On the Decoupling of a Class of Linear Functional Systems

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Abstract

Multivariate polynomial matrices arise from the treatment of functional linear systems such as systems described by partial differential equations, delay-differential equations or linear multidimensional discrete equations. In this paper we present conditions under which a class of multivariate polynomial matrices is equivalent to a block diagonal form. The conditions correspond to the decomposition of the associated linear systems of functional equations. The constructive method which can be easily implemented on a computer algebra system is illustrated by an example.

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1 Introduction

Over the recent years, the theory of linear functional systems e.g. systems of partial differential or delay-differential equations, has attracted many researchers. In his pioneering work, Rosenbrock [8] used matrices over $\mathbb{R}[s]$, $s \equiv d/dt$ to represent linear systems of ordinary differential equations. This ring is a principal ideal domain with the Euclidean division property thus allowing the establishment of canonical forms e.g. the Smith normal form which
allows the complete decoupling of such systems. In the case of functional systems, the resulting system matrices have their elements in multivariate polynomial rings which are not principal ideal rings and do not have a Euclidean division. Several authors attempted to extend some of the results obtained for matrices from the univariate to the multivariate case. One interesting problem is to reduce a given system into an equivalent simpler one in order to simplify its analysis and synthesis. The problem of reducing linear system of functional equations to a simpler form containing fewer equations and unknowns has been considered by a number of authors, eg. Boudellioua and Quadrat [1], Cluzeau and Quadrat [4], etc. In particular, in the case where the reduced system contains only one equation in a single unknown, this corresponds to the reduction of a multivariate polynomial matrix to a particular Smith normal form, see for example Frost and Boudellioua [5], Lee and Zak [6] and Boudellioua and Quadrat [1], etc. The latter results mentioned previously correspond to the complete decomposition of a linear functional system which requires quite strong conditions. Cluzeau and Quadrat [3] presented conditions for the reduction of system to a block-triangular or block diagonal form. In this paper we present a constructive method for the reduction by equivalence of system in block-triangular form to a block diagonal form. First we present some definitions which will be used in the paper.

**Definition 1.1** Let $D = K[x_1, \ldots, x_n]$ be a polynomial ring with indeterminates $x_1, \ldots, x_n$ and coefficients over a field $K$. The general linear group $GL_p(D)$ is defined by

$$GL_p(D) = \{ M \in D^{p \times p} \mid \exists N \in D^{p \times p} : MN = NM = I_p \}$$

An element $M \in GL_p(D)$ is called a unimodular matrix. It follows that $M$ is unimodular if and only if $|M| \in K \setminus \{0\}$.

**Definition 1.2** Let $T_1$ and $T_2$ denote two matrices in $D^{q \times p}$ then $T_1$ and $T_2$ are said to be unimodular-equivalent if there exist two matrices $M \in GL_q(D)$ and $N \in GL_p(D)$ such that $T_2 = MT_1N$. In case $N = M^{-1}$, the matrices $T_1$ and $T_2$ are said to be similar over $D$.

The transformation of unimodular equivalence has been shown to exhibit fundamental algebraic properties amongst its invariants. In particular, it preserves the zero structure of the original matrix, see for example Pugh et al. [7].

**2 Reduction by Unimodular Equivalence**

The aim of the following result is to present conditions for which a block upper triangular matrix with elements in $D$ is unimodular equivalent to a block diagonal matrix.
Theorem 2.1 Let \( T \in D^{p \times p} \) be a block upper triangular matrix given by:

\[
T = \begin{pmatrix}
T_{11} & T_{12} \\
0 & T_{22}
\end{pmatrix}
\]  

(2)

in which the ideal \( \langle |T_{11}|, |T_{22}| \rangle \) generated by determinants of \( T_{11} \) and \( T_{22} \) is equal to \( \langle 1 \rangle \), then \( T \) is unimodular-equivalent over \( D \) to the block diagonal form

\[
T_D = \begin{pmatrix}
T_{11} & 0 \\
0 & T_{22}
\end{pmatrix}
\]  

(3)

Proof. Since the ideal \( \langle |T_{11}|, |T_{22}| \rangle = \langle 1 \rangle \) then by Bezout’s identity, there exist polynomials \( \alpha, \beta \in D \) such that \( \alpha |T_{11}| + \beta |T_{22}| = 1 \). Then considering the unimodular transformation:

\[
\begin{pmatrix}
I_r & -\beta \cdot T_{12} \cdot \text{adj}(T_{22}) \\
0 & I_s
\end{pmatrix}
\begin{pmatrix}
T_{11} & T_{12} \\
0 & T_{22}
\end{pmatrix}
\begin{pmatrix}
I_r & -\alpha \cdot \text{adj}(T_{11}) \cdot T_{12} \\
0 & I_s
\end{pmatrix} = \begin{pmatrix}
T_{11} & 0 \\
0 & T_{22}
\end{pmatrix}
\]  

(4)

where \( r \) and \( s \) depend on the sizes of \( T_{11} \) and \( T_{22} \) respectively, establishes that the matrix \( T \) is unimodular-equivalent over \( D \) to the matrix \( T_D \) in (3).

In the case of a similarity transformation, we can state the following result.

Lemma 2.2 Let \( T \in D^{p \times p} \) be a polynomial matrix in block upper triangular form (2). Then a sufficient condition for \( T \) to be similar over \( D \) to \( T_D \) in (3) is the existence of a matrix \( X \) over \( D \) which satisfies the matrix equation:

\[
T_{11}X - XT_{22} = T_{12}
\]  

(5)

Proof. The result follows by considering the similarity transformation

\[
\begin{pmatrix}
I_r & -X \\
0 & I_s
\end{pmatrix}
\begin{pmatrix}
T_{11} & 0 \\
0 & T_{22}
\end{pmatrix}
\begin{pmatrix}
I_r & X \\
0 & I_s
\end{pmatrix} = \begin{pmatrix}
T_{11} & T_{12} \\
0 & T_{22}
\end{pmatrix}
\]  

(6)

Similar results to Theorem 2.1 and Lemma 2.2 can be obtained for matrices that are block lower triangular.

Example 2.3 Consider the following system of delay-differential equations:

\[
\begin{align*}
\ddot{r}(t-h) + r(t) + \ddot{v}(t) - v(t) + w(t-h) - w(t) &= 0 \\
-r(t-2h) - \ddot{r}(t-2h) - \ddot{r}(t-h) + \ddot{u}(t-h) - u(t) + 2\ddot{v}(t-h) - \dddot{v}(t-h) + 3\ddot{w}(t-h) - w(t) - \ddot{w}(t-2h) &= 0 \\
\dot{v}(t) + v(t-h) + w(t-h) &= 0 \\
-v(t-h) + \dddot{v}(t) + \dot{v}(t-h) + \dddot{w}(t) - w(t-h) + \dddot{w}(t-h) &= 0
\end{align*}
\]  

(7)
where \( r(t), u(t), v(t), w(t) \) are unknown functions of \( t \). The system (7) can be written in matrix form \( T(s, z)\rho(t) = 0 \), where \( s = \partial/\partial t \) and \( z \) is a backward shift operator i.e. \( zf(t) = f(t - h) \) with \( h \in \mathbb{R}^+ \). \( \rho(t) \) is the vector of the functions \( r, u, v, w \) and \( T \) is given by:

\[
T = \begin{pmatrix}
zs^2 + 1 & 0 & s^2 - 1 & z - 1 \\
-zs^2 - sz(zs^2 + 1) & zs^2 - 1 & sz(2 - s^2) & sz(3 - z) - 1 \\
0 & 0 & s + z & z \\
0 & 0 & -z + s(s + z) & s - z + sz
\end{pmatrix}
\]

Writing \( T \) in the form (2) yields

\[
T_{11} = \begin{pmatrix}
zs^2 + 1 & 0 \\
-zs^2 - sz(zs^2 + 1) & zs^2 - 1
\end{pmatrix},
T_{22} = \begin{pmatrix}
s + z & z \\
-z + s(s + z) & s - z + sz
\end{pmatrix},

|T_{11}| = s^4z^2 - 1, \quad |T_{22}| = s^2.
\]

Using the Maple package OreModules [2], we can find \( \alpha = -1 \) and \( \beta = s^2z^2 \) satisfying \( \alpha|T_{11}| + \beta|T_{22}| = 1 \) and it follows that the unimodular transformation (4) with

\[
M(s, z) = \begin{pmatrix}
I_2 & G \\
0 & I_2
\end{pmatrix}, \quad N(s, z) = \begin{pmatrix}
I_2 & Q \\
0 & I_2
\end{pmatrix}
\]

where,

\[
G(1, 1) = (-z^2 - z^3)s^5 + (-z^2 + 2z^3)s^4 + (z^4 + z^2)s^3 - s^2z^4
\]

\[
G(1, 2) = s^4z^3 + (-z^3 + z^2)s^3 - s^2z^4
\]

\[
G(2, 1) = \left(\frac{9}{2} z^4 + \frac{9}{2} z^5\right)s^8 + \left(-9 z^5 + \frac{9}{2} z^4\right)s^7 + \left(\frac{11}{2} z^3 + z^4 - \frac{9}{2} z^6\right)s^6
\]

\[
+ \left(\frac{15}{2} z^3 - \frac{13}{2} z^4 + \frac{9}{2} z^6 + \frac{9}{2} z^5\right)s^5 + \left(-\frac{29}{2} z^5 - \frac{13}{2} z^3 - z^2 + \frac{11}{2} z^4\right)s^4 + \left(\frac{11}{2} z^5 - z^3 - \frac{9}{2} z^6 - \frac{11}{2} z^4\right)s^3 + \left(z^2 + \frac{9}{2} z^6\right)s^2
\]

\[
G(2, 2) = -\frac{9}{2} z^5s^7 + \left(\frac{9}{2} z^5 - \frac{9}{2} z^4\right)s^6 + \left(-\frac{11}{2} z^4 + \frac{9}{2} z^6\right)s^5
\]

\[
+ \left(-\frac{9}{2} z^5 + \frac{11}{2} z^4 - \frac{15}{2} z^3\right)s^4 + \left(10 z^5 - \frac{11}{2} z^4 + z^2\right)s^3 + \left(z^3 + \frac{9}{2} z^6\right)s^2
\]

and

\[
Q = \begin{pmatrix}
1 + zs^4 + (-z - 1)s^2 & (z^2 - z)s^2 - z + 1 \\
zs^2 + z^2s^3 + zs & z^3 - z^2 + 2z^2s^3 + 2zs - zs^2 - 1
\end{pmatrix}
\]
reduces the matrix \( T(s, z) \) to the block diagonal form (3):

\[
T_D = \begin{pmatrix}
zs^2 + 1 & 0 & 0 & 0 \\
-z^2 - sz (zs^2 + 1) & zs^2 - 1 & 0 & 0 \\
0 & 0 & s + z & z \\
0 & 0 & -z + s (s + z) & s - z + sz
\end{pmatrix}
\]  

Now the block matrix in the (1,1) position in \( T_D(s, z) \), i.e. the matrix

\[
T_{D1} = \begin{pmatrix}
zs^2 + 1 & 0 \\
-z^2 - sz (zs^2 + 1) & zs^2 - 1
\end{pmatrix}
\]  

can be reduced to a diagonal form using the same procedure applied before or alternatively, using Lemma 2.2. In fact taking

\[
X = -\frac{1}{2} z (zs^3 + s + z)
\]

satisfies the equation (5), i.e. \( T_{D1}(1, 1)X - XT_{D1}(2, 2) = T_{D1}(2, 1) \). It follows that the similarity transformation:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
-X & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \quad \begin{pmatrix}
1 & 0 & 0 & 0 \\
X & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} = \tilde{T}_D
\]  

yields the matrix:

\[
\tilde{T}_D = \begin{pmatrix}
zs^2 + 1 & 0 & 0 & 0 \\
0 & zs^2 - 1 & 0 & 0 \\
0 & 0 & s + z & z \\
0 & 0 & -z + s (s + z) & s - z + sz
\end{pmatrix}
\]  

A further simplification is possible by applying an elementary row operation

Row4 = \(-s\times Row3\) which reduces the matrix \( \tilde{T}_D \) to

\[
\tilde{T}_F = \begin{pmatrix}
zs^2 + 1 & 0 & 0 & 0 \\
0 & zs^2 - 1 & 0 & 0 \\
0 & 0 & s + z & z \\
0 & 0 & -z & s - z
\end{pmatrix}
\]  

The original matrix \( T \) is connected with the reduced matrix \( \tilde{T}_F \) by the unimodular transformation \( \tilde{T}_D = \bar{M}T\bar{N} \), in which \( \bar{M} \) and \( \bar{N} \) are given by:

\[
\bar{M} = \begin{pmatrix}
1 & 0 \\
-X & 1 \\
0 & 1 \\
0 & -s
\end{pmatrix} \quad G, \quad \bar{N} = \begin{pmatrix}
1 & 0 \\
X & 1 \\
0 & I_2
\end{pmatrix}
\]
i.e., the system in (7) is equivalent to the simplified decomposed system:

\[
\begin{align*}
\dot{f}(t-h) + f(t) &= 0, \\
\dot{g}(t-h) - g(t) &= 0
\end{align*}
\]

\[
\begin{align*}
\dot{l}(t) + l(t-h) + k(t-h) &= 0 \\
\dot{k}(t) - \dot{k}(t) + k(t-h) &= 0
\end{align*}
\]

where \( f(t), g(t), l(t), k(t) \) are functions in \( t \) related to \( r(t), u(t), v(t), w(t) \) by

\[
\begin{bmatrix}
f(t) & g(t) & l(t) & k(t)
\end{bmatrix}^T = \begin{bmatrix}
r(t) & u(t) & v(t) & w(t)
\end{bmatrix}^T (\bar{N}^{-1})^T \tag{18}
\]

### 3 Conclusions

We have presented a constructive result for the decomposition of a class of linear functional systems. The result can be easily implemented on modern computer algebra systems such as Maple.

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### References


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