A Closed-Form Solution for a Dynamic Pricing Model with Reference Price Effect

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Abstract

This paper addresses the problem of pricing for an airline or a retailer selling one product to a stream of repeated customers over an infinite time horizon. We propose a dynamic pricing model that incorporates the concept of customers’ reference price. We derive an optimal pricing policy and prove its monotone convergence in a monopoly context. We derive an analytical result for the optimal prices. Our pricing policy is a function of the current market condition, represented by the reference price, as opposed to the previous studies where the pricing policy is a function of time.

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1 Introduction

Dynamic pricing, also known as time-based pricing, is a strategy that adjusts the product price in real-time in order to maximize the profit. Dynamic pricing is particularly suitable for the travel industries, such as airlines, because inventory cannot be replenished and unsold products have little salvage value. An airline has incentive to dynamically adapt its pricing to the demand. Interested readers can refer to Elmaghraby and Keskinocak (2003), McGill and van Ryzin (1999), or Weatherford and Bodily (1992) for a general survey on dynamic pricing and its role in revenue management.

Traditional literature in airline pricing assumes that there is no interaction between the firms and their customers. However, customers have a memory of past ticket prices and they develop a kind of price expectations. These price expectations are called reference prices. They become benchmark prices against which customers compare the current tickets prices before they make purchases. If the reference price is below the observed price, customers perceive gains and purchasing becomes more attractive (Hardie et al. (1993)). For a review of the theoretical foundations and modeling of the reference price concept we refer the reader to Winer (1988).

This paper presents a dynamic pricing model in the airline industry incorporating the concept of customer’s reference price.

2 Dynamic Pricing Formulation

We are making the assumption that the airline is only selling one class category at a time and we will consider the demand function as being generated by:

\[ D(p, r) = a - bp + c(r - p) \]

like in Kopalle (96), Fibich et al. (2003).

The reference price is following the dynamic:

\[ r(t + 1) := g(p(t), r(t)) = \alpha r(t) + (1 - \alpha) p(t) \]

The airline decision problem takes the following form:
Find:

\[ V^*(r_0) = \sup_{(p_0, p_1, ...)} \sum_{t=0}^{\infty} \beta^t D(p(t), r(t)) p(t) \]

s.t.

\[ r(t + 1) = g(p(t), r(t)) \]
We can write the simplified Bellman equation dropping the time subscript:

\[ V^*(r) = \sup_p D(p, r)p + \beta V^*(g(p, r)) \] (1)

**Theorem 2.1** The optimal value function that solves the Bellman equation (1) is given by:

\[ V^*(r) = \gamma r^2 + \delta r + \epsilon \] (2)

which is attained by the following pricing policy

\[ p^*(r) = \frac{a + \beta \delta (1 - \alpha) + r(c + 2\beta \gamma \alpha (1 - \alpha))}{2(b + c - \beta \gamma (1 - \alpha)^2)} \] (3)

where

\[ \gamma = \frac{b(1 - \alpha^2 \beta) + c(1 - \alpha \beta) - \sqrt{(b(1 - \alpha^2 \beta) + c(1 - \alpha \beta))^2 - c^2(1 - \alpha)^2 \beta}}{2(1 - \alpha)^2 \beta} \] (4)

\[ \delta = \frac{a(c + 2(1 - \alpha)\alpha \beta \gamma)}{2b(1 - \alpha \beta) + c(2 - \beta - \alpha \beta) - 2(1 - \alpha)^2 \beta \gamma} \] (5)

\[ \epsilon = \frac{(a + (1 - \alpha)\beta \delta)^2}{4(1 - \beta)(b + c - (1 - \alpha)^2 \beta \gamma)} \] (6)

In order to solve equation (1) we state the postulate that the optimal value function is quadratic and given by:

\[ V^*(r) = \gamma r^2 + \delta r + \epsilon \]

where \( \gamma, \delta \) and \( \epsilon \) are to be found.

Hence (1) becomes:

\[ V^*(r) = \sup_p (a - bp + c(r - p))p + \beta ((\alpha r + (1 - \alpha)p)^2 \gamma + (\alpha r + (1 - \alpha)p)\delta + \epsilon) \] (7)

The optimal price \( p^*(r) \) results from writing the partial derivative of \( V(r) \) with respect to \( p \) and setting it equal to 0. Solving \( \frac{\partial V(r)}{\partial p} = 0 \) for \( p \) yields the optimal pricing policy

\[ p^*(r) = \frac{a + \beta \delta (1 - \alpha) + r(c + 2\beta \gamma \alpha (1 - \alpha))}{2(b + c - \beta \gamma (1 - \alpha)^2)} \]

By plugging (3) into (7) and then collecting the coefficients of the polynomial we determine \( \gamma, \delta \) and \( \epsilon \):
Later, we will show that this choice of 
\[ \alpha = \frac{1}{1696} \]

enures, it must be positive and increasing in \( r \) for all values of \( r \).

Since the value function \( V \) is always nonnegative, this is true because:

To show that the roots are real, we only need to show that the term inside the quadratic and the constant terms are positive, whereas the coefficient of the quadratic equation (8). Suppose that the quadratic equation has real roots, then it is easy to see that both roots are positive since the coefficients of the quadratic and the constant terms are positive, whereas the coefficient of the linear term is negative.

First, we will establish that our postulate is valid.

Therefore, \( \gamma \) is the smaller root of the quadratic equation:

\[
(1 - \alpha)^2 \beta^2 - (b(1 - \alpha^2 \beta) + c(1 - \alpha \beta)) \gamma + c^2/4 = 0 \quad (8)
\]

Therefore,

\[
\gamma = \frac{b(1 - \alpha^2 \beta) + c(1 - \alpha \beta) - \sqrt{(b^2(1 - \alpha^2 \beta) + c(1 - \alpha \beta))^2 - c^2(1 - \alpha)^2 \beta}}{2(1 - \alpha)^2 \beta}
\]

Later, we will show that this choice of \( \gamma \) guarantees that \( \delta \) is positive.

Since the value function \( V^*(r) \) represents the present value of the total revenue, it must be positive and increasing in \( r \) for all values of \( r \geq 0 \). Given the uniqueness of \( V^*(r) \) (Stockey et al. 1989), it suffices to show that the coefficients \( \gamma, \delta \) and \( \epsilon \) are nonnegative real numbers in order to establish that our postulate is valid.

First, we will establish that \( \gamma \) is positive. Recall that \( \gamma \) is a solution of the quadratic equation (8). Suppose that the quadratic equation has real roots, then it is easy to see that both roots are positive since the coefficients of the quadratic and the constant terms are positive, whereas the coefficient of the linear term is negative.

To show that the roots are real, we only need to show that the term inside the square-root in (4) is always nonnegative. this is true because:

\[
(b(1 - \alpha^2 \beta) + c(1 - \alpha \beta))^2 - c^2(1 - \alpha)^2 \beta = (1 - \alpha^2 \beta)(b^2(1 - \alpha^2 \beta + 2bc(1 - \alpha \beta) + c^2(1 - \beta)))
\]
Next, to show that \( \delta \) is well defined and positive, we need to show that

\[
\gamma < \frac{2b(1 - \alpha \beta) + c(2 - \beta - \alpha \beta)}{2(1 - \alpha)^2 \beta}
\]

This is true if and only if

\[
b(1 - \alpha^2 \beta) + c(1 - \alpha \beta) - \sqrt{(b(1 - \alpha^2 \beta) + c(1 - \alpha \beta))^2 - c^2(1 - \alpha)^2 \beta} \\
\leq 2b(1 - \alpha \beta) + c(2 - \beta - \alpha \beta)
\]

\( \Leftrightarrow b((1 - (1 - \alpha)^2 \beta - 1) + c(\beta - 1) < \sqrt{(b(1 - \alpha^2 \beta) + c(1 - \alpha \beta))^2 - c^2(1 - \alpha)^2 \beta}
\]

which holds because the left hand side is negative.

Finally, \( \epsilon \) is well-defined and positive if and only if

\[
\gamma < \frac{b + c}{(1 - \alpha)^2 \beta}
\]

which is again true because \( b \) and \( c \) are positive and \( b(1 - \alpha^2 \beta) + c(1 - \alpha \beta) \leq 2(b + c) \).

Now that we established the validity of the postulate, the correctness of \( p^*(r) \), and proved the result of the Theorem, we can derive the expression for the transition of the reference price under the optimal pricing policy, \( g^*(r) = g(p^*(r), r) \). It is given by:

\[
g^*(r) = \frac{(2\alpha(b + c) + c(1 - \alpha))r + (1 - \alpha)(a + (1 - \alpha)\beta \delta)}{2(b + c - (1 - \alpha)^2 \beta \gamma}
\]

Finally, the steady state reference price \( r^{**} \) is determined by solving \( r^{**} = g^*(r^{**}) \).

\[
r^{**} = \frac{a + (1 - \alpha)\beta \delta}{2b + c - 2(1 - \alpha)\beta \gamma} = \frac{a(1 - \alpha \beta)}{c(1 - \beta) + 2b(1 - \alpha \beta)}
\]

The second equality is not straightforward and follows from tedious calculations that involve substituting the expression for \( \delta \) as given in (5), and using (8) to complete the square in the denominator.

The expression of \( r^{**} \) in (11) is the same equation (13) of Fibich et al. (2003). Because of the dynamic of the reference price, it is easy to see that the optimal price at the steady state must be the same as the reference price, i.e. \( p^{**} := p^*(r^{**}) = r^{**} \).
3 Monotonicity and Convergence Results

Let’s first define the sequences of optimal reference prices \( r^* \) and optimal prices \( p^* \) as follow:

\[
\begin{align*}
  r^*(t) &= g^*(r^*(t-1)) \\
  p^*(t) &= p^*(r^*(t))
\end{align*}
\]

3.1 Proposition 1

Given an initial reference price \( r_0 \), the sequence of optimal reference prices \( r^* \) converges monotonically to the steady state \( r^{**} \).

Condition (9) implies that \( p^*(r) \) in (3) is positive for all \( r \geq 0 \), and \( g^*(r) \) in (10) is strictly increasing in \( r \). Since \( g^*(0) = (1-\alpha)p^*(0) \), \( g^*(0) \) is also positive. Furthermore, from (11), it is easy to see that the steady state reference price is positive and finite. It then follows that \( \frac{dg^*(r)}{dr} < 1 \).

Therefore we have:

\[
\begin{align*}
  g^*(r) > r, & \text{ for } 0 \leq r < r^{**} \\
  g^*(r) < r, & \text{ for } r > r^{**}
\end{align*}
\]

Then, we can show by a simple induction that \( (r^*) \) converges monotonically to \( r^{**} \).

Proposition 1 implies that if \( r_0 < r^{**} \), then the reference price under optimal pricing policy, \( r^*(t) \), will be increasing over time until the steady state is reached. The converse is also true when \( r_0 > r^{**} \). In the next proposition, we will show that the same result also holds for the sequence of optimal prices.

3.2 Proposition 2

Given an initial reference price \( r_0 \), the sequence of optimal prices \( p^* \) converges monotonically to the steady state \( p^{**} \).

First, it is easy to see that the steady state price is the same as the steady state reference price, because \( r^{**} = \alpha r^{**} + (1-\alpha)p^{**} \). Condition (9) implies that the optimal price \( p^*(r) \) is strictly increasing in \( r \). Then, proposition 1 implies that \( (p^*) \) must also converge monotonically to \( r^{**} \).

3.3 Proposition 3

If the initial reference price is lower than the steady state price (respectively higher) then the customers always perceive a surcharge (respectively a discount), until the steady state is reached. In other words,

\[
p^*(r) > r, \text{ for } 0 \leq r < r^{**}
\]
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\[ p^*(r) < r, \text{ for } r > r^{**} \]

This is similar to the proof of proposition 1. The fact that \( p^*(r) \) is positive, linear, and strictly increasing in \( r \) for \( r \geq 0 \), and the fact that the steady state price \( p^{**} = r^{**} = p^*(r^{**}) \) is positive and finite imply that \( \frac{dp^*(r)}{dr} < 1 \).

Figure 1 plots \( p^*(r) \) and \( g^*(r) \), and Figure 2 shows how the optimal and reference prices converge to the steady state for two cases where \( r_0 < r^{**} \) and \( r_0 > r^{**} \).

Proposition 3 allows the extension of this result to asymmetric reference price effect. Since the customers always perceive either a surcharge or a discount, the effect of the reference price never switches between the two sides of the asymmetry.
Conclusion

In our current economic environment where internet made it easy to access the information, shoppers are becoming more and more sophisticated and aware of prices. In this paper, we analyzed an optimal dynamic pricing model under repeated customer interactions for monopoly markets. Fibich et al. (2003) solved for an explicit pricing formula in continuous-time, while claiming that an explicit solution in discrete-time is cumbersome. As a result, the optimal pricing policy they derived is a function of time, which is quite unnatural and difficult to implement in practice. Our formula in discrete-time allowed us to derive a closed-form solution for the optimal pricing policy as a function of the state of the system, which is the current level of the reference price. This provides a more comprehensible tool to managers, from which they can derive economic insights as well as implement the policy in practice.

The economics literature models competition in the market by assuming that the demand of a product is a function of it’s competitor’s price in addition to its own price. We intend to extend our result to a multiple-product scenario under competition. The concept of reference price would provide a different framework for modeling competition where firms interact through their influence on the customers’ reference price.
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References


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