Special Numerical Analysis Methods
for Mathematical Models of Gas Flow
in Trunkline Network

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Abstract
We consider two new methods for numerical solution of a complete system of partial differential equations describing the flow of a gas mixture in pipeline systems. The first method tracks Lagrangian particles as they move together with the flow of transported fluid. When implementing this method, the flow parameters are found by means of a difference scheme, and the distribution of mass fractions of components and enthalpy of matter along the pipeline, by analyzing the motion of the Lagrangian particles. If we ignore the processes of diffusion, these particles must preserve their composition. The energy equation without diffusion and heat conduction reduces to an ordinary differential equation. The method proposed is free of artificial viscosity, because, for example, when considering the equation of continuity of components, variations in their specific mass fractions at any point in space are related only to physically meaningful processes, namely, to the inflow of “new” particles (with “new” specific mass fractions of the components). The second method includes constructing spline functions along the space and time coordinates of the computational mesh subject
to the fulfillment of differential equations at its nodes. The use of splines of high orders of approximation improves the accuracy of modeling. In addition, the spline schemes are fully conservative, which implies the possibility of satisfying not only the major integral conservation laws, but also all kinds of physically meaningful consequences of the system (for example, the laws of conservation of kinetic energy, internal energy, enthalpy etc.). Thus, the spline schemes enable producing credible solutions on difference schemes with a coarse space step, which makes them suitable for high-accuracy real-time modeling of multi-component gas mixture transport processes in complex-geometry branched pipeline systems. The approach used in constructing the spline schemes can be extended to the case of numerical modeling of a system of continuum mechanics equations in two- and three-dimensional settings.

Keywords: Lagrangian particle method, spline scheme, computer gas dynamics simulator, multi-component gas mixture, trunkline network.

1 Introductory Notes

Publications [1–6] propose industry-oriented approaches to numerical modeling of operating conditions of complex trunkline networks that transport process gases, liquids and multiphase fluids. These publications repeatedly pointed at the necessity of improving the accuracy of simulations using the mathematical models proposed, first of all, to enable credible simulation-aided commercial accounting of gas supply to consumers. As we know, one of the central accuracy improvement prerequisites is a science-based choice of appropriate techniques for the numerical analysis of the mathematical models developed. The present paper discusses current versions of two efficient techniques of numerical analysis of mathematical gas flow models. They have demonstrated their efficiency in practice as applied to commercial natural gas accounting and detection of discrepancy origins in estimated gas supply volumes of gas distribution networks of Gazprom’s subsidiaries.

2 Description of the first numerical analysis method for mathematical models of gas flow in trunkline networks

An essential part in solving the mathematical problem of commercial gas accounting is numerical monitoring of the share of gas distribution network suppliers in gas supply to specific consumers. This problem gains in complexity under the deficiency of field measurements of gas supply volumes and in the presence of network ring collectors operating under non-steady-state non-isothermal conditions.
Let natural gas, which in the first approximation can be treated as a single-component gas with known physical and mechanical properties, be transported through a network of branched pipelines with absolutely stiff rough heat conducting walls.

To solve the problem, we use the following computational approach: we conventionally assume that each supplier (in our case, “supplier” refers to a dispatching control station) introduces a unique grade of a single-component chemically inert gas, the properties of which are exactly the same as those of natural gas, to the common collector. In this case, as a result of non-isothermal mixing in the common collector and downstream pipelines, a multi-grade homogeneous chemically inert gas mixture will form, possessing physical and mechanical properties of the initially transported natural gas. Its grade (or component) composition will vary with time only due to corresponding variations in the gas supply conditions and parameters of gas consumption from the given pipeline system.

Based on the time variations of the gas mixture composition at the outlet boundary of the downstream pipeline associated with a certain consumer, one can conclude, which particular suppliers influence the volume of gas supply to the given consumer. Constructing the target functions of gas grade shares versus time reduces to numerical modeling of non-steady-state non-isothermal homogeneous multi-component time-variant-composition gas mixture transport through the given gas distribution network of long branched pipelines. Here, considering the natural gas flow velocities in common collectors and downstream pipelines, one can ignore the influence of diffusion and heat conductivity in the gas flow direction without loss of accuracy of simulation results [7].

One of the most promising approaches to credible numerical evaluation of space-time distribution of physical flow parameters in pipeline systems involves using the high-accuracy computer gas dynamic simulator to model the performance of the system of interest (GDS) [1, 7, 8]. The gas dynamics model of non-steady-state non-isothermal turbulent flow (without diffusion and heat conductivity along the gas flow) of a multi-component homogeneous mixture of viscous chemically inert compressible gases through a branched system of long varied round cross-section graded pipelines with absolutely stiff rough heat conducting walls, as implemented in the computational core of GDS, can be represented as follows:

for each pipe (bend or non-branched segment of the common collector):

\[
\frac{\partial}{\partial t} (\rho f) + \frac{\partial}{\partial x} (\rho wf) = 0; \\
\frac{\partial}{\partial t} (\rho Y_m f) + \frac{\partial}{\partial x} (\rho Y_m wf) = 0, \quad m = 1, N_s - 1, \quad Y_{N_s} = 1 - \sum_{m=1}^{N_s-1} Y_m; \\
\frac{\partial}{\partial t} (\rho w^2 f) + \frac{\partial}{\partial x} \left[ \frac{\partial p}{\partial x} + g \rho \frac{\partial z_1}{\partial x} \right] = -f \left[ \frac{\partial p}{\partial x} + g \rho \frac{\partial z_1}{\partial x} \right]_{x} \pi \frac{\lambda w}{w |R|};
\]
\[
\frac{\partial}{\partial t} \left[ \rho f \left( h + \frac{w^2}{2} \right) \right] + \frac{\partial}{\partial x} \left[ \rho w f \left( h + \frac{w^2}{2} \right) \right] = f \frac{\partial p}{\partial t} - \rho w f g \frac{\partial z}{\partial x} + Q f - \Phi (t, T_{\text{am}}); \tag{4}
\]

for each junction (boundary sections of pipelines adjoining the junction):

\[
Y_{m, \text{Joint}} = \sum_{n \in \Gamma_{\text{IN}}} \left[ (n) \rho \right] \left[ (n) w \right] \left[ (n) f \right] \left[ \sum_{n \in \Gamma_{\text{IN}}} \left[ (n) \rho \right] \left[ (n) w \right] \left[ (n) f \right] \right]^{-1} , \quad m = 1, N_S - 1; \quad Y_{S} = 1 - \sum_{n=1}^{N_S-1} Y_{m}; \tag{5}
\]

\[
h_{\text{Joint}} = \sum_{n \in \Gamma_{\text{IN}}} \left[ (n) \rho \right] \left[ (n) w \right] \left[ (n) f \right] \left[ \sum_{n \in \Gamma_{\text{IN}}} \left[ (n) \rho \right] \left[ (n) w \right] \left[ (n) f \right] \right]^{-1} ; \tag{6}
\]

\[
n \in \begin{cases} Y_{\text{IN}} \in \overline{1, M}, & \text{if} \quad (n) \left[ (n) w \right] S \geq 0; \\ Y_{\text{OUT}} \in \overline{1, M}, & \text{if} \quad (n) \left[ (n) w \right] S < 0; \end{cases} \tag{7}
\]

\[
T_{\text{Joint}} = T \left( p_{\text{Joint}}, h_{\text{Joint}}, \left\{ Y_{m, \text{Joint}} \left| m = 1, N_S \right. \right\} \right); \tag{8}
\]

\[
\rho_{\text{Joint}} = \rho \left( p_{\text{Joint}}, T_{\text{Joint}}, \left\{ Y_{m, \text{Joint}} \left| m = 1, N_S \right. \right\} \right); \tag{9}
\]

\[
(n) p = p_{\text{Joint}}, \quad n = 1, M; \quad \rho_{\text{Joint}}, \quad n \in Y_{\text{OUT}} ; \quad (n) T = T_{\text{Joint}}, \quad n \in Y_{\text{OUT}} ; \quad (n) h = h_{\text{Joint}}, \quad n \in Y_{\text{OUT}} ; \quad \left\{ (n) Y_m \right\} = Y_{m, \text{Joint}}, \quad n \in Y_{\text{OUT}}, \quad m = 1, N_S ; \tag{10}
\]

\[
\sum_{n \in \Gamma_{\text{IN}}} \left[ (n) \rho \right] \left[ (n) w \right] \left[ (n) f \right] = \sum_{n \in \Gamma_{\text{OUT}}} \left[ (n) \rho \right] \left[ (n) w \right] \left[ (n) f \right]; \tag{11}
\]

\[
(n)_S = \begin{cases} 1, & \text{if} \ n^{th} \text{pipeline adjoins joining knot the right section;} \\ -1, & \text{if} \ n^{th} \text{pipeline adjoins joining knot the left section;} \end{cases} \tag{12}
\]

for each junction (boundary sections of pipelines adjoining the junction):

\[
(n) p = p_{\text{Joint}}, \quad n = 1, M; \quad \rho_{\text{Joint}}, \quad n \in Y_{\text{OUT}} ; \quad (n) T = T_{\text{Joint}}, \quad n \in Y_{\text{OUT}} ; \quad (n) h = h_{\text{Joint}}, \quad n \in Y_{\text{OUT}} ; \quad \left\{ (n) Y_m \right\} = Y_{m, \text{Joint}}, \quad n \in Y_{\text{OUT}}, \quad m = 1, N_S ; \tag{13}
\]

\[
\sum_{n \in \Gamma_{\text{IN}}} \left[ (n) \rho \right] \left[ (n) w \right] \left[ (n) f \right] = \sum_{n \in \Gamma_{\text{OUT}}} \left[ (n) \rho \right] \left[ (n) w \right] \left[ (n) f \right]; \tag{14}
\]

where \( \rho \) is the gas mixture density; \( f \) is the pipeline flow section area; \( t \) is the time (marching variable); \( x \) is the spatial coordinate along the pipeline axis; \( w \) is the projection of the vector of gas velocity averaged over the pipeline cross section to the pipeline axis (on the assumption of developed flow turbulence); \( Y_m = \rho_m / \rho \) is the mass fraction of the \( m \)-th component; \( \rho_m \) is the reduced density of the \( m \)-th component (mass of the \( m \)-th component in unit volume of
the mixture); \( N_g \) is the number of mixture components; \( p \) is the static gas mixture pressure; \( g \) is the gravity acceleration modulus; \( z_i \) is the spatial coordinate of the point in the pipeline axis reckoned from an arbitrary horizontal plane vertically upward (for trunklines, along the Earth radius); \( \pi \) is the Pythagorean number; \( \lambda \) is the hydraulic friction coefficient in the Darcy–Weisbach formula; \( R = \sqrt{\rho/\pi} \) is the inner pipe radius; \( h \) is specific (per unit mass) enthalpy of the mixture; \( Q \) is specific (per unit volume) power of heat sources; \( T \) is the gas mixture temperature; \( M \) is the number of pipes constituting the given junction; \( \gamma_{IN} \) is the subset of upstream pipes of the given junction (upstream pipes with respect to the junction are the pipes, through which the gas enters the junction); \( \gamma_{OUT} \) is the subset of downstream pipes of the given junction (downstream pipes with respect to the junction are the pipes, through which the gas leaves the junction); \( p_{\text{joint}} \) is the static gas mixture pressure in the given pipeline junction; \( T_{\text{joint}} \) is the gas mixture temperature in the pipeline junction (i.e. in the inner space of the junction); \( \gamma_{m,\text{joint}} \) is the mass fraction of the \( m \)-th gas mixture component in the junction; \( h_{\text{joint}} \) is the specific (per unit mass) enthalpy of the gas mixture in the junction of interest. The function \( \Phi(T, T_{am}) \) characterizes the heat exchange of the gas flow core through the boundary gas layer, pipe wall and insulation with the environment. It expresses the total specific (per unit length) thermal flux along the perimeter \( \chi \) of the cross section having an area of \( f \) from the transported gas to the environment (\( \Phi(T, T_{am}) > 0 \) means heat removal; \( T_{am} \) is the space-time distribution of ambient temperature at the domain boundary). To indicate that a quantity belongs to the \( n \)-th pipe, we use a superscript in parentheses left of the quantity, for example: \( ^{(n)} \rho \). The system of equations (1–14) is supplemented with boundary conditions and conjugation conditions. As conjugation conditions we can define boundary conditions simulating a complete rupture of the pipeline and/or its shutoff, operation of valves, etc.

To numerically solve the system of equations (1–14), the computational core of GDS usually employs grid methods. Unfortunately, distribution trunkline networks contain a large number of pipe joints distributed on general collectors extremely nonuniformly. In our case, this results in the necessity of considerable spatial mesh refinement and, consequently, in a much longer runtime. Such an increase in the runtime is intolerable in simulations done to provide computer-aided support of pipeline dispatcher decisions [10, 11].

To overcome this situation, the computational core of GDS employs a hybrid modification of the known integro-interpolation method developed by A. N. Tikhonov and A.A. Samarsky [12–14] and S.N. Pryalov’s Lagrangian particle method [7, 15, 16]. The Lagrangian particle method is applied only to the equations of component continuity (2, 5, 7) and energy equations (4, 6, 7). Note
that this Lagrangian particle method is essentially a dedicated modification of the
approach for solving hyperbolic partial differential equations by the well-known
method of characteristics [17-19]. Let us consider the algorithm of applying this
method to the component continuity equations.

As we know, in a non-steady-state flow of a gas mixture through a branched
network system, regions (moving together with the gas flow) can occur, where
concentrations of the mixture components can differ significantly from those in
neighbor regions. The use of difference schemes may lead to substantial front
“smearing”. Given that the difference schemes [12] are conservative, the total
mass of the mixture components in the pipeline network will not vary with time.
However, the values of component concentrations can decrease (or increase)
unreasonably, and as a result, the fluid of these materials will get non-physically
“smeared” along the length of the pipeline. The idea of the Lagrangian particle
method in this case rests upon the known fact that if we separate a small particle
of matter (a small moving volume of matter), the continuity equations of the
components will actually describe the motion of this particle (with some set of
components) with the gas flow. Accordingly, it is suggested that the component
continuity equation be solved by analyzing the motion of some set of particles in
the gas flow. Flow parameters in this case can be defined in any manner enabling
simulation of a single-component gas flow (for example, by finite differences).
Owing to the way of introducing these particles, they are conventionally called
Lagrangian.

The general algorithm of flow modeling without multicomponent gas mixture
diffusion in branched pipeline systems using the Lagrangian particle method can
be described as follows (S.N. Pryalov’s algorithm):

1. Initially, along the length of every \( n \)-th pipeline \( (n = 1, M) \) we “place” a
   set of Lagrangian particles ordered by increasing coordinates (with numbering
   \( \hat{p} = 1 \), \( \text{base} N_{Lagr} \)) in the amount of \( \text{base} N_{Lagr} \) at a distance on the order of the
   length of a spatial mesh cell. A mandatory requirement is that the Lagrangian
   particles with numbers \( \hat{p} = 1 \) and \( \hat{p} = \text{base} N_{Lagr} \) lie on the left (coordinate \( x = 0 \))
   and on the right (coordinate \( x = \text{Length} \)), where \( \text{Length} \) is the length of the
   \( n \)-th pipeline) pipeline boundaries, respectively. Specific fractions of the
   components for the particles are defined by approximation of initial conditions.
   Let \( x_{(p)}^{(n)} \) be the coordinate of the \( \hat{p} \)-th Lagrangian particle belonging to the
   \( n \)-th pipeline for the \( j \)-th time step.

2. Gas dynamic variables of the gas mixture flow at the next time step \( t_{j+1} \) are
   calculated (using difference equations approximating the gas dynamics equations
   (1–14), except for the component continuity equations (2, 5, 7) and energy
   equations (4, 6, 7)). The values of the specific component fractions at mesh nodes
   are defined by interpolation between these values for the Lagrangian particles
   adjacent to the mesh node.
3. For each $p$-th Lagrangian particle of each $n$-th pipeline, we update the coordinate for the new time step $t_{j+1}$ using the formula:

$$
(n)_{\rho} x'_{j+1} = (n)_{\rho} x' + (n)_{\rho} \Delta x'_{j+1} = (n)_{\rho} x' + (n)_{\rho} \left\{\left(\frac{n}{\rho}\right) x', t_j\right\} \Delta t_{j+1},
$$

(15)

4. For each pipeline, we delete the Lagrangian particles that leave the pipeline. At the same time, a new particle is generated at the corresponding “outlet” boundary (the boundary, through which the gas leaves the pipe) with specific component fractions $\{Y_m \mid m = 1, N_s\}$ equal to the interpolated values of the particles closest to the boundary: inside the pipe and outside the pipe (the latter particle is deleted with respect to the pipe).

5. For each pipeline junction, specific component fractions are calculated by the following formula:

$$
(Y_m)_{j+1}^{(n)} = \sum_{n=1}^{(n)} \left(\frac{n}{L} (Y_m) \rho_L^{(n)} | w_L^{(n)} | f_L^{(n)}\right) \left[\sum_{n=1}^{(n)} \left(\rho_L^{(n)} | w_L^{(n)} | f_L^{(n)}\right)^{-1}, m = 1, N_s\right],
$$

(16)

where $(Y_m)_{j+1}^{(n)}$ is the specific mass fraction of the $m$-th component of the particle located at the outlet boundary of the upstream pipeline; $\rho_L$, $w_L$ and $f_L$ are the gas mixture density, velocity and cross section area corresponding to the outlet boundary of the upstream pipeline.

6. For each inlet boundary of each pipeline, a new Lagrangian particle is generated with specific component fractions $\{Y_m \mid m = 1, N_s\}$ corresponding to the boundary conditions (if the boundary is inlet for the given pipeline system) or equal to the specific component fractions in the pipeline junction (if the boundary adjoins the junction). At the next time step, if the distance between this particle and the following (downstream) particle is smaller than the given distance (on the order of the spatial cell length), this particle is deleted.

7. If $t_{j+1}$ reaches the limiting value, the simulation is completed. Otherwise, we assume that $j \leftarrow j + 1$ and proceed to step 2.

Now, let us consider the application of the Lagrangian particle method to the energy equations (4, 6, 7). As applied to this case, its essentials can be summarized as follows. We imagine Lagrangian particles distributed along the pipe length. They are assumed to be weightless. This allows the Lagrangian particles to move together with matter. Due to its small size, each particle can instantaneously acquire the temperature of its environment. Thus, by monitoring the motion of the Lagrangian particles together with matter and their temperature, one can analyze the process of heat transport in the pipelines.

One should note at this point that in order to make the paper content more illustrative, the term “Lagrangian particles” as applied to the energy equations was interpreted differently than as applied to the component continuity equations. However, the mathematical essence of the Lagrangian particle method remains unchanged.
Without loss of generality and avoiding bulky manipulations, let us provide a more detailed description of the mentioned application of the Lagrangian particle method for the case of an energy equation for a single-component gas transported in a constant-section non-branched pipeline. This equation can be easily derived by simplifying Eq. (4) accounting for other equations of the system (1–14). The simplified equation will have the following form:

\[
\frac{\partial h}{\partial t} + w \frac{\partial h}{\partial x} = \frac{w}{\rho} \frac{\partial p}{\partial t} + \frac{1}{\rho} \frac{\partial p}{\partial t} + \frac{\lambda}{4R} \frac{\Phi(T,T_{am})}{f \rho}
\]

or

\[
\frac{ Dh }{ Dt } = G[x, t, T(h)],
\]

where

\[
G(x, t, T) = \frac{1}{\rho} \frac{Dp}{Dt} + \frac{\lambda}{4R} \frac{\Phi(T,T_{am})}{f \rho};
\]

\[
\frac{D\xi}{Dt} = \frac{\partial \xi}{\partial t} + w \frac{\partial \xi}{\partial x}
\]

is a derivative of the arbitrary function \( \xi \) over \( t \) in the direction

\[
\frac{ds}{dt} = w(x,t).
\]

This direction is called characteristic, and the equation is called the characteristics direction equation. Eq. (18) is called the characteristic form of Eq. (17) or characteristics-based differential relationship.

In terms of physics, the derivative \( Dh/Dt \) corresponds to the material derivative, and the solution of Eq. (20) defines the coordinate of a continuum particle (in our case, the spatial coordinate of the fluid flow cross section) at each time point.

Considering the known thermodynamic relationship

\[
dh = c_p dT - \mu c_p dp,
\]

where \( c_p \) is heat capacity at constant pressure, and \( \mu \) is the Joule–Thomson coefficient, Eq. (18) can be transformed into

\[
\frac{DT}{Dt} = \left( \mu + \frac{1}{\rho c_p} \right) \frac{Dp}{Dt} + \frac{\lambda}{4Rc_p} \frac{\Phi(T,T_{am})}{f \rho c_p}.
\]

Eq. (22) is satisfied along each characteristic curve (20). These curves in fact describe the motion trajectory of matter particles. In other words, these equations describe the temperature variation of matter for each cross section of the transported fluid flow.

When implementing the Lagrangian particle method, the flow variables (such as pressure and velocity) are found using a difference scheme, and the gas temperature distribution, by analyzing the motion of Lagrangian particles. In this case, Eq. (22) is solved for each particle. The form of this equation enables such calculations, because it corresponds to the temporal variation of the fluid temperature in each cross section of the flow. In the present problem statement
(i.e. in the Lagrangian problem statement with respect to each particle), Eq. (22) transforms into an ordinary differential equation with respect to the marching variable:

$$\frac{dT}{dt} - \left[ \mu + \left( \rho \epsilon \right) \right] \frac{dp}{dt} = \frac{\lambda |w|^3}{4R_{c_p}} - \Phi(T, T_{am}) / f \rho c_p. \tag{23}$$

Numerical analysis of the resulting equation can be performed by different methods for solving ordinary differential equations, such as the widely-known Runge–Kutta method with controlled accuracy [17, 19].

As initial temperatures of the Lagrange particles we use corresponding values from the given boundary conditions (i.e. the temperature of each Lagrangian particle is assumed to be equal to the temperature of matter at the location of the particle).

As far as the particles move with matter toward the pipe's outlet boundary, new Lagrangian particles need to be introduced at the pipe’s inlet boundary at some intervals. The value of the initial temperature of each introduced particle should be defined based on the boundary conditions at the pipe’s inlet boundary. The Lagrangian particles that leave the pipe are deleted. As for the inlet boundaries of downstream pipelines of each junction, temperature values of introduced particles need to be defined in accordance with Eqs. (8) and (10).

Since the Lagrangian particle method for the energy equation is not related directly to the finite difference mesh employed for solving the continuity and motion equations, this mesh has almost no effect on the accuracy of the proposed method. Thus, high-accuracy calculated values of gas temperature are obtained without mesh refinement, which speeds up the calculations significantly.

In addition, due to the absence of direct connection between the Lagrangian particle method and the finite difference mesh, this method is free of the so-called artificial viscosity (see above and [12–14]). As a result, the method makes it possible to obtain solutions without artificial smoothing of temperature fronts, which corresponds to real physical processes. This significantly increases the credibility of simulations compared to the use of difference schemes for the energy equation.

3 Description of the second numerical analysis method for mathematical models of gas flow in trunkline networks

To improve the credibility of numerical modeling of trunkline network operation, it is reasonable to use conservative difference schemes (see, e.g., [14]). This property is inherent in the schemes, in which the integral notation of the major conservation laws is satisfied for each computational cell. Such schemes prevent the formation of purely difference-related fictitious sources of mass, momentum and energy, which in dynamic processes can commensurate with the physically justified terms. As a next stage in enhancing the credibility of modeling one can consider the use of fully conservative difference schemes.
These schemes additionally provide a balance between each type of energy preventing the physically invalid (artificial) "conversion" of one type of energy into another (for example, internal energy into kinetic and vice versa).

This section of the paper considers an approach that uses fully conservative spline schemes for solving a complete system of gas dynamics equations for the case of a model describing the flow of a single-component gas in a trunkline. The consequence of (1–14) for an unbranched trunkline, considering the adopted assumptions, is the system of equations [7]:

\[
\rho \frac{\partial \rho}{\partial t} + f \frac{\partial \rho}{\partial x} + \rho w \frac{\partial f}{\partial x} + \rho f \frac{\partial w}{\partial x} = 0; \\
\rho f \frac{\partial w}{\partial t} + \rho w f \frac{\partial w}{\partial x} = -f \left[ \frac{\partial p}{\partial x} + g \rho \frac{\partial \varepsilon_z}{\partial x} \right] - \frac{\pi}{4} \lambda \rho w | w | R; \\
\rho f \frac{\partial \varepsilon}{\partial t} + \rho w f \frac{\partial \varepsilon}{\partial x} = -p \frac{\partial (\rho w)}{\partial x} - p \frac{\partial f}{\partial t} + \frac{\pi}{4} \lambda \rho w | w | R + Qf - \Phi(T, T_{num}); \\
p = p(\rho, T), \\
\varepsilon = \varepsilon(h - p/\rho)
\]

where \( \varepsilon = h - p/\rho \) is the specific (per unit mass) internal energy.

As we know, for systems of nonlinear partial differential equations (to which the system (24-28) belongs), in the general case, it is impossible to obtain an analytical solution. There exist various methods for constructing difference counterparts of original differential equations that enable obtaining difference equations possessing different properties. To produce a credible solution, the difference counterpart (the difference scheme) should be convergent [13, 14]. This property holds true if the difference scheme possesses approximation and robustness. In addition, the scheme should preferably be conservative and have an increased order of approximation (these properties allow the scheme to produce more accurate numerical solutions).

This section of the paper proposes employing spline schemes for solving a complete system of gas dynamics differential equations [7, 20, 21]. This method involves searching for target values of the gas dynamic parameters at the nodes of the space-time mesh such that the gas dynamic differential equations at the difference mesh nodes are satisfied, when approximating the distributions of these parameters by splines (along the space and time coordinates).

This method possesses an increased order of approximation; it is proven to be fully conservative, which provides higher credibility of process modeling when analyzing the parameters of gas transport in trunklines. On the other hand, spline schemes are implicit, which makes them more robust compared to explicit and semi-implicit schemes.

Let us begin considering this type of schemes from the simplest case, when splines are used to solve ordinary differential equations (see, e.g., [22]).

Suppose we should solve the following equation in the range of \( x_L \leq x \leq x_R \):
\[
\frac{dy}{dx} = p\left(x, y(x)\right),
\]
where \( p\left(x, y(x)\right) \) is a given function. Integration of Eq. (29) gives a constant defined by means of a given boundary condition. For definiteness, but without loss of generality, we assume that a first-kind boundary condition is specified on the left boundary \( (x_L) \):
\[
y(x_L) = Y_L,
\]
where \( Y_L \) is a given constant.

We construct a nonuniform finite-difference mesh \( \Sigma_h = \{x_i\} \), where \( x_i \) is the coordinate of the \( i \)-th mesh node, \( i = 0, N - 1 \); \( N \) is the number of nodes of the finite-difference mesh; and \( h_i = x_i - x_{i-1} \) is a fixed step backward for the \( i \)-th node, \( i = 1, N - 1 \) (Fig. 1).

![Fig. 1 Fragment of the finite-difference mesh](image)

To solve Eq. (29) numerically, we find the values of the parameter \( y \) at the nodes of the difference scheme, such that the following equation is satisfied for the spline constructed based on these values:
\[
\left( \frac{dy}{dx} \right)_i = p\left(x_i, y(x_i)\right) \quad \text{(31)}
\]

To indicate this approximation of the derivative \( dy/dx \), we introduce the notation
\[
x^\Delta_{\{\text{Spline}, <...>\}}(y),
\]
where \( \{\text{Spline}, <...>\} \) means that a spline interpolation is used, and the number substituted for <...> corresponds to the order of the spline. The notation \( (\text{DER} = <...>) \) corresponds to the order of the derivative (in our case, it is equal to 1). Considering (32), Eq. (31) can be rewritten as
\[
x^\Delta_{\{\text{Spline}, <...>\}}(y_i) = p\left(x_i, y(x_i)\right) \quad \text{(33a)}
\]
or, in short,
\[
x^\Delta_{\{\text{DER}=1\}}(y) = p\left[x, y(x)\right]. \quad \text{(33b)}
\]

To construct a spline, we must find the coefficients of polynomials for each mesh cell \( \Delta x_{i-0.5} = [x_{i-1}, x_i] \) assuming a number of conditions that will be discussed below.
Let the order of spline polynomials be $K \geq 2$. The latter condition is required to ensure that the first derivative of $y(x)$ is continuous. Then, to each mesh cell $\Delta x_{i-0.5}$, $i = 1, N - 1$ there corresponds the polynomial

$$y_{i-0.5}(x) = \sum_{k=0}^{K} a_{k, i-0.5} x^k,$$

where $a_{k, i-0.5}$, $k = 0, K$, $i = 1, N - 1$ is a set of spline constants. In our case, the spline constants are found from the condition that Eq. (30) should be satisfied at the mesh nodes and that the spline should be smooth:

$$y^{(1)}_{i-0.5}(x_{i-1}) = p\left[ x_{i-1}, y_{i-0.5}(x_{i-1}) \right], \quad y^{(1)}_{i-0.5}(x_i) = p\left[ x_i, y_{i-0.5}(x_i) \right], \quad i = 1, N - 1; \tag{35}$$

$$y^{(m)}_{i-0.5}(x_i) = y^{(m)}_{i+0.5}(x_i), \quad m = 0, 2, 3, ..., K - 1, \quad i = 1, N - 2, \tag{36}$$

where $y^{(m)}_{i-0.5}(x)$ is the $m$-th derivative of the function $y_{i-0.5}(x)$.

The system of equations (35, 36) has $2(N - 1)(K - 1)(N - 2)$ equations per $[N - 1](K + 1)$ variables ($N - 1$ is the number of mesh cells; $(K + 1)$ is the number of constants in each polynomial). The difference between these quantities is $(K - 1)$, i.e. for the spline to be constructed uniquely, the system (35, 36) must be additionally closed by the $(K - 1)$-th equation.

As applied to the problem of interest, one of the closing equations will be Eq. (30):

$$y_{0.5}(x_0) = Y_0.$$

To obtain the remaining $(K - 2)$ relations, one can use the so-called neutrality conditions, or the higher-order derivatives for the system’s boundary points being equal to zero (see, e.g., [7]).

We next analyze the conservatism of the spline scheme constructed. As we know [14], the difference schemes that express the principal conservation laws on a mesh are called conservative difference schemes. In other words, the principal conservation laws (in our case, Eq. (29)) must be satisfied for each mesh cell. This implies that approximation of the quantities under total derivatives on common boundaries must be the same (in our case, approximation of $y(x)$), i.e. for a difference scheme to be conservative, divergent derivatives must be approximated by divergent difference counterparts.

Otherwise, there will be a “nonphysical” jump in the differentiated function (in our case, the parameter $y$) when passing from one cell to another. This jump can be considered a fictitious source (sink) of the function on the common boundary between adjacent cells. In fact, balance is violated in the scheme, namely, the quantity under consideration ($y$) varies not only due to the presence of the physically justified term ($p$), but also due to the balance-violating summand.
To consider conservatism of the method proposed (the spline scheme), we next pass to the difference counterpart of the spline scheme and explore this issue as applied to it.

**Remark.** Let us show that there exists a partition \( \{ \bar{x}_{i+0.5}(y) \} \) of the original integration domain (here, the nodes \( \bar{x}_{i+0.5} \) do not necessarily lie at the centers of the cells \( h_i \) ) and a corresponding piecewise-linear approximation of the function \( y(x) \) in the form \( \{ \bar{y}_{i+0.5} \} \), such that

\[
\frac{\bar{y}_{i+0.5} - \bar{y}_{i-0.5}}{\bar{x}_{i+0.5} - \bar{x}_{i-0.5}} = \left( \frac{dy}{dx} \right)_{\text{Spline, } < \ldots >} = \frac{\Delta x}{(DER-1)} (y_i). \tag{38}
\]

For this, it suffices to draw tangents to the numerical solution at the mesh nodes (Fig. 2). The points of their intersection are then just the required partition \( \{ \bar{x}_{i+0.5}(y) \} \).

![Diagram](image-url)

**Fig. 2 Example of partition \( \{ \bar{x}_{i+0.5}(y) \} \) of the integration domain for the function \( y(x) \).**

The difference ratio (38) is divergent by construction, because the values of the function \( \bar{y} \) on the common boundaries of adjacent cells coincide. Replacing the
left-hand side in Eq. (33) with (38), we obtain the conservative difference equation
\[
\frac{\bar{Y}_{i+0.5} - \bar{Y}_{i-0.5}}{\bar{x}_{i+0.5} - \bar{x}_{i-0.5}} = p_i.
\] (39)

Because the values of both summands in Eq. (39) for each segment \([\bar{x}_{i-0.5}, \bar{x}_{i+0.5}]\) are absolutely the same as respective summands in Eq. (33), one can consider the resulting \textit{spline scheme conservative in the sense that there exists its conservative discrete counterpart.}

Based on the proof of the above remark, one can draw the following conclusion. \textit{If some parametric dependence (not necessarily a spline; it can be, e.g., a polynomial of the order \((N + 1)\) or a Fourier series with a finite number of terms, etc.) is used for solving a differential equation, and its parameters are chosen such that the differential equation is satisfied at mesh nodes, then ensuring that the chosen approximation is conservative requires the functions under the derivative sign to be continuous and differentiable.}

From the standpoint of the classical notation of conservative difference schemes, the fact that a spline scheme is conservative can be interpreted as follows. A spline scheme is conservative because, for an equivalent discrete counterpart, the chosen conservation law is satisfied for the cells \([\bar{x}_{i-0.5}(y), \bar{x}_{i+0.5}(y)]\).

However, it should be noted that in the presence of several divergent differential derivatives in the original conservation law, the requirement that the spline scheme should be conservative contradicts the classical notion of conservative difference schemes. Let us consider this in more detail for the case of the equation
\[
\frac{dy}{dx} + \frac{d}{dx}g(y) = p[x, y(x)].
\] (40)

The auxiliary partition cells, on which the divergent difference counterparts of the derivatives of interest \([\bar{x}_{i-0.5}(y), \bar{x}_{i+0.5}(y)]\) and \([\bar{x}_{i-0.5}(g), \bar{x}_{i+0.5}(g)]\) are constructed, are different. Therefore (unlike the example above), in the general case, there does not exist a cell containing a mesh node such that Eq. (40), valid for this node, could be considered satisfied. That is, the classical definition of the notion of conservatism is not satisfied. However, as has been shown before, when using a spline scheme, there are no balance-violating terms of a purely difference nature. For this reason, the method proposed can be considered conservative.

Let us discuss application of a spline scheme for solving the system (24–28). Let the spline order be \(K_t\) with respect to the time coordinate and \(K_x\) with respect to the spatial coordinate. The spline scheme for solving (1) can then be represented in the form proposed by S.N. Pryalov [7]:
The spline scheme (41) approximates a nondivergent form of mass, momentum and energy conservation laws (24–28). The differential forms of these laws are satisfied at the nodes of the space-time mesh. Accordingly, the divergent forms of the principal conservation laws are also satisfied at the mesh nodes. If the spline order ensures the continuity and differentiability of the functions under the derivative sign in (24–28) (which requires that the conditions $K_t \geq 2$, $K_x \geq 2$ hold), then there exists a divergent discrete counterpart for each divergent derivative. Consequently, the spline scheme is also conservative for systems of partial differential equations.

As the conservation laws are satisfied in the differential form at the mesh nodes, it follows that all possible representations of a given system are satisfied at these nodes (for example, equations for kinetic energy, total energy, entropy etc.). The existence of divergent discrete counterparts of all the divergent differential derivatives in this case ensures that the spline scheme is not only conservative but completely conservative [12–14].

This conclusion is extended similarly to the continuum mechanics equations of any dimensionality. Thus, we show that a spline scheme is completely conservative. This enables modeling gas and hydrodynamics processes with higher credibility and accuracy due to the correct modeling of all possible conservation laws. On the other hand, spline schemes are implicit, which makes them more robust compared to explicit and semi-implicit schemes [12–14]. Increasing the order of the splines used makes it possible to increase the order of approximation of gas- and hydrodynamic equations, which also improves the accuracy of simulations.

4 Examples of production simulations

Below we illustrate production simulations of non-steady-state non-isothermal
analysis of the effects of dispatching control stations of the Moscow circular gas trunkline on gas supply through each branch of the common ring collector connected to a specific group of natural gas consumers by means of the trunkline (Fig. 3).

![Image](image.png)

**Fig. 3. Example of results of numerical estimation of the share of dispatching control station 10 in non-steady-state non-isothermal gas supply to a specific consumer connected to the Moscow circular gas trunkline with reference to the map (time slice)**

The simulation was done using the AlfaRus/Mosregiongaz computer analytical system (CAS) by dispatchers of Gazprom Mezhregiongaz Moscow LLC supported by the CAS developers from Physical and Technical Center LLC. Different colors in Fig. 4 show shares of gas supply from the dispatching control station under consideration. The influence of the dispatching control station on a specific group of consumers is denoted qualitatively by the color of the associated branch, and quantitatively, by the representation of simulation results in the form of tables indicating gas proportions (in percent) consumed by a specific group of consumers from each dispatching control station of the Moscow circular gas trunkline.
5 Conclusion

The above numerical analysis methods for mathematical models of commercial gas transport through trunklines represent essentially a step towards solving the pressing task of commercial accounting of gas supply through distribution network. With minor modifications, they can be used for the numerical analysis of pipeline networks at power engineering facilities, which transport commercial gases and water vapor.

In 2008–2014 these methods demonstrated their efficiency as applied to production simulations done to validate discrepancy mechanisms in natural gas supply through the Moscow circular gas trunkline. Application of these methods in practice does not require any special high-performance computers.

References


Received: February 19, 2014