Anosov Diffeomorphisms on Real Bott Manifolds

Admi Nazra

Department of Mathematics, Andalas University
Kampus Unand Limau Manis Padang 25163, Indonesia

Copyright © 2014 Admi Nazra. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

In this paper we study a relation between the Anosov relation of a Anosov diffeomorphism of an \( n \)-dimensional real Bott manifold \( M(A) = T^k \times (\mathbb{Z}_2)^s \) and that of an \( (n-k) \)-dimensional real Bott manifold \( M(B) \).

Mathematics Subject Classification: MSC2010 Primary 37D20; Secondary 57S25

Keywords: Bieberbach group, Flat Riemannian manifold, Real Bott manifold, Seifert fibration, Anosov diffeomorphism, Anosov relation

1 Introduction

A real Bott manifold is the total manifold \( B_n \) in a sequence of \( \mathbb{R}P^1 \)-bundles starting with a point:

\[
B_n \to B_{n-1} \to \cdots \to B_2 \to B_1 \to \{ \text{a point} \}
\]

where each \( \mathbb{R}P^1 \)-bundle \( B_i \to B_{i-1} \) is the projectivization of the Whitney sum of a real line bundle \( L_i \) and the trivial line bundle over \( B_{i-1} \).

An \( n \)-dimensional real Bott manifold, from the viewpoint of group actions, is the quotient of the \( n \)-dimensional torus \( T^n = S^1 \times \cdots \times S^1 \) by the product \( (\mathbb{Z}_2)^n \) of cyclic group of order 2. The free action of \( (\mathbb{Z}_2)^n \) on \( T^n \) can be expressed by an \( n \)-th upper triangular matrix \( A \) whose diagonal entries are 1 and the other entries are either 1 or 0. The orbit space \( M(A) = T^n/(\mathbb{Z}_2)^n \) is the
n—dimensional real Bott manifold and we call $A$ a Bott matrix of size $n$. A real Bott manifold $M(A)$ also provides an example of a flat Riemannian manifold. For more background we refer to [2].

In [2] it is also proved that every $n$-dimensional real Bott manifold $M(A)$ admits an injective Seifert fibred structure which has the form $M(A) = T^k \times_{(\mathbb{Z}_2)^s} M(B)$, that is there is a $k$-torus action on $M(A)$ whose quotient space is an $(n - k)$-dimensional real Bott orbifold $M(B)/(\mathbb{Z}_2)^s$ by some $(\mathbb{Z}_2)^s$-action $(1 \leq s \leq k)$. Here, $M(B)$ is an $(n - k)$—dimensional real Bott manifold corresponding to a Bott matrix $B$ of size $(n - k)$. By this result, when the low dimensional real Bott manifolds are classified, we can determine the diffeomorphism classes of higher dimensional ones.

Let $f: M \to M$ be a continuous self-map of a smooth closed manifold $M$. In fixed point theory, there are two numbers associated with $f$, namely the Lefschetz number $L(f)$ and the Nielsen number $N(f)$, to provide information on its fixed points. Since the information obtained from $N(f)$ is more than that of $L(f)$, and unfortunately $N(f)$ is not readily computable whereas $L(f)$ is easier to calculate, then to compute $N(f)$ via $L(f)$ we use the Anosov relation $N(f) = |L(f)|$ (see [1]). Recall that a diffeomorphism $f$ of a compact differentiable manifold $M$ is an Anosov diffeomorphism if and only if at any point of $M$ the tangent space decomposes as a direct sum of a contracting and an expanding part (see [3]).

In this article we proved that if a real Bott manifold $M(A) = T^k \times_{(\mathbb{Z}_2)^s} M(B)$ admitting an Anosov diffeomorphism $f: M(A) \to M(A)$ satisfies the Anosov relation $N(f) = |L(f)|$ then there exists an Anosov diffeomorphism $g: M(B) \to M(B)$ satisfying the Anosov relation $N(g) = |L(g)|$.

## 2 Main Results

These are the main results of the paper.

**Theorem 2.1.** If an $n$-dimensional real Bott manifold $M(A) = T^k \times_{(\mathbb{Z}_2)^s} M(B)$ admits an Anosov diffeomorphism then the $(n - k)$-dimensional real Bott manifold $M(B)$ also admits an Anosov diffeomorphism.

**Proof.** Let $f: M(A) \to M(A)$ be an Anosov diffeomorphism. Then there exists an isomorphism $f_*: \pi(A) \to \pi(A)$ where $\pi(A)$ is the fundamental group of $M(A)$. Since $\pi(A)$ is a crystallographic group, the Bieberbach theorem implies that there exists an affine map $\varphi = (d, D) \in Aff(\mathbb{R}^n) = \mathbb{R}^n \rtimes GL(n, \mathbb{R})$ such that

$$f_*(\gamma) = \varphi \gamma \varphi^{-1}, \quad (\forall \gamma \in \pi(A)).$$

(2)
Note that \( \varphi(\gamma x) = \varphi \gamma \varphi^{-1} \varphi x = f_\ast(\gamma) \varphi(x) \). Then \( \varphi: \mathbb{R}^n \to \mathbb{R}^n \) induces a diffeomorphism \( \hat{\varphi}: M(A) \to M(A) \) by \( \hat{\varphi}[x] = [\varphi x] \ (\forall x \in M(A)) \). Therefore we have the following commutative diagram

\[
\begin{array}{c}
\pi(A) \xrightarrow{f_\ast} \pi(A) \\
\downarrow \quad \downarrow \\
\mathbb{R}^n \xrightarrow{\varphi} \mathbb{R}^n \\
\downarrow \quad \downarrow \\
M(A) \xrightarrow{\hat{\varphi}} M(A).
\end{array}
\]

From this diagram, it also follows that \( \varphi(\gamma x) = \hat{\varphi}_\ast(\gamma) \varphi(x) \). Therefore \( f_\ast = \hat{\varphi}_\ast: \pi(A) \to \pi(A) \). By Whitehead theorem, since \( M(A) \) is aspherical, \( f \) is homotopic to \( \hat{\varphi} \).

On the other hand, by Lemma 2.1 in [1], \( f \) is homotopic to a hyperbolic diffeomorphism \( g: M(A) \to M(A) \). Hence we may put \( g = \hat{\varphi} \). Moreover \( D \) is hyperbolic.

By considering (2), equations (2,2), (2,3), (2,9) and Theorem I in [2], one can prove that

\[
\varphi = \left( \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}, \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} \right),
\]

d_1 \in \mathbb{R}^k, d_2 \in \mathbb{R}^{n-k}, D_1 \in GL(k, \mathbb{R}), D_2 \in GL(n - k, \mathbb{R}) \) (see (3.5) in the proof of Theorem 3.1[2]). So, \( D_2 \) is also hyperbolic.

Now take

\[
\gamma_B = (a_2, A_2) = \left( \begin{pmatrix} 0 \\ a_2 \end{pmatrix}, \begin{pmatrix} I_k & 0 \\ 0 & A_2 \end{pmatrix} \right) \in \pi(B), \ a_2 \in \mathbb{R}^{n-k}, A_2 \in O(n - k).
\]

Consider

\[
f_\ast \left( \begin{pmatrix} 0 \\ a_2 \end{pmatrix}, \begin{pmatrix} I_k & 0 \\ 0 & A_2 \end{pmatrix} \right) = \left( \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}, \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} \right) \left( \begin{pmatrix} 0 \\ a_2 \end{pmatrix}, \begin{pmatrix} I_k & 0 \\ 0 & A_2 \end{pmatrix} \right) \left( \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}, \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} \right)^{-1}.
\]

\( f_\ast \) induces an isomorphism \( h_\ast: \pi(B) \to \pi(B) \) where \( h_\ast(\gamma_B) = (d_2, D_2) \gamma_B (d_2, D_2)^{-1} \). Therefore, by Bieberbach theorem, there exists a diffeomorphism \( h: M(B) \to M(B) \) which is induced by \((d_2, D_2)\) where \( D_2 \) is hyperbolic. Hence
h is a hyperbolic diffeomorphism. Since each hyperbolic diffeomorphism of a flat manifold \( M \) defines an Anosov diffeomorphism on \( M \), \( h \) is an Anosov diffeomorphism.

**Theorem 2.2.** Given a real Bott manifold \( M(A) = T^k \times (\mathbb{Z}_2)^n M(B) \). Let \( f: M(A) \to M(A) \) be an Anosov diffeomorphism which admits the Anosov relation \( N(f) = |L(f)| \). Then there exists an Anosov diffeomorphism \( h: M(B) \to M(B) \) with \( N(h) = |L(h)| \).

**Proof.** Let \( f: M(A) \to M(A) \) be an Anosov diffeomorphism. In the proof of Theorem 2.1, we have obtained that \( f \) is homotopic to a hyperbolic diffeomorphism \( \varphi \) of \( M(A) \) with \( f_* = \varphi_* \). Then there exists the affine lift \( \varphi = (d, D) \in \text{Aff}(\mathbb{R}^n) \) of \( f \) where

\[
d = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}, \quad D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}, \quad D_1 \in \text{GL}(k, \mathbb{R}), \quad D_2 \in \text{GL}(n-k, \mathbb{R})
\]

with \( D, D_1, D_2 \) are hyperbolic. Moreover, \( (d_2, D_2) \) induces an Anosov diffeomorphism \( h \) of \( M(B) \).

Let \( T: F \to GL(n, \mathbb{Z}) \) and \( T': F' \to GL(n-k, \mathbb{Z}) \) be the holonomy representations of \( M(A) \) and \( M(B) \) respectively. Since \( \pi(B) \) is a subgroup of \( \pi(A) \) (see Proposition 2.4[2]), there is a projection \( p: F \to F' \). So, for each \( x' \in F' \), there is an element \( x \in F \) such that

\[
T(x) = \begin{pmatrix} I_k & 0 \\ 0 & T'(p(x)) \end{pmatrix} = \begin{pmatrix} I_k & 0 \\ 0 & T'(x') \end{pmatrix}.
\]

Now, since \( N(f) = |L(f)| \), by Theorem 2.3 in [1], for each \( x' \in F' \), we assume that \( 0 \leq \det(I_n - T(x)D) = \det(I_k - D_1)\det(I_{n-k} - T'(x')D_2) \). So, for each \( x' \in F' \), \( \det(I_{n-k} - T'(x')D_2) \geq 0 \) or \( \det(I_{n-k} - T'(x')D_2) \leq 0 \) depending on whether \( \det(I_k - D_1) > 0 \) or \( \det(I_k - D_1) < 0 \) respectively. Note that \( \det(I_k - D_1) \neq 0 \), because \( D_1 \) is hyperbolic. Similarly, if \( 0 \geq \det(I_n - T(x)D) \), we also get the same result. Hence, again by Theorem 2.3 in [1], \( N(h) = |L(h)| \).

By this theorem, we can obtain a real Bott manifold \( M(A) = T^k \times (\mathbb{Z}_2)^n M(B) \) satisfying the Anosov relation \( N(f) = |L(f)| \) with \( f: M(A) \to M(A) \) an Anosov diffeomorphism, from a real Bott manifold \( M(B) \) satisfying the Anosov relation \( N(g) = |L(g)| \) with \( g: M(B) \to M(B) \) an Anosov diffeomorphism.

**References**


**Received: February 21, 2014**