Statement of the Problem of Numerical Modelling of Finite Deformations

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Abstract

In operation the fundamentals of a technique of numerical research of finite strains of the isotropic hyperelastic bodies, oriented on application FEM are stated. The first section is devoted kinematics of finite strains in the Lagrangian frame, tensors and measures of the strain, defining are entered speeds, various tensors of stress are reduced. In the second section physical parities of the hyperelastic isotropic environment are formulated, using the thermodynamics equations. In the third section it is reduced resolving the equation in a current configuration and the parities defining speed of change of a stress tensor of Cauchy-Euler as linear function from a tensor of a spatial gradient of speed are output.

Keywords: a method of finite elements, finite strains, metric tensor.

Introduction

Modeling stress-strain conditions of difficult technical products from the elastic materials supposing considerable deformations, it is possible only with application of modern computing technologies on the basis of FEM. Variants of statement of
a problem and possible algorithms of the decision are presented in works [1-3]. They are based on fundamental results of nonlinear mechanics of the elastic environments stated in known monographies and manuals. As a rule, the Lagrange description of kinematics of environment and three-dimensional statement is used. As Lagrange coordinates in [2-3] are used Cartesian coordinates of material points in an initial configuration. In this case structural parities have the elementary appearance. However at finite element digitization for each finite element (FE) the local curvilinear system of coordinates as each material point of environment is unequivocally identified by values of these local coordinates and number FE is entered. Therefore it is represented expedient to formulate a problem in these coordinates, generally curvilinear and not orthogonal.

In the present work statement of a problem of numerical modeling of finite deformations of hyperelastic environments focused on application FEM and isoparametric approximation is given. In particular in the first section the basic parities of kinematics of finite deformations with representation of tensors in various bases are resulted. The second section is devoted construction of defining parities for hyperelastic isotropic environments with use spatial tensors. In the third section it is resulted linearize the equation on a step loading, and expressions for speed stress tensor as linearly depending from spatial gradient tensor of speed are under construction.

1. Kinematics of finite deformations

Deformation process we will consider in some inertial system of readout into which we will enter Cartesian system of coordinates with orts $\hat{e}_i, \hat{e}_j, \hat{e}_k$.

Let in an initial configuration radius – the vector of a material point looks like

$$\bar{R} = X^r(\xi^1, \xi^2, \xi^3)\hat{e}_i,$$

where - $\xi^1, \xi^2, \xi^3$ curvilinear Lagrangian coordinates.

Let’s define
- Basic vectors

$$\bar{R}_i = \frac{\partial X^r}{\partial \xi^i} \hat{e}_i = R'_i \hat{e}_i,$$

- Metric tensor

$$(G) = G_{ij} (\bar{R} \bar{R}') = G''_{ij} (\bar{R} \bar{R}') = G_{ij} (\hat{e}_i \hat{e}_j),$$

By analogy in an actual condition we will enter
- Radius – a vector of a material point

$$\bar{r} = X^r(\xi^1, \xi^2, \xi^3)\hat{e}_i,$$

- Basic vectors

$$\bar{r}_i = \frac{\partial X^r}{\partial \xi^i} \hat{e}_i = \hat{r}'_i \hat{e}_i,$$

-Metric tensor

$$(g) = g_{ij} (\bar{r} \bar{r}') = g''_{ij} (\bar{r} \bar{r}') = \sum_{i,j} \hat{g}_{ij} (\hat{e}_i \hat{e}_j),$$
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Further the vector of speed which $\dot{\upsilon}$ we will define in a following kind is required

$$\dot{\upsilon} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}} = \dot{\upsilon}^i \left( \xi^1, \xi^2, \xi^3 \right) \mathbf{e}_i,$$

Let’s enter into consideration group tensors, describing kinematics of environment at its deformation.

- Deformation gradient tensor

$$ (F) = \left( \mathbf{R} \mathbf{r} \right) = G_{ij} \left( \mathbf{R} \mathbf{r} \right) = g^{\upsilon i} \left( \mathbf{R} \mathbf{r} \right) = \sum_{i,j} F_{ij} \left( \mathbf{e}_i \mathbf{e}_j \right),$$

- Return deformation gradient tensor

$$ (F^{-1}) = \left( \mathbf{R} \mathbf{r} \right) = g_{ij} \left( \mathbf{R} \mathbf{r} \right) = G_{ij} \left( \mathbf{R} \mathbf{r} \right) = \sum_{i,j} F_{ij} \left( \mathbf{e}_i \mathbf{e}_j \right) = \frac{\partial X^i}{\partial x^j} \left( \mathbf{e}_i \mathbf{e}_j \right),$$

Further at the deformation description we will use

- Measure of deformation Cauchy – Green

$$ (C) = (F)^T \cdot (g) \cdot (F) = g_{ij} \left( \mathbf{R} \mathbf{R} \right) = \sum_{i,j} \tilde{g}_{ij} \left( \mathbf{e}_i \mathbf{e}_j \right),$$

- Measure of deformation Almansi

$$ (B^{-1}) = (F^{-1})^T \cdot (g) \cdot (F^{-1}) = G_{ij} \left( \mathbf{R} \mathbf{R} \right) = \sum_{i,j} \tilde{B}_{ij} \left( \mathbf{e}_i \mathbf{e}_j \right),$$

- Deformation tensor Cauchy – Green

$$ (E) = \frac{1}{2} \left[ (C) - (G) \right] = \frac{1}{2} \left[ G_{ij} \right] \left( \mathbf{R} \mathbf{R} \right) = \sum_{i,j} \tilde{E}_{ij} \left( \mathbf{e}_i \mathbf{e}_j \right),$$

- Deformation tensor Almansi

$$ (A) = \frac{1}{2} \left[ (g) - (B^{-1}) \right] = \frac{1}{2} \left[ g_{ij} \right] \left( \mathbf{R} \mathbf{R} \right) = \sum_{i,j} \tilde{A}_{ij} \left( \mathbf{e}_i \mathbf{e}_j \right),$$

(1)

The parity connecting tensors deformations Cauchy – Green and Almansi is fair

$$ (E) = (F)^T \cdot (A) \cdot (F),$$

(2)

Return transformation, that is

$$ (A) = (F^{-1})^T \cdot (E) \cdot (F^{-1}).$$

Now we will consider tensors, currents of environment used for the description. Basic tensors here are

- A spatial gradient tensor of speed

$$ (h) = (\dot{F}) \cdot (F^{-1}) = \left( \dot{\upsilon} \mathbf{r} \right) = \sum_{i,j} \tilde{h}_{ij} \left( \mathbf{e}_i \mathbf{e}_j \right),$$

(3)

where

$$ \tilde{\upsilon}_i = \frac{\partial \upsilon^i}{\partial \xi^j} \frac{\partial \xi^j}{\partial \mathbf{r}},$$

(4)

- Deformation tensor of speed

$$ (d) = \frac{1}{2} \left[ (h) + (h)^T \right] = \sum_{i,j} \tilde{d}_{ij} \left( \mathbf{e}_i \mathbf{e}_j \right),$$

(5)

- Speed tensor of rotation

$$ (\omega) = \frac{1}{2} \left[ (h) - (h)^T \right] = \sum_{i,j} \tilde{\omega}_{ij} \left( \mathbf{e}_i \mathbf{e}_j \right),$$
Let's enter into consideration material derivative (a full derivative on time) deformation tensor Cauchy – Green

\[
\left( \dot{E} \right) = \frac{1}{2} g_{ij} \left( \ddot{R}^i \ddot{R}^j \right) = \sum_{i,j} \dot{E}_{ij} \left( \ddot{e}_i \ddot{e}_j \right),
\]

(6)

From (4) follows

\[
(d) = \frac{1}{2} \sum_{\alpha} \left[ \frac{\partial v^{\alpha}}{\partial \xi^i} \dot{\xi}^j + \frac{\partial v^{\alpha}}{\partial \xi^j} \dot{\xi}^i \right] \left( \ddot{r}^i \ddot{r}^j \right) = \frac{1}{2} \sum_{i,j} \dot{g}_{ij} \left( \ddot{r}^i \ddot{r}^j \right).
\]

(7)

Thus, speed of change deformation tensor Cauchy – Green (6) and deformation tensor of speed (5) in curvilinear bases identical values have a component. Hence, fairly

\[
(d) = \left( F^{-1} \right)^T \cdot \left( \dot{E} \right) \cdot \left( F^{-1} \right).
\]

(8)

Let's write down (8) taking into account (2), (3) in a kind

\[
(d) = \left( F^{-1} \right)^T \cdot \frac{d}{dt} \left[ \left( F \right)^T \cdot (A) \cdot (F) \right] \cdot \left( F^{-1} \right) = \left( F^{-1} \right)^T \left[ \left( F \right)^T \cdot (\dot{A}) \cdot (F) + \left( F \right)^T \cdot (A) \cdot (F) \right] + \left( F \right)^T \cdot (A) \cdot (F) = (A^k).
\]

(9)

Whether expression in the right part (9) name a derivative (Lie Rate) for which, taking into account (7), expression is fair

\[
\left( A^k \right) = (d) = \frac{1}{2} \sum_{i,j} \dot{g}_{ij} \left( \ddot{r}^i \ddot{r}^j \right).
\]

(10)

By analogy to differentiation operation on time we will define variations of the basic tensors.

\[
(\delta E) = \frac{1}{2} \delta g_{ij} \left( \ddot{R}^i \ddot{R}^j \right) = \sum_{i,j} \delta \dot{E}_{ij} \left( \ddot{e}_i \ddot{e}_j \right),
\]

From (8) follows

\[
(\delta d) = \left( F^{-1} \right)^T \cdot (\delta E) \cdot \left( F^{-1} \right) = \frac{1}{2} \delta g_{ij} \left( \ddot{r}^i \ddot{r}^j \right) = \sum_{i,j} \delta \dot{d}_{ij} \left( \ddot{e}_i \ddot{e}_j \right),
\]

From (3) follows

\[
(\delta h) = \sum_k \delta v^k \dot{h}_j \left( \ddot{r}^j \ddot{r}^j \right) = \sum_{i,j} \delta \dot{h}_{ij} \left( \ddot{e}_i \ddot{e}_j \right).
\]

2. Physical model of a hyperelastic body

For construction of defining parities we will take advantage of the second equation of thermodynamics for isothermal deformation of the elastic isotropic environment in a kind

\[
\rho \psi - (\sigma) \cdot (d) = 0.
\]

(11)

Here \( \rho \) - material density, \( \psi \) - function of free energy, \( (\sigma) \) - stress tensor Cauchy, \( (d) \) - deformation tensor of speed which is defined by various forms (5), (7-10).

Let's construct the general structural parities, accepting as base expression – deformation tensor Almansi (1). The choice basic deformation tensor assumes that
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function of free energy depends from a component this tensor. Thus, we will consider set function

\[ \psi = \psi(A). \]  

(12)

According to a rule of differentiation of scalar function on tensor is had

\[ \psi = \left( \frac{\partial \psi}{\partial A} \right) : (A). \]  

(13)

From (12) follows

\[ \dot{A} = (d) - (A) : (h) - (h)^T : (A) = \left[ (d) - (A) : (d) - (d) : (A) \right] - \left[ (A) : (\omega) - (\omega) : (A) \right]. \]  

(14)

Let's substitute (14) in (13) and further in the equation (11) and we will receive

\[ \rho \left( \frac{\partial \psi}{\partial A} \right) : \left[ (d) - (A) : (d) - (d) : (A) \right] - (\sigma) : (d) = 0. \]  

(15)

Let's transform the equation (15) to the form

\[ \sigma = \rho \left( \frac{\partial \psi}{\partial A} \right) : \left[ (g) - 2(A) \right] = \rho \left( \frac{\partial \psi}{\partial A} \right) : (B^{-1}). \]  

(17)

(18)

Taking into account (18) defining parities (17) it is possible to write down or in a kind

\[ (\sigma) = \rho \left( \frac{\partial \psi}{\partial A} \right) : \left[ (g) - 2(A) \right] = \rho \left( \frac{\partial \psi}{\partial A} \right) : (B^{-1}) \]

or

\[ (\sigma) = \rho \left[ (g) - 2(A) \right] \left( \frac{\partial \psi}{\partial A} \right) = (B^{-1}) \left( \frac{\partial \psi}{\partial A} \right) \rho. \]

Both expressions are equivalent. For the isotropic material expression (12) becomes simpler before the scalar function depending on the main invariants.

\[ \psi = \psi(I_{12}, I_{23}, I_{31}). \]

Here, and further, a designation of scalars we will simplify

\[ I_1 = I_{12}, I_2 = I_{23}, I_3 = I_{31}. \]

In result, after simple transformations, we will receive

\[ (\sigma) = \left[ \psi_1 + \psi_2 I_1 \right] (g) - \psi_2 (A) + \psi_3 I_3 (A^{-1}), \]

where

\[ \psi_1 = \rho \frac{\partial \psi}{\partial I_1}, \psi_2 = \rho \frac{\partial \psi}{\partial I_2}, \psi_3 = \rho \frac{\partial \psi}{\partial I_3}. \]

Let's define stress tensor

\[ (\sigma) = \sum_{i,j} \sigma_{ij} (\varepsilon_i \varepsilon_j) \]

and

\[ \sigma_{ij} = [\psi_1 + \psi_2 I_1] \delta_{ij} - \psi_2 \lambda_{ij} - \psi_3 I_3 \lambda_{ij}. \]
where

\[ A_j = g_j - G_j, A_j' (\vec{r}^i \vec{r}^{-i}) = \sum_{m,n} \vec{A}_j' (\vec{e}_m \vec{e}_n) = (A^{-i})^\prime, A_j' = \vec{r} \cdot \sum_{m,n} \vec{A}_j' (\vec{e}_m \vec{e}_n) \cdot \vec{r} = \vec{A}_j' \vec{r} \vec{r}^\prime. \]

Thus, the presented parities allow calculating components stress tensor in various bases at the known initial and deformed configurations.

### 3. Numerical example

As a basic ratio the variation equation of the principle of virtual speeds in the current configuration which for problems of a statics can be written down in a look is accepted

\[ f \sigma \cdot dV = \int f^* \cdot \delta \nu dV + \int \bar{t}_e \delta \nu dS, \]

where \( f^* \) - a vector of set external volume forces, \( \bar{t}_e \) - a vector of the set tension on part of a surface on which power boundary conditions are defined. The technology of calculations represents a method of consecutive loadings with definition of the current metrics, as the main for calculations. Sampling of settlement area is carried out on the basis of a method of final elements.

We consider the problem of bending test beam into the ring. Consider first the one-dimensional case of this problem (see Fig. 1a).

We write the basic kinematic relations

\[ \pi R = L, R = \frac{L}{\alpha}, \varphi = \frac{\pi}{L} \alpha. \]

Enter the radius - vector of initial configuration

\[ \vec{R}_0 = \alpha \vec{e}_i \]

and radius - vector of the current configuration

\[ \vec{r}_e = R \sin \varphi \vec{e}_i + R (1 - \cos \varphi) \vec{e}_i. \]

Find the basis vectors

\[ \frac{\partial \vec{R}_0}{\partial \alpha} = \vec{e}_i, \frac{\partial \vec{r}_e}{\partial \alpha} = \cos \varphi \vec{e}_i + \sin \varphi \vec{e}_j, \quad 0 \leq \alpha \leq L, \vec{R}_0 = -\sin \varphi \vec{e}_i + \cos \varphi \vec{e}_j. \]

Then, using the above relations, we turn to the three-dimensional case (see Fig. 1b). Consider radius - vector of initial configuration

\[ \vec{R} = \alpha \vec{e}_i + \gamma \vec{e}_j + \beta \vec{e}_k. \]
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where

\[ 0 \leq \alpha \leq L, \quad 0 \leq \gamma \leq b, \quad \frac{-h}{2} \leq \beta \leq \frac{h}{2}. \]

For the deformed configuration (see Fig. 1c) holds

\[ \bar{\rho} = \alpha \left[ \sin \phi \bar{e}_3 + (1 - \cos \phi) \bar{e}_1 \right] + \gamma \bar{e}_2 + \beta \left[ -\sin \phi \bar{e}_1 + \cos \phi \bar{e}_3 \right]. \]

After obvious transformations we obtain the covariant and contravariant components of the metric tensor

\[ g_{11} = \left( 1 - \frac{\pi}{L} \beta \right)^2, \quad g_{22} = g_{33} = 1, \quad g_{ij} = 0, \quad g^{11} = \left( 1 - \frac{\pi}{L} \beta \right)^2, \quad g^{22} = g^{33} = 1, \quad g^{ij} = 0, \quad i \neq j. \]

Then Almansi tensor will have the form

\[ A_{11} = \frac{1}{2} [g_{11} - G_{11}] = \frac{1}{2} \left[ \left( 1 - \frac{\pi}{L} \beta \right)^2 - 1 \right] = \frac{1}{2} \left[ 1 - 2 \frac{\pi}{L} \beta + \left( \frac{\pi}{L} \beta \right)^2 \right] - 1 = \frac{\pi}{L} \beta \left( -1 + \frac{\pi}{2L} \beta \right). \]

Other \( A_{ij} = 0. \) For the material in Seth we have therefore

\[ \sigma_{i1} = \lambda g_{i1} A_{11} + 2 \mu A_{i1}; \quad I_{11} = g^{ii} A_{ii}; \]

then

\[ \sigma_{i1} = \lambda g_{i1} A_{11} + 2 \mu A_{i1} = A_{i1} \left[ 2 \mu + \lambda g_{i1} \right] = \frac{\pi}{L} \beta \left( \frac{\pi}{L} \beta \right)^2 - 1 \left[ 2 \mu + \lambda \left( 1 - \frac{\pi}{L} \beta \right)^2 \right]. \]

Compute \( \sigma_{i1}^{\min} \) and \( \sigma_{i1}^{\max} \) in nodes and the free edge of the beam

\[ \sigma_{i1}^{\min} = \frac{\pi h}{2L} \left( \frac{\pi h}{4L} - 1 \right) \left[ 2 \mu + \lambda \left( 1 - \frac{\pi h}{2L} \right)^2 \right], \sigma_{i1}^{\max} = \frac{\pi h}{2L} \left( \frac{\pi h}{4L} + 1 \right) \left[ 2 \mu + \lambda \left( 1 + \frac{\pi h}{2L} \right)^2 \right]. \]

To take into account the geometry of the shell \( \frac{\pi h}{4L} << 1 \), then

\[ \sigma_{i1}^{\min} = -\frac{\pi h}{2L} (\lambda + 2 \mu); \quad \sigma_{i1}^{\max} = \frac{\pi h}{2L} (\lambda + 2 \mu); \quad \sigma_{22} = \lambda g_{22} A_{11} + 2 \mu A_{22} = \lambda A_{11} = \frac{\lambda \pi \beta}{L} \left( -1 + \frac{\pi \beta}{2L} \right); \]

\[ \sigma_{33} = \lambda g_{33} A_{11} + 2 \mu A_{33} = \lambda A_{11} \left( \text{with the hypothesis } \sigma_{33} = 0 \right) \]

![Fig. 2.](image)

Problem is calculated using the methodology proposed above. The length \( L = 200 \text{ cm} \) of beam thickness \( h = 1 \text{ cm} \), width \( b = 5 \text{ cm} \), modulus of elasticity \( E = 20000 \frac{kg}{cm^2} \), Poisson's ratio \( \nu = 0 \). Fig.2 shows the deformed state of the beam and several intermediate stages of loading. In the case where the applied
load is split into 1000 steps of loading, error numerical solutions less than 1 %. Error could occur due to the finite-element approximation of the study area.

The conclusion

The received parities represent a theoretical basic algorithm of research of finite deformations of non-linear elastic bodies. It is necessary to add only concrete physical model in the form of expression function free energy, fair for a corresponding material. Some recommendations about their choice are given in works [4-7].

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References


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