Hydromagnetic Self-gravitating Stability of Streaming Fluid Cylinder With Longitudinal Magnetic Field

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Abstract

Hydromagnetic Self-gravitating Stability of Streaming Fluid Cylinder With Longitudinal Magnetic Forces is developed. A general eigenvalue relation is derived studied analytically and results are confirmed numerically. In absence of the effect of the electromagnetic forces interior and exterior the fluid, so the model is only subjected to the capillary force. It is found that the model is unstable in the region $0 < x \leq 1.07$. While it is stable in the region $1.07 \leq x < \infty$. This means that model is just unstable in small domains of axisymmetric perturbation but it stable in all other domains. For very high intensity of magnetic field the model is completely stable for all values of wave lengths. The Lorentz
force for some states has the character of enlarging the gravitational unstable domain and decreasing the stable regions in all possible modes of perturbations. In other states the Lorentz force is totally stabilizing but no magnetic field, however strong, can suppress the gravitational instability because the gravitational instability of sufficiently long waves will persist. The streaming has strong destabilizing effect for all short and long wave lengths, that effect is independent of the kind of perturbation.

1 Introduction

The dynamical oscillation of cylindrical jet endowed with surface tension and acted upon by the electromagnetic force has been documented in several reported works (see Rayleigh [15], Abramowitz and Stgun [1], Drazin [5,7], Drazin and Reid [6], Radwan et al [13,14], Chandrasekhar [2], Korovin [8], Samia et al [16], Radwan [10], Radwan and Callebaut [9], Radwan and Hasan [11,12]), based on the linear perturbation technique of a small disturbance. The stability of the cylindrical jet acting upon the capillary force only may be traced in the documented texts by Wang [17], Donnelly and Glaberson [4], Chandrasekhar [2] and Radwan [10]. The self-gravitating instability of a full fluid cylinder is studied for first time in the scientific province by Chandrasekhar and Fermi [3] for symmetric mode, on utilizing the energy conservation principle. The instability analysis of that model for all possible axisymmetric and non-axisymmetric modes has been developed and the complete analysis. His pioneering investigations predicted that for \( x \) (the dimensionless longitudinal wave number) less than the critical cutoff wave number \( x_c \), the axisymmetric perturbations render the model gravitationally unstable, leading to the break-up of the fluid jet. This problem is of a considerable interest in describing the appearance of condensation within astronomical bodies. It is also fundamental in understanding the crucial dynamical behavior of the spiral arm of galaxy. The self-gravitating instability of fluid jet ambient with an infinite (radially) self-gravitating fluid has been developed by Radwan [10] and Radwan and Hasan [11,12]. The stability of self gravitating full fluid cylinder in a tenuous medium pervaded by and ambient with constant magnetic field, is studied for axisymmetric perturbation by Chandrasekhar [2], on utilizing the technique of presentation of the solenoidal vectors in terms of toroidal and poloidal quantities. The aim of the present work is discussing the stability of a fluid cylinder subject to self-gravitating force in axisymmetric perturbation for all long and short wave lengths.
2 Formulation of the Problem

We consider a self-gravitating fluid cylinder of radius $R$ ambient with a self-gravitating vacuum. This system is acting upon the gravitational, inertia and electromagnetic forces. The fluid is considered to be incompressible, non-viscid and perfectly conducting. We shall use a cylindrical polar coordinate $(r, \phi, z)$ system with $z$-axis coinciding with the axis of the cylinder. Under the present envelopes, the basic equations for the fluid are:

$$\rho \left( \frac{\partial \bold{u}}{\partial t} + (\bold{u} \cdot \nabla) \bold{u} \right) = \rho \nabla \phi - \nabla P + \mu (\nabla \wedge \bold{H}) \wedge \bold{H}$$  \hspace{1cm} (1)

$$\nabla \cdot \bold{u} = 0 \hspace{1cm} (2)$$

$$\frac{\partial \bold{H}}{\partial t} = \nabla \wedge (\bold{u} \wedge \bold{H}) \hspace{1cm} (3)$$

$$\nabla \cdot \bold{H} = 0 \hspace{1cm} (4)$$

$$\nabla^2 \phi = -4\pi \rho G \hspace{1cm} (5)$$

While in the vacuum,

$$\nabla \cdot H_{\text{vac}} = 0 \hspace{1cm} (6)$$

$$\nabla \wedge H_{\text{vac}} = 0 \hspace{1cm} (7)$$

$$\nabla^2 \phi_{\text{vac}} = 0 \hspace{1cm} (8)$$

Where $\rho$, $\bold{u}$ and $P$ are the fluid mass density, velocity vector and static pressure, $\mu$ and $\bold{H}$ are the coefficient of the magnetic field permeability and intensity, respectively, and $H_{\text{vac}}$ is the vacuum magnetic field excitation. Note that, since the fluid is acting upon its own attraction a new variable is introduced which is the gravitational potential $\phi$ (the self-gravitating force in equation (1) given by $\rho \nabla \phi$) that satisfies the Poisson’s equation (equation (5)) i.e., the field equation for Newtonian gravitation equations (4), (6) are the conservation of the magnetic fluxes interior and exterior the fluid cylinder, equation (7) is getting from Maxwell’s equations with the fact that there is no current flow in the vacuum. In deriving the evolution equation of magnetic field (equation (3)) from Maxwell’s equations it was assumed that the displacement current vanishes which is true for slowly varying fields (velocities are very small compared with the light velocity).

The unperturbed state is supported by,

$$H_0 = (0, 0, H_0) \hspace{1cm} (9)$$

$$H_{\text{vac}} = (0, 0, \alpha H_0) \hspace{1cm} (10)$$

$$\bold{u}_0 = (0, 0, U) \hspace{1cm} (11)$$
Where $\alpha$ is parameter that must satisfy some physical conditions. The self-gravitating potential $\phi_o, \phi_o^{\text{vac}}$ in the equilibrium state satisfy inside cylinder
\[ \nabla^2 \phi_o = -4\pi \rho G \tag{12} \]
and outside cylinder
\[ \nabla^2 \phi_o^{\text{vac}} = 0 \tag{13} \]
from the infinite solution, the solutions of equations (12),(13) are given by
\[ \phi_o = -\pi \rho G r^2 + C_2 \tag{14} \]
\[ \phi_o^{\text{vac}} = C_3 \ln r + C_4 \tag{15} \]
Where $C_2, C_3$ and $C_4$ are constant of integration to be identified upon applying appropriate boundary Conditions, these conditions are
(i) The self-gravitating potential $\phi_o$ must be continuous across the boundary surface.
(ii) The derivative of the self-gravitating $\phi_o$ must be continuous across fluid-tenuous interface at $r = R_o$, these conditions, yield
\[ C_2 = 0 \tag{16} \]
\[ C_3 = -2\pi \rho G R_o^2 \tag{17} \]
\[ C_4 = -\pi \rho G R_o^2 + 2\pi \rho G R_o^2 \ln R_o \tag{18} \]
so,
\[ \phi_o = -\pi \rho G r^2 \tag{19} \]
\[ \phi_o^{\text{vac}} = -\pi \rho G R_o^2 [1 + 2 \ln \left( \frac{r}{R_o} \right)] \tag{20} \]
Solving equation (1) for the problem under consideration applying the balance of the pressure at $r = R_o$ and utilizing the foregoing basic results in equation (9),(20) the equilibrium kinetic fluid pressure $P_o$ is given by
\[ P_o = \pi \rho^2 G (R_o^2 - r^2) + \frac{\mu H_o^2}{2} (\alpha^2 - 1) \tag{21} \]
where
\[ \alpha^2 \geq 1 \tag{22} \]
Under the condition that $P_o \geq 0$ at $r = R_o$, where the equality holds as a limiting case with zero fluid pressure.
3 Linear Perturbation Analysis

For small departures from the equilibrium state due to an infinitesimal perturbation, every perturbed quantity \( Q(r, \phi, z; t) \) can be expressed as

\[
Q(r, 0, z; t) = Q_o(r) + \varepsilon(t)Q_1(r, 0, z)
\]  

Where the quantities with subscript zero are those of equilibrium while those with 1 are small increments due to the perturbation. Here \( Q \) stands for each of \( u, \rho, \phi, H, \phi^{\text{vac}}, H^{\text{vac}} \) and the perturbed radial distance of the fluid cylinder is given by

\[
r = R_o + \varepsilon(t)R_o \exp(ikz)
\]  

Where \( k \) real is the longitudinal wave number and where the second term on the right-hand side of equation (24) is the elevation of the surface wave normalized with respect to \( R \) and measured from the equilibrium position. The amplitude of the perturbed wave at time \( t \) is,

\[
\varepsilon(t) = \varepsilon_o \exp(\sigma t)
\]  

Where \( \varepsilon_o \) is the initial amplitude and \( \sigma \) is the temporal amplification.

By an appeal to the expansion (23), the linear perturbation equation deduced from the basic equations (1)-(8) are given by

\[
\left( \frac{\partial u_1}{\partial t} + U \frac{\partial u_1}{\partial t} \right) - \frac{\mu}{\rho} (H_o \cdot \nabla)H_1 = -\nabla \left[ \frac{P_1}{\rho} - \phi_1 + \frac{\mu}{\rho} (H_o \cdot H_1) \right]
\]  

\[
\nabla \cdot u_1 = 0
\]  

\[
\frac{\partial H_1}{\partial t} = (H_o \cdot \nabla)u_1 - (u_o \cdot \nabla)H_1
\]  

\[
\nabla \cdot H_1 = 0
\]  

\[
\nabla^2 \phi_1 = 0
\]

while in the vacuum,

\[
\nabla \cdot H_1^{\text{vac}} = 0
\]  

\[
\nabla \wedge H_1^{\text{vac}} = 0
\]  

\[
\nabla^2 \phi_1^{\text{vac}} = 0
\]

by taking into account the time dependence (25) and according to the linear perturbation technique used for solving the stability problems of cylindrical models (Chandrasekhar [2], Radwan [10]), every fluctuating quantity \( Q_1(r, z, t) \) may be expressed as

\[
Q_1(r, z, t) = \varepsilon_o Q_1^*(r) \exp(\sigma t + ikz)
\]  

where the asterisk denotes complex conjugate.
The system of linearized equations (26)-(33) is simplified in the forms

\[ u_1 = \frac{-(\sigma + ikU)}{(\sigma + ikU)^2 + \Omega_A^2} \nabla \Pi_1 \]  

\[ H_1 = \frac{ikH_o}{(\sigma + ikU)} u_1 \] 

\[ H_1^{vac} = -\nabla \psi_1^{vac} \]  

\[ \nabla^2 \psi_1^{vac} = 0 \]

Here \( \Pi_1 + \phi_1 = \frac{P_o}{\rho} + \frac{\mu}{\rho} (H_o \cdot H_1) \) is the total MHD pressure, \( \psi_1^{vac} \) is the magnetic potential in the vacuum region, while \( \Omega_A = \sqrt{\frac{\mu k^2 H_o^2}{\rho}} \) is the Alfven wave frequency defined in terms of \( H_o \).

By taking the divergence of equation (35) and using equation (27) we get

\[ \nabla^2 \Pi_1 = 0 \]  

As we have seen the linearized system of equation (26)-(33) could be solved if laplace equation

\[ \nabla^2 F_1(r, z) = 0 \] 

is solved where \( F_1 \) stands for \( \Pi_1, \phi_1, \phi_1^{vac} \) and \( \psi_1^{vac} \). By the use of the time-space dependence (34), equation (40) takes the form

\[ \frac{1}{r} \frac{d}{dr} (r \frac{dF_1}{dr}) - k^2 F_1 = 0 \]

this is the modified Bessel equation, its solution is given in terms of Bessel functions of imaginary argument. Under the present circumstances, the non-singular solution for the different variables in this case is given by

\[ \phi_1 = C_1 I_o(kr) \exp(\sigma t + ikz) \]  

\[ H_1 = C_2 \nabla I_o(kr) \exp(\sigma t + ikz) \] 

\[ \Pi_1 = C_3 I_o(kr) \exp(\sigma t + ikz) \]  

\[ H_1^{vac} = C_4 \nabla K_o(kr) \exp(\sigma t + ikz) \] 

\[ \phi_1^{vac} = C_5 K_o(kr) \exp(\sigma t + ikz) \]

The solution of the unperturbed and perturbed sates equation (42)-(46) must satisfy appropriate boundary conditions across the fluid interface at \( r = R_o \), these conditions are given as follows.

1) The normal component of the velocity vector \( u \) must be compatible with the velocity of the fluid particles across the boundary surface, equation (24) at
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\( r = R_0. \)

(2) The gravitational potential and its derivative must be continuous across the perturbed interface (24) at the initial position \( r = R_0. \)

(3) The normal component of the magnetic field must be also continuous across the interface (24) at \( r = R_0. \)

(4) The normal component of the total stress tensor must be continuous across the interface (24) at \( r = R_0. \) consequently, after cumbersome calculations, we obtain,

\[
\begin{align*}
\phi_1 &= 4\pi\rho G R K_0(x) I_o(kr) \varepsilon(t) \exp(ikz) \\
H_1 &= \frac{\alpha H_o}{K_o(x)} \nabla K_o(kr) \varepsilon(t) \exp(ikz) \\
\phi_{1 vac} &= 4\pi\rho G R K_0(kr) I_o(x) \varepsilon(t) \exp(ikz) \\
P &= -\frac{R}{x I'_o(kr)} ((\sigma + ikU)^2 + \Omega^2_\lambda) I_o(kr) \varepsilon(t) \exp(ikz)
\end{align*}
\]

where \( P \) is the total magneto-fluid dynamic which is a sum of the kinetic pressure \( P_s \) of the fluid and the magnetic pressure \( \frac{1}{2} \mu (H_o \cdot H_1) \) inside the fluid, \( (x = kr) \) is the longitudinal dimensionless wave number \( I_o(kr) \) and \( K_o(kr) \) are the modified Bessel functions of the first and second kind of the order zero.

After lengthy calculations, the following dispersion relation is derived;

\[
(\sigma + ikU)^2 = 4\pi\rho G \left( \frac{x I'_o(x)}{I_o(x)} \right) |I_o(x) K_o(x) - \frac{1}{2}| + \frac{\mu H_o^2}{\rho R^2} (-x^2 + x^2 \alpha^2 \frac{K_o(x) I'_o(x)}{K'_o(x) I_o(x)})
\]

\[ (53) \]

4 General Discussion

The relation (53) is the desires dispersion equation of a fluid cylinder acting upon the electromagnetic and self-gravitating forces. It relates the growth rate \( \sigma \) with the cylindrical functions \( I_o(x) \) and \( K_o(x) \), the entity \( (4\pi\rho G) \) as well as \( \left( \frac{\mu H_o^2}{\rho R^2} \right) \) as a unit of \( (time)^{-2} \) and with the vacuum magnetic field parameters \( \alpha \). Relation (53) is a simple linear combination of eigenvalue relations of a fluid cylinder subject to the electromagnetic force only and that one subject to self-gravitating force only. It is remarkable that in the axisymmetric mode \( m = 0 \) for a fluid cylindrical jet pervaded by and ambient with constant axial fields, subject to its own attraction and lorentz forces. Chandrasekhar [2](page.534)
has also obtained just the simple linear combination. This character of a simple linear combination is also true if the acting forces are the capillary and electromagnetic forces. Whether the models is a full liquid jet (Chandrasekhar [2], page 545) or a hollow jet (Radwan [10]). The physical interpretation of such phenomenon is given explicitly in this paper reference. Since the relation (53) is somewhat more general several stability criteria can be obtained as limiting cases from the eigenvalue relation (53) if we put $H_o = 0$, the eigenvalue relation (53) degenerates to

$$ (\sigma + ikU)^2 = 4\pi \rho G \left( \frac{xI_o'(x)}{I_o(x)} \right) [I_o(x)K_o(x) - \frac{1}{2}] $$

(54)

this produces the result by Chandrasekhar [2] (page 518), which has been derived by utilizing the normal mode analysis for all possible modes of perturbations. If we suppose that $\alpha = 1$, the eigenvalues relation (53) will be,

$$ (\sigma + ikU)^2 = 4\pi \rho G \left( \frac{xI_1'(x)}{I_o(x)} \right) [I_o(x)K_o(x) - \frac{1}{2}] + \left[ \frac{\mu H_o^2}{\rho R^2} \right] \left( -x^2 + x^2 \frac{K_o(x)I_o'(x)}{K_o(x)I_o(x)} \right) $$

(55)

by use the wronskian relation

$$ W_o(I_o(x), K_o(x)) = I_o(x)K_o'(x) - K_o(x)I_o'(x) = -x^{-1} $$

(56)

$$ I_o'(x) = I_1(x) $$

(57)

$$ K_o'(x) = -K_1(x) $$

(58)

the relation (55) reduces to

$$ (\sigma + ikU)^2 = 4\pi \rho G \left( \frac{xI_1(x)}{I_o(x)} \right) [I_o(x)K_o(x) - \frac{1}{2}] + \left[ \frac{\mu H_o^2}{\rho R^2} \right] \left( \frac{-x}{K_1(x)I_o(x)} \right) $$

(59)

this coincides with Chandrasekhar [2] (page 634 (equ 127)) result in investigating the effect of a uniform axial magnetic field on the gravitational instability of an infinite cylinder by utilizing the method of representing solenoidal vectors in terms of toroidal and poloidal vector fields. If the electromagnetic force influence is superior to that of the self-gravitating force (such that $G = 0$), the general dispersion equation (53) reduces to,

$$ (\sigma + ikU)^2 = \left[ \frac{\mu H_o^2}{\rho R^2} \right] \left( -x^2 + x^2 \alpha^2 \frac{K_o(x)I_o'(x)}{K_o'(x)I_o(x)} \right) $$

(60)

the self-gravitating stability in the absence of the electromagnetic force, the self-gravitating dispersion relation is given by equation (54). In the axisymmetric disturbance $m = 0$, the analytical and numerical discussions of the relation (54) shows that the fluid cylinder is gravitationally stable for all values of $x$ except in the domain, the marginal stability is obtained at the critical wave number $x_c$.
4.1 Numerical discussions

In this general case the fluid cylinder is acting upon the combined effect of the self-gravitating and electromagnetic forces, its stability criterion is given by the (53) in its general form. If we combine the results of section (4) we may conclude that there are unstable domains whether the disturbance is axisymmetric or non-axisymmetric also we conclude that the magnetic fields (especially for \( m = 0 \)) will enlarge the gravitationally unstable domain and decreases all the gravitationally stable domains. In order to confirm this analytical results and also to determine the effect of the electromagnetic force on the gravitational stability for different values of \( \alpha \), the dimensionless eigenvalue relation in numerically analyzing,

\[
\frac{(\sigma + ikU)^2}{4\pi \rho G} = \left[ x \frac{I_1(x)}{I_0(x)} \right] (I_0(x)K_0(x) - \frac{1}{2}) + \left( \frac{H_o}{H_G} \right)^2 \left[ x^2 \frac{I_0(x)K_1(x) + \alpha^2 K_o(x)I_1(x)}{I_o(x)K_1(x)} \right]
\]

\[
\sigma^* = \sqrt{\left[ x \frac{I_1(x)}{I_0(x)} (I_0(x)K_0(x) - \frac{1}{2}) \right] + \left( \frac{H_o}{H_G} \right)^2 \left[ x^2 \frac{I_0(x)K_1(x) + \alpha^2 K_o(x)I_1(x)}{I_o(x)K_1(x)} \right]} - U^*
\]

when \( \sigma^* = \sigma \sqrt{4\pi \rho G} \) and \( U^* = \frac{IKU}{\sqrt{4\pi \rho G}} \) the relation (62) is investigated numerically.

5 Conclusion

From the foregoing discussions, we may conclude the following
1- In the absence of the effect of the electromagnetic force, the model is self-gravitating unstable in small domains of axisymmetric perturbation but it is stable in all other domains.
2- The axial magnetic field interior the fluid has a strong stabilizing effect for all short and long wave lengths. Such effect is independent of the kind of perturbation whether the perturbed wave length is shorter or longer the circumference of the fluid cylinder.
3- The axial magnetic field pervaded in the region surrounding the fluid cylinder has strong stabilizing effect for all short and long wavelengths.
4- For very high intensity of magnetic field the destabilizing character of the model could be suppressed completely and stable seta in.
5- The streaming has strong destabilizing effect for all short and long wave-length, that effect is independent of the kind of perturbation.

References

1- Abramowitz, M and Stgun, I. ”Handbook of mathematical functions”, Dover