_1 z_1} \left( j f_2(z) + k f_3(z) \right) \left( -i F_q\{g_3\}(\omega_1, -\omega_2) + k F_q\{g_1\}(-\omega) \right) e^{-j \omega_2 z_2} \ d^2z \]

\[ = \int_{\mathbb{R}^2} e^{-i \omega_1 z_1} \left( f_0(z) + i f_1(z) \right) e^{-j \omega_2 z_2} \ d^2z \left( F_q\{g_0\}(-\omega) - j F_q\{g_2\}(\omega_1, -\omega_2) \right) \]
As a consequence of the above theorem, we immediately obtain

**Lemma 2.3** Given any two quaternion-valued functions \( f, g \in L^2(\mathbb{R}^2; \mathbb{H}) \). If we assume that \( \mathcal{F}_q\{g\}(\omega) \in L^2(\mathbb{R}^2; \mathbb{R}) \), then

\[
\mathcal{F}_q\{f \otimes g\}(\omega) = \mathcal{F}_q\{f\}(\omega_1, -\omega_2) \left( \mathcal{F}_q\{g\}(\omega) - i \mathcal{F}_q\{g\}_1(\omega) \right) - \mathcal{F}_q\{f\}(\omega_1, -\omega_2) \left( j \mathcal{F}_q\{g\}_2(\omega_1, -\omega_2) + k \mathcal{F}_q\{g\}_3(\omega_1, -\omega_2) \right).
\]

**Proof.** Straightforward computations show that

\[
\mathcal{F}_q\{f \otimes g\}(\omega) = \int_{\mathbb{R}^2} e^{-i\omega_1 x_1} \left( \int_{\mathbb{R}^2} f(x+y) \overline{g(y)} \ d^2y \right) e^{-j\omega_2 z_2} d^2x.
\]

\[
= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-i\omega_1 (x_1 - y_1)} \left( f(x)(g(y) - ig_1(y)) - f(x)(j g_2(y) + k g_3(y)) \right) e^{-j\omega_2 (z_2 - y_2)} d^2y \ d^2z.
\]

\[
= \int_{\mathbb{R}^2} e^{-i\omega_1 (x_1 - y_1)} f(x) \int_{\mathbb{R}^2} (g_0(y) - ig_1(y)) e^{j\omega_2 y_2} d^2y \ e^{-j\omega_2 z_2} d^2z.
\]

\[
- \int_{\mathbb{R}^2} e^{-i\omega_1 (x_1 - y_1)} f(x) \int_{\mathbb{R}^2} (j g_2(y) + k g_3(y)) e^{j\omega_2 y_2} d^2y \ e^{-j\omega_2 z_2} d^2z.
\]

\[
= \mathcal{F}_q\{f\}(\omega_1, -\omega_2) \int_{\mathbb{R}^2} e^{-i\omega_1 y_1} \left( g_0(y) - ig_1(y) \right) e^{-j\omega_2 y_2} d^2y.
\]
\[-F_q \{ f \}(\omega_1, -\omega_2) \int_{\mathbb{R}^2} e^{-i\omega_1y_1} (jg_2(y) + kg_3(y)) e^{-i\omega_2y_2} \, d^2y.\]

We further obtain
\[
F_q \{ f \otimes g \}(\omega) = F_q \{ f \}(\omega_1, -\omega_2) \int_{\mathbb{R}^2} e^{i\omega_1y_1} (g_0(y) - ig_1(y)) e^{i\omega_2y_2} \, d^2y
\]
\[
- F_q \{ f \}(\omega_1, -\omega_2) \int_{\mathbb{R}^2} e^{i\omega_1y_1} (jg_2(y) + kg_3(y)) e^{i\omega_2y_2} \, d^2y
\]
\[
= F_q \{ f \}(\omega_1, -\omega_2) \left( \int_{\mathbb{R}^2} e^{i\omega_1y_1} g_0(y) e^{i\omega_2y_2} \, d^2y - \int_{\mathbb{R}^2} e^{i\omega_1y_1} ig_1(y) e^{i\omega_2y_2} \, d^2y \right)
\]
\[
- F_q \{ f \}(\omega_1, -\omega_2) \left( \int_{\mathbb{R}^2} e^{i\omega_1y_1} jg_2(y) e^{i\omega_2y_2} \, d^2y + \int_{\mathbb{R}^2} e^{i\omega_1y_1} kg_3(y) e^{i\omega_2y_2} \, d^2y \right)
\]
\[
= F_q \{ f \}(\omega_1, -\omega_2) \left( F_q \{ g_0 \}(\omega) - i F_q \{ g_1 \}(\omega) \right)
\]
\[
- F_q \{ f \}(\omega_1, -\omega_2) \left( jF_q \{ g_2 \}(\omega_1, -\omega_2) + kF_q \{ g_3 \}(\omega_1, \omega_2) \right).
\]

which was to be proved.

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**References**


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