An Iterative Method of Order Four
Based on Power Series
for Solving a Nonlinear Equation

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Abstract

In this paper, a one-point iteration formula for solving a nonlinear
equation is presented. The formula is derived by finding a coefficient of
a power series for updating Newton method. We show analytically that
the method is of order four for a simple root. We verify the theoretical
result on relevant numerical problems and compare the behavior of the
propose method with some existing methods.

Mathematics Subject Classification: 65D99, 65H05

Keywords: a four order formula, nonlinear equation, Newton method,
Chebyshev method

1 Introduction

Solving a nonlinear equation

\[ f(x) = 0, \]  \hspace{1cm} (1)

is an active research in numerical analysis. In this paper, we consider an
iterative method to find a simple root of a non-linear equation (1), where
\( f : D \subset \mathbb{R} \rightarrow \mathbb{R} \) for an open interval \( D \) is a scalar function. The case of
multiple roots is not treated.
One of the classic and most advantage technique to derive a numerical method is using the Taylor expansion. Applying a linear Taylor expansion of $f(x)$ about $x = x_n$ and substituting $x = x_{n+1}$, we end up with an iterative formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots$$

(2)

This is Newton method which is quadratically convergent for a simple root, provided that the initial guess closed enough to the root $x^*$[4, p.68].

If we expand $f(x)$ about $x = x_n$ using Taylor expansion and ignoring the term of containing the third order, we obtain Chebyshev method of the form

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f''(x_n)}{2 f'(x_n)} \left( \frac{f(x_n)}{f'(x_n)} \right)^2, \quad n = 0, 1, 2, \ldots$$

(3)

This method is cubicly convergence[3]. Using the same strategy we can also obtain an iterative method of third order in the form

$$x_{n+1} = x_n - \frac{2 f(x_n) f'(x_n)}{2 (f'(x_n))^2 - f''(x_n) f(x_n)}, \quad n = 0, 1, 2, \ldots$$

(4)

This method is known as Halley method [1, 5].

Germani et. al. [6] derive an iterative method of order $n+1$, by expanding $f, f', \ldots, f^{(n)}$ by Taylor expansion and solve the system using $LU$ decomposition [2, p.104] to obtain an iterative method. For $n = 3$, we can express their formula as

$$x_{n+1} = x_n - \frac{f(x_k)}{f'(x_k)} - \frac{1}{2} \frac{f''(x_k)}{f'(x_k)} \left( \frac{f(x_k)}{f'(x_k)} \right)^2$$

$$- \left( \frac{1}{2} \left( \frac{f''(x_k)}{f'(x_k)} \right)^2 - \frac{1}{6} \frac{f'''(x_k)}{f''(x_k)} \right) \left( \frac{f(x_k)}{f'(x_k)} \right)^3.$$ 

(5)

In this paper, we consider deriving a formula by updating the each step iterations by a power series of $f(x_n)/f'(x_n)$. The new method, which is simpler in terms of formula than those of (5), is proved to be convergence of order four. The performance of the new method is compared with some discussed methods through numerical examples.

## 2 Proposed Method

We consider an iterative method of the form

$$x_{n+1} = x_n - \sum_{j=1}^{n} \alpha_j \left( \frac{f(x_n)}{f'(x_n)} \right)^j, \quad n = 1, 2, 3,$$

(6)
where

\[ \alpha_j = \frac{f^{(j)}(x_n)}{f'(x_n)}, \quad j = 1, 2, 3. \]  

(7)

In (6), for \( n = 1 \) and \( n = 2 \) we obtain respectively Newton method (2) and Chebyshev method (3). For \( n = 3 \) the iterative formula becomes

\[ x_{n+1} = x_n - \left( \alpha_1 \frac{f(x_k)}{f'(x_k)} + \alpha_2 \left( \frac{f(x_k)}{f'(x_k)} \right)^2 + \alpha_3 \left( \frac{f(x_k)}{f'(x_k)} \right)^3 \right), \]  

(8)

with \( \alpha_j \) as in (7).

In the following we show that the iterative formula (8) is of order four.

**Theorem 2.1.** Let \( f : D \subset \mathbb{R} \to \mathbb{R} \) for an open interval \( D \). Assume that \( f \) has sufficiently continuous derivatives in the interval \( D \). If \( f \) has a simple root at \( x^* \in D \) and \( x_0 \) is sufficiently close to \( x^* \), then the new method defined by (8) satisfies the following error equation:

\[ e_{n+1} = - \left( \frac{17 F_2^3}{8 F_1^3} - \frac{1}{4} \frac{F_3 F_2}{F_1^2} - \frac{1}{8} \frac{F_4}{F_1} \right) e_n^4 + O(e_n^5), \]

where \( e_n = x_n - x^* \) and

\[ F_j = f^{(j)}(x^*), \quad j = 1, 2, 3, 4. \]

**Proof.** Let \( x^* \) be a simple root of \( f(x) = 0 \), then \( f'(x^*) \neq 0 \) and \( f(x^*) = 0 \). Let \( e_n = x_n - x^* \). Taylor expansion of \( f(x_n) \) and \( f'(x_n) \) about \( x_n = x^* \) are given respectively by

\[ f(x_n) = F_1 e_n + \frac{F_2}{2} e_n^2 + \frac{F_3}{6} e_n^3 + \frac{F_4}{24} e_n^4 + \frac{F_5}{120} e_n^5, \]  

(9)

and

\[ f'(x_n) = F_1 + F_2 e_n + \frac{F_3}{2} e_n^2 + \frac{F_4}{6} e_n^3 + \frac{F_5}{24} e_n^4 + \frac{F_6}{120} e_n^5. \]  

(10)

In writing \( f'(x_n) \) as

\[ f'(x_n) = F_1 \left( 1 + \frac{F_2}{F_1} e_n + \frac{F_3}{2 F_1} e_n^2 + \frac{F_4}{6 F_1} e_n^3 + \frac{F_5}{24 F_1} e_n^4 + \frac{F_6}{120 F_1} e_n^5 \right), \]  

(11)

using a geometric series and computing \( \frac{f(x_n)}{f'(x_n)} \), we obtain after simplifying

\[ \frac{f(x_n)}{f'(x_n)} = e_n \left( 1 - \frac{F_2}{2 F_1} e_n^2 + \left( \frac{F_2^2}{2 F_1^2} - \frac{1}{3} \frac{F_3}{F_1} \right) e_n^3 + \left( -\frac{1}{8} \frac{F_4}{F_1} - \frac{1}{2} \frac{F_3^2}{F_1^2} + \frac{7}{12} \frac{F_3 F_2}{F_1^2} \right) e_n^4 \right. \]

\[ + \left. \left( \frac{1}{6} \frac{F_3^2}{F_1^2} + \frac{1}{30} \frac{F_5}{F_1} + \frac{1}{24} \frac{F_4 F_2}{F_1^2} + \frac{5}{24} \frac{F_4 F_2}{F_1^2} - \frac{5}{6} \frac{F_3 F_2^2}{F_1^3} \right) e_n^5. \]  

(12)
On computing \( \left( \frac{f(x_n)}{f'(x_n)} \right)^2 \) and \( \left( \frac{f(x_n)}{f'(x_n)} \right)^3 \), using (12), we get respectively

\[
\left( \frac{f(x_n)}{f'(x_n)} \right)^2 = e_n^2 - \frac{F_2 e_n^3}{F_1} + \left( \frac{5 F_2^2}{4 F_1^2} - 2/3 \frac{F_3}{F_1} \right) e_n^4 + \left( -\frac{1}{4 F_1} + \frac{3 F_3 F_2}{2 F_1^2} - \frac{3 F_2^3}{2 F_1^3} \right) e_n^5,
\]

and

\[
\left( \frac{f(x_n)}{f'(x_n)} \right)^3 = e_n^3 - \frac{3 F_2 e_n^4}{2 F_1} + \left( \frac{9 F_2^2}{4 F_1^2} - \frac{F_3}{F_1} \right) e_n^5.
\]

On using (12), (13) and (14), we compute \( \alpha_1 \frac{f(x_n)}{f'(x_n)} + \alpha_2 \left( \frac{f(x_n)}{f'(x_n)} \right)^2 + \alpha_3 \left( \frac{f(x_n)}{f'(x_n)} \right)^3 \) and after simplifying, we obtain

\[
\alpha_1 \frac{f(x_n)}{f'(x_n)} + \alpha_2 \left( \frac{f(x_n)}{f'(x_n)} \right)^2 + \alpha_3 \left( \frac{f(x_n)}{f'(x_n)} \right)^3
= \alpha_1 e_n + \left( \alpha_2 - \frac{\alpha_1 F_2}{2 F_1} \right) e_n^2 + \left( -\frac{\alpha_2 F_2}{F_1} + \frac{\alpha_1 F_2^2}{2 F_1} - \frac{\alpha_1 F_3}{3 F_1} + \alpha_3 \right) e_n^3
+ \left( -\frac{2 \alpha_2 F_3}{3 F_1} - \frac{3 \alpha_3 F_2}{2 F_1} + \frac{5 \alpha_2 F_2^2}{4 F_1^2} + \frac{7 \alpha_1 F_3 F_2}{12 F_1^2} - \frac{\alpha_1 F_2^3}{2 F_1^3} - \frac{\alpha_1 F_4}{8 F_1} \right) e_n^4
+ \left( \frac{\alpha_1 F_3^2}{6 F_1^2} - \frac{\alpha_2 F_4}{4 F_1} - \frac{\alpha_1 F_5}{30 F_1} + \frac{\alpha_1 F_2^4}{2 F_1^4} + \frac{5 \alpha_1 F_4 F_2}{24 F_1^2}
+ \frac{3 \alpha_2 F_3 F_2}{2 F_1^2} + \frac{9 \alpha_3 F_2^2}{4 F_1^2} - \frac{\alpha_3 F_3}{6 F_1} - \frac{5 \alpha_1 F_3 F_2^2}{6 F_1^3} - \frac{3 \alpha_2 F_2^3}{2 F_1^3} \right) e_n^5.
\]

On substituting \( \alpha_1 = 1 \), into the righthand side of (15) we obtain after simplifying

\[
\alpha_1 \frac{f(x_n)}{f'(x_n)} + \alpha_2 \left( \frac{f(x_n)}{f'(x_n)} \right)^2 + \alpha_3 \left( \frac{f(x_n)}{f'(x_n)} \right)^3
= e_n + \left( \alpha_2 - \frac{1}{2 F_1} \right) e_n^2 + \left( -\frac{\alpha_2 F_2}{F_1} + \frac{1}{2 F_1^2} - \frac{1}{3 F_1} + \alpha_3 \right) e_n^3
+ \left( -\frac{2 \alpha_2 F_3}{3 F_1} - \frac{3 \alpha_3 F_2}{2 F_1} + \frac{5 \alpha_2 F_2^2}{4 F_1^2} + \frac{7 F_3 F_2}{12 F_1^2} - \frac{1}{2 F_1^3} - \frac{1}{8 F_1} \right) e_n^4
+ \left( \frac{1}{6 F_1^2} - \frac{\alpha_2 F_4}{4 F_1} + \frac{1}{2 F_1^4} + \frac{5 F_1 F_2}{24 F_1^2}
+ \frac{3 \alpha_2 F_3 F_2}{2 F_1^2} + \frac{9 \alpha_3 F_2^2}{4 F_1^2} - \frac{\alpha_3 F_3}{6 F_1} - \frac{5 F_3 F_2^2}{6 F_1^3} - \frac{3 \alpha_2 F_2^3}{2 F_1^3} \right) e_n^5.
\]
From (16), we see that by choosing $\alpha_2 = \frac{F_2}{2 F_1}$, we can remove the $e_n^2$-term from (16). So that we obtain

$$\alpha_1 \frac{f(x_n)}{f'(x_n)} + \alpha_2 \left( \frac{f(x_n)}{f'(x_n)} \right)^2 + \alpha_3 \left( \frac{f(x_n)}{f'(x_n)} \right)^3 = e_n + \left( \alpha_3 - \frac{1}{3} \frac{F_3}{F_1} \right) e_n^3 + \left( \frac{17 F_2^3}{8 F_1^3} + \frac{1}{4} \frac{F_3 F_2}{F_1^2} - \frac{1}{8} \frac{F_4}{F_1} \right) e_n^4 + \left( \frac{35 F_3 F_2^2}{12 F_1^3} + \frac{1}{12} \frac{F_4 F_2}{F_1^2} - \frac{1}{30} \frac{F_5}{F_1} + \frac{3}{4} \frac{F_2^4}{F_1^3} \right) e_n^5. \quad (17)$$

In the same way, we can remove the $e_n^3$-term from (16), by choosing $\alpha_3 = \frac{F_3}{3 F_1}$. Thus, we get

$$\alpha_1 \frac{f(x_n)}{f'(x_n)} + \alpha_2 \left( \frac{f(x_n)}{f'(x_n)} \right)^2 + \alpha_3 \left( \frac{f(x_n)}{f'(x_n)} \right)^3 = e_n + \left( \frac{17 F_2^3}{8 F_1^3} - \frac{1}{4} \frac{F_3 F_2}{F_1^2} - \frac{1}{8} \frac{F_4}{F_1} \right) e_n^4 + \left( \frac{11 F_3 F_2^2}{3 F_1^3} + \frac{1}{12} \frac{F_4 F_2}{F_1^2} - \frac{1}{30} \frac{F_5}{F_1} + \frac{3}{4} \frac{F_2^4}{F_1^3} - \frac{1}{6} \frac{F_3^2}{F_1^3} \right) e_n^5. \quad (18)$$

On substituting (18) into (8), noting $x_n - e_n = x^*$, we get

$$x_{n+1} = x^* - \left( \frac{17 F_2^3}{8 F_1^3} - \frac{1}{4} \frac{F_3 F_2}{F_1^2} - \frac{1}{8} \frac{F_4}{F_1} \right) e_n^4 - \left( \frac{11 F_3 F_2^2}{3 F_1^3} + \frac{1}{12} \frac{F_4 F_2}{F_1^2} - \frac{1}{30} \frac{F_5}{F_1} + \frac{3}{4} \frac{F_2^4}{F_1^3} - \frac{1}{6} \frac{F_3^2}{F_1^3} \right) e_n^5, \quad (19)$$

and after simplifying, we end up

$$e_{n+1} = - \left( \frac{17 F_2^3}{8 F_1^3} - \frac{1}{4} \frac{F_3 F_2}{F_1^2} - \frac{1}{8} \frac{F_4}{F_1} \right) e_n^4 - \left( \frac{11 F_3 F_2^2}{3 F_1^3} + \frac{1}{12} \frac{F_4 F_2}{F_1^2} - \frac{1}{30} \frac{F_5}{F_1} + \frac{3}{4} \frac{F_2^4}{F_1^3} - \frac{1}{6} \frac{F_3^2}{F_1^3} \right) e_n^5.$$

This ends the proof $\square$.

We remarks from the error analysis that we cannot obtain an order five in the formula (8) by choosing $n = 4$. The best we can do to get fifth order is if $f''(x_n) = 0$, then $\alpha_4 = \frac{1}{8} \frac{F_3}{F_1}$. $\square$
3 Numerical Experiments

In this section, we give the results of some numerical examples to demonstrate the efficiency of the method we proposed. All the computation is done in Matlab. We stop the iteration process by the following criteria

(a) \(|f'(x_n)| < \text{eps},\)
(b) If \(|f(x_{n+1})| < \text{eps},\)
(c) If \(|x_{n+1} - x_n| < 1.0E - 15,\)
(d) Maximum iteration=100.

We compare Newton method (NM), Chebyshev Method (CM), Halley method (HM), Germani Method (GM) defined by (5) and the proposed method (PSM) defined by (8). We use the following test functions,

\[
\begin{align*}
  f_1(x) &= x^3 - x + 3, \quad x^* = -1.67169988165716097[6], \\
  f_2(x) &= x^3 - 3x^2 + 2x + 0.4, \quad x^* = -0.159704852764861765[6], \\
  f_3(x) &= x^7 + 2x^5 + 3x^3 + x^2 + x + 1, \quad x^* = -0.584114422468403061[6], \\
  f_4(x) &= \sin(x^2) - x^2 + 1, \quad x^* = -1.39088576481033275[6], \\
  f_5(x) &= x(x - 3) - 4.0 \sin(x)^2, \quad x^* = 0.0, [7, 8] \\
  f_6(x) &= \sin(x) - 0.5, \quad x^* = 0.523598775598298873, [7, 8] \\
  f_7(x) &= x \exp(-x), \quad x^* = 0.0[7, 8].
\end{align*}
\]

In Table 1, 100+ indicates that the method does not convergence after the maximum iteration allowed is reached. (a) indicates that the stopping criteria (a) is reached. Star(*) in the number of iterations indicates that the method converges to a different root. Div indicates that the method is divergent. From the Table 1, as far as the numerical results are concerned, for most of the functions we tested, the presented methods can be competitive with the methods we are comparing.
Table 1: Comparisons of the discussed methods

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<th>$x_0$</th>
<th>The number of iteration of</th>
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References


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