Viscous Oscillatory Stratified Flow in a Vertical Narrow Channel with a Porous Wall

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Abstract

Viscous oscillatory fully developed flow of stratified fluid in a long vertical narrow rectangular channel with one side being porous and the other two sides impervious is considered in this paper. Closed form solutions are obtained using lubrication approximation and similarity transformation. The effects of various parameters pertaining to the problem on transverse & axial velocity profiles, density distribution and pressure drop are illustrated in the figures. The corresponding steady flow is deduced in the limit as the frequency of oscillation tending to be zero.

Keywords: Viscous, Oscillatory Flow, Stratification Parameter, Stokes Flow, Similarity Transformation, Channel flow, Lubrication Approximation
1 Introduction

The two dimensional Oscillatory flow in thin channels with a porous wall has numerous applications in various branches of Engineering and Technology such as filtration, boundary layer control and Lubrication approximation problems. It plays an important role in the study of problems which involve porous media, adhesion, biological flows and some manufacturing flows. Krishnambal S and Ganesh S discussed analytical solution to unsteady stokes flow of viscous fluid between two parallel porous plates [2]. Forced Oscillation in an inviscid stratified fluid have been considered by many authors Krishna & Sharma [6], Hendershot [4], Sharma and Naidu [11], Naidu [7], Prasanna Venkatesh [10] and Channabassappa and Ranganna [1]. But very little work has been done on the oscillations in stratified viscous fluid. MHD flow of stratified fluid through porous medium has been discussed by Gupta and Goyal [3], Khandelwal and Jain [5] discussed unsteady MHD flow of stratified fluid through porous medium over moving plate in slip flow regime. Similarity solution was obtained by defining the pressure as a known function of x and y and an unknown function of ‘t’. Heat transfer in MHD flow of dusty viscoelastic stratified fluid in porous medium under variable viscosity was discussed by Om Prakash [9] by presenting the similarity solution to velocity profiles by assuming both axial and transverse velocity profiles as a function of y and t only. R.L. Panton [8] considered small geometry parameter flow where the flow region is thin transverse to the flow direction with a prescribed Poiseuilli flow as entrance velocity with time independence. Complete solution of the problem was obtained by applying lubrication approximation with the withdrawal velocity $v_w$ through the porous channel which is a function of x only. In present investigation we consider the viscous oscillatory flow vertically stratified fluid through a vertical narrow channel with a porous wall through which the fluid is withdrawn with a constant velocity. The resultant partial differential equations are transformed into ordinary differential equations by using similarity transformation. The solutions were presented and interpreted graphically. For $N = 0$ and a prescribed initial velocity the problem reduces to that of oscillatory viscous flow through narrow channel.

2 Mathematical Formulation and solution of the Problem

We consider the flow in a channel of width $h_0$ where fluid is withdrawn along a porous right side wall. The impervious plate is placed on y – axis with coordinate system origin at the lower end and the porous plate at a distance $h_0$ parallel to y – axis. The fluid is withdrawn with a constant suction velocity $u_1$ through this plate for a length of L. The walls are solid prior to this point and the flow is assumed fully developed. If the channel is thin such that $h_0/L$ tends to zero then the inertial terms in the momentum equations can be neglected with a small suction velocity
compared with average longitudinal velocity. The fluid is taken to be vertically stratified with varying density. The density in unperturbed state is assumed to be linearly distributed and in the perturbed state is taken to be a function of x, y and t. The mathematical model of the problem is as follows.

Equation of continuity:
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$  \hspace{1cm} (1)

Equation of Incompressibility for stratified incompressible flow:
$$\frac{d}{dt} \frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} = 0$$  \hspace{1cm} (2)

Equation of motion for Stokes’ flow
$$\rho \frac{du}{dt} = -\frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$  \hspace{1cm} (3)

$$\rho \frac{dv}{dt} = -\frac{\partial p}{\partial y} + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) - \rho g$$  \hspace{1cm} (4)

Here $\mu$ represents the coefficient of viscosity and $\rho$ the density of the fluid. The density distribution in the undisturbed state is taken as $\rho = \rho_0(y) + \rho'(x, y, t)$

$$\rho_0(y) = \rho_0(1 - \beta y)$$ \hspace{1cm} (5)

$\rho_0'$ is a constant, density, $\rho_0(y)$ is linearly distributed, $\rho'(x, y, t)$ is perturbation density, $\beta$ stratification parameter (a constant) so that Brunt – Vaisala frequency N becomes $N^2 = \beta g$

$$\frac{\partial \rho}{\partial t} = \frac{\partial \rho'}{\partial t} = \rho_0' \beta v$$ \hspace{1cm} (7)

Let $v_0$ be the initial average velocity and taken as a scale for the ‘v’ profiles. By continuity equation the horizontal velocity ‘u’ scale is given by $u_0 = v_0 h_0/L$.

Define nondimensional variables as follows

$X = x/h_0, \ Y = y/L, \ U = Lu_0/v_0, \ V = v/v_0, \ P = (p - p_{ref}) h_0^2/L\mu u_0, \ Re = h_0 v_0/v$

As $\frac{h_0}{L} \to 0$, neglecting the inertial terms in the momentum equations and applying Bousinesq’s approximation we get,

$$\rho_0 \frac{\partial U}{\partial t} = -\frac{\partial P}{\partial X}$$ \hspace{1cm} (8)

$$\rho_0 \frac{\partial V}{\partial t} = -\frac{\partial P}{\partial Y} + \mu \frac{\partial^2 V}{\partial X^2} - \rho g$$ \hspace{1cm} (9)

Differentiating (8) with respect to t and Y partially and (9) with respect to t and X partially and then subtracting we get
\[ \rho_0 \frac{\partial}{\partial t} \left( \frac{\partial U}{\partial Y} - \frac{\partial V}{\partial X} \right) = -\mu \frac{\partial}{\partial t} \frac{\partial^3 V}{\partial X^3} + \rho_0' N^2 \frac{\partial V}{\partial X} \]

(10)

\[ U = U(X,Y)e^{iaX}, \quad V = V(X,Y)e^{iaX}, \quad \text{and} \quad P = P(X,Y)e^{iaX} \]

\[ \rho_0 \left( \frac{\partial U}{\partial Y} - \frac{\partial V}{\partial X} \right) = -\mu \frac{\partial^3 V}{\partial X^3} - i \frac{\rho_0' N^2}{\omega} \frac{\partial V}{\partial X} \]

(11)

The boundary conditions of the problem are

\[ U(0, Y) = 0, \quad U(1, Y) = u_1 \]

\[ V(0, Y) = 0, \quad V(1, Y) = 0 \]

(12)

We define Stream Function \( \psi \) such that

\[ U = \frac{\partial \psi}{\partial Y} \quad \text{and} \quad V = -\frac{\partial \psi}{\partial X} \]

(13)

Equation (12) becomes

\[ \rho_0' \nabla^2 \psi = \mu \frac{\partial^4 \psi}{\partial X^4} + i \frac{\rho_0' N^2}{\omega} \frac{\partial^2 \psi}{\partial X^2} \]

(14)

The function \( f(Y) \) is introduced as follows:

\[ \Psi = (v_0 - u_1 Y)f(X) \]

(15)

Equation (14) becomes

\[ \left( D^4 - \frac{1}{\mu} \left( \rho_0' \frac{1}{\omega} - i \frac{\rho_0' N^2}{\omega} \right) \right)f(X) = 0 \]

(16)

Where \( D^2 = \frac{d^2}{dX^2} \), \( \alpha^2 = \frac{1}{\mu} \left( \rho_0' \frac{1}{\omega} - i \frac{\rho_0' N^2}{\omega} \right) \Rightarrow \alpha = \sqrt{\frac{1}{\mu} \left( \rho_0' \frac{1}{\omega} - i \frac{\rho_0' N^2}{\omega} \right)} \)

(17)

Equation (16) is a differential equation of order 4 with constant co-efficient for which the solution is given by

\[ f(Y) = c_1 + c_2 X + c_3 e^{\alpha X} + c_4 e^{-\alpha X} \]

(18)

The Boundary Conditions are transformed in terms of \( f(X) \) are as follows

\[ f(0) = 0, \quad f(1) = -1 \quad \text{and} \quad f'(0) = f'(1) = 0 \]

(19)

The results obtained by applying the boundary conditions on \( f(X) \) are

\[ f(0) = 0 = c_1 + c_3 + c_4 \]

(20)

\[ f(1) = -1 = c_1 + c_2 + c_3 e^\alpha + c_4 e^{-\alpha} \]

(21)

\[ f'(0) = 0 = c_2 + \alpha c_3 - \alpha c_4 \]

(22)

and

\[ f'(1) = 0 = c_2 + \alpha c_3 e^\alpha - \alpha c_4 e^{-\alpha} \]

(23)

Solving the above equations, the values of the constants are as follows.

\[ c_1 = \frac{e^{\alpha - 1}}{(\alpha + 2) + (\alpha - 2)e^\alpha} \]

(24)
Viscous oscillatory stratified flow

\[ c_2 = \frac{-\alpha(1+e^{\alpha})}{(a+2)+(a-2)e^{\alpha}} \]  

(25)

\[ c_3 = \frac{1}{(a+2)+(a-2)e^{\alpha}} \]  

(26)

\[ c_4 = \frac{-e^{\alpha}}{(a+2)+(a-2)e^{\alpha}} \]  

(27)

Using the expression for \( \psi \) and equation (18) the velocity components are obtained. By Substituting (28) and (29) in (8) and (9) we get the expression for pressure drop and using (5), (6) and (7) density distribution is obtained. The corresponding expressions are as follows.

\[ U(X,Y) = -u_1 \left( \frac{e^{\alpha(1-\alpha(1+e^{\alpha}))X + e^{\alpha X} - e^{\alpha} e^{-\alpha X}}}{(a+2)+(a-2)e^{\alpha}} \right) \]  

(28)

\[ V(X,Y) = -(v_0 - u_1 Y) \left( \frac{-\alpha(1+e^{\alpha})+ae^{\alpha X} + ae^{\alpha} e^{\alpha X}}{(a+2)+(a-2)e^{\alpha}} \right) \]  

(29)

\[ P(X,Y) = i\omega \rho_0 \int_0^X f(X) dX + \left( v_0 Y - u_1 \frac{v^2}{2} \right) \left[ \frac{1}{\mu} f''(X) - i \left( \frac{\omega v_0 + \rho_0 N^2}{\omega} \right) f'(X) \right] \]  

(30)

\[ \rho = \rho_0 \left( 1 - \beta Y \right) + \frac{\rho_0 \beta}{\omega} \left( 1 - e^{i\alpha} \right) \psi(x,y) \]  

(31)

4. Results and Discussion

The effects of various parameters namely time (\( \omega t \)), height (Y), Brunt – Vaisala frequency (N) and constant density (\( \rho_0 \)) on Transverse velocity, Axial Velocity, Density Distribution and Pressure drop are presented graphically. The range of values of X and Y is assumed to be 0 to 1 while that of each of the other parameters is varied. Figure 1 shows that the transverse velocity decreases as time increases and also it decreases when height increases. Figure 2, 3 and 4 supports the above observation as it depicts transverse velocity at \( \omega t = 0 \) for various values of \( y \) ranging from 0 to 1 for \( N=5, \ N=10 \) and \( N=20 \) respectively. There are significant effects of stratification parameter N on transverse velocity throughout the channel at specific heights. Figure 5 indicates that increase in stratification parameter N results in decrease of transverse velocity. Figure 6 represents the transverse velocity for different values of constant density \( \rho_0 \) varying from 1 to 5. It is observed that transverse velocity decreases as the constant density \( \rho_0 \) increases. Figure 7 and Figure 8 represent axial velocity for \( \omega t \) varying from 0 to \( \pi \) and varying stratification parameter N (\( N=0 \) to 10). It is observed that the axial velocity increases with time and symmetric about a central point of the channel.
with increase in stratification parameter N. Pressure drop at different time is given by Figure 9 while Figure 10 displays the pressure drop for different heights from \( y = 0 \) to \( y = 1 \). As height increases there is a variable effect on pressure drop while pressure drop increases with time. The effects on Density Distribution by time and height are depicted by Figure 11 and Figure 12. It is evident from the above figures that density distribution decreases with time and height.

5. Conclusion

Analysis of viscous oscillatory flow of stratified fluid through vertically narrow channel with a porous wall is presented in this paper. Lubrication approximation is applied in the momentum equation and the system of partial differential equation is solved analytically using similarity transformation. The following conclusions are obtained as a result of the investigation.

1. Transverse velocity decreases with time and height
2. Increase in stratification parameter N results in linear flow around the center of the channel while increased non linear flow near the boundary.
3. The effect of both stratification and constant density are similar and both decreases transverse velocity for an increase in their corresponding values
4. Pressure drop is non linear for higher values of stratification parameter N
5. Density distribution decreases both with time and height.
Viscous oscillatory stratified flow
References


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