Upper Bounds for Ruin Probability in a Generalized Risk Process under Rates of Interest with Homogenous Markov Chain Claims and Homogenous Markov Chain Premiums

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Abstract

In this study, we built upper bounds for ruin probabilities of generalized risk processes under rates of interest with homogenous Markov chain claims and homogenous Markov chain Premiums. Generalized Lundberg inequalities for ruin probabilities of these processes are derived by the martingale approach.

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1. Introduction

As one of the most important topics in risk theory, the ruin problem in stochastic environments has been studied by many researchers [4], [5]. In classical risk model, the claim number process was assumed to be a Poisson process and the individual claim amounts were described as independent and identically distributed random variables. In recent years, the classical risk process has been...
extended to more practical and real situations. For most of the investigations treated in risk theory, it is very significant to deal with the risks that rise from monetary inflation in the insurance and finance market, and also to consider the operation uncertainties in administration of financial capital. Teugels and Sundt [7], [8] studied ruin probability under the compound Poisson risk model with the effects of constant rate. Yang [10] given both exponential and non–exponential upper bounds for ruin probabilities in a risk model with constant interest force and independent premiums and claims. Xu and Wang [9] given upper bounds for ruin probabilities in a risk model with interest force and independent premiums and claims with Markov chain interest rate. Cai [1], [2] considered the ruin probabilities in two risk models, with independent premiums and claims and used a first – order autoregressive process to model the rates of in interest. Cai and Dickson [3] built Lundberg inequalities for ruin probabilities in two discrete-time risk process with a Markov chain interest model and independent premiums and claims.

In this paper, we study the models considered by Cai and Dickson [3] to the case homogenous markov chain claims, homogenous markov chain premiums and independent interests. The main difference between the model in our paper and the one in Cai and Dickson [3] is that claims, premiums in our model are assumed to follow homogeneous Markov chains. Generalized Lundberg inequalities for ruin probabilities of these processes are derived by the martingale approach.

To establish probability inequalities for ruin probabilities of these models, we study two style of premium collections. On one hand of the premiums are collected at the begining of each period then the surplus process \( \{U(t)\}_{t \geq 1} \) with initial surplus \( u \) can be written as

\[ U(t) = U_{t-1}(1 + I_n) + X_n - Y_n, \]

which can be rearranged as

\[ U(t) = u \prod_{k=1}^{n} (1 + I_k) + \sum_{k=1}^{n} (X_k - Y_k) \prod_{j=1}^{n} (1 + I_j). \]

On the other hand, if the premiums are collected at the end of each period, then the surplus process \( \{U(t)\}_{t \geq 1} \) with initial surplus \( u \) can be written as

\[ U(t) = (U_{t-1} + X_n)(1 + I_n) - Y_n, \]

which is equivalent to

\[ U(t) = u \prod_{k=1}^{n} (1 + I_k) + \sum_{k=1}^{n} [X_k(1 + I_k) - Y_k] \prod_{j=1}^{n} (1 + I_j). \]

where throughout this paper, we denote \( \prod_{i=a}^{b} x_i = 1 \) and \( \sum_{i=a}^{b} x_i = 0 \) if \( a > b \).

We assume that:

**Assumption 1.1** \( U_{t-1}^{(1)} = U_{t-1}^{(2)} = u > 0 \).
**Assumption 1.2** $X = \{X_n\}_{n\geq0}$ is homogeneous Markov chain, $X_n$ take values in a set of non-negative numbers $E_X = \{x_1, x_2, \ldots, x_M\}$ with $X_0 = x_i \in E_X$ and

$$p_j = P\left[X_{m+1} = x_j \mid X_m = x_i\right], (m \in N), x_j, x_i \in E_X,$$

where $0 \leq p_j \leq 1, \sum_{j=1}^{M} p_j = 1.$

**Assumption 1.3** $Y = \{Y_n\}_{n\geq0}$ is a homogeneous Markov chain, $Y_n$ take values in a set of non-negative numbers $E_Y = \{y_1, y_2, \ldots, y_K\}$ with $Y_0 = y_i \in E_Y$ and

$$q_s = P\left[Y_{m+1} = y_s \mid Y_m = y_r\right], (m \in N), y_s, y_r \in E_Y,$$

where $0 \leq q_s \leq 1, \sum_{s=1}^{K} q_s = 1.$

**Assumption 1.4** $I = \{I_n\}_{n\geq0}$ is sequence of independent and identically distributed non-negative continuous random variables with the same distribution function $F(t) = P(I_t \leq t).$

**Assumption 1.5** $X, Y$ and $I$ are assumed to be independent.

We define the finite time and ultimate ruin probabilities in model (1) with assumption 1.1 to assumption 1.5, respectively, by

$$\psi^{(1)}(u, x_i, y_r) = P\left(\bigcup_{k=1}^{n} (U^{(1)}_k < 0) \mid U^{(1)}_0 = u, X_0 = x, Y_0 = y_r\right),$$

$$\psi^{(1)}(u, x_i, y_r) = \lim_{n \to \infty} \psi^{(1)}(u, x_i, y_r) = P\left(\bigcup_{k=1}^{n} (U^{(1)}_k < 0) \mid U^{(1)}_0 = u, X_0 = x, I_0 = y_r\right).$$

Similarly, we define the finite time and ultimate ruin probabilities in model (3) with assumption 1.1 to assumption 1.5, respectively, by

$$\psi^{(2)}(u, x_i, y_r) = P\left(\bigcup_{k=1}^{n} (U^{(2)}_k < 0) \mid U^{(2)}_0 = u, Y_0 = x, I_0 = y_r\right),$$

$$\psi^{(2)}(u, x_i, y_r) = \lim_{n \to \infty} \psi^{(2)}(u, x_i, y_r) = P\left(\bigcup_{k=1}^{n} (U^{(2)}_k < 0) \mid U^{(2)}_0 = u, Y_0 = x, I_0 = y_r\right).$$

In this study, we derive probability inequalities for $\psi^{(1)}(u, x_i, y_r)$ and $\psi^{(2)}(u, x_i, y_r)$ by the martingale approach. The paper is organized as follows: in section 2, we derive probability inequalities for $\psi^{(1)}(u, x_i, y_r)$ and $\psi^{(2)}(u, x_i, y_r)$ by the martingale approach. Finally, we conclude our paper in Section 3.

### 2. Upper Bounds for Ruin Probability by the Martingale approach

To establish probability inequalities for ruin probabilities of model (1), we first prove the following Lemma.
Lemma 2.1. Let model (1) satisfy assumptions 1.1 to 1.5. If any \( x_i, y_i \in E_x, y_r \in E_y \),
\[ E(Y|Y_o = y_r) < E(X|X_o = x_o) \] and \( P\left((Y_i - X_i)(1 + I_i)^{-1} > 0|X_o = x_o, Y_o = y_r\right) > 0 \), then there exists a unique positive constant \( R_o \) satisfying:
\[ E\left(e^{R_o(Y_i - X_i)(1 + I_i)^{-1}}\right)|X_o = x_o, Y_o = y_r\right) = 1. \quad (10) \]

Proof.
Define
\[ f_o(t) = E\left(e^{(Y_i - X_i)(1 + I_i)^{-1}}\right)X_o = x_o, Y_o = y_r\right) - 1, t \in (0, +\infty). \]
with \( g_o(t) = E\left(e^{(Y_i - X_i)(1 + I_i)^{-1}}\right)X_o = x_o, Y_o = y_r\right) \).
Let \( C_1 = \max\{x_i : x_i \in E_x\} \geq 0, C_2 = \max\{y_r : y_r \in E_y\} \geq 0. \)
With any \( t \in (0, +\infty) \), from \( X_i \geq 0, I_i \geq 0, Y_i \geq 0 \), we have
\[ 0 \leq g_o(t) = E\left(e^{(Y_i - X_i)(1 + I_i)^{-1}}\right)|X_o = x_o, Y_o = y_r\right) \leq E\left(e^{c_2}\right)|X_o = x_o, Y_o = y_r\right) < +\infty, \]
\[ E\left(Y_i - X_i \left(\frac{(Y_i - X_i)}{1 + I_i}\right)\right)|X_o = x_o, Y_o = y_r\right) \leq E\left(C_2 e^{c_2}\right) |X_o = x_o, Y_o = y_r\right) \leq C_2 e^{c_2} < +\infty. \]
In addition, we also have
\[ E\left(Y_i - X_i \left(\frac{(Y_i - X_i)}{1 + I_i}\right)\right)|X_o = x_o, Y_o = y_r\right) \geq -C_2 e^{c_2} E\left(\frac{1}{1 + I_i}\right) \geq -C_2 e^{c_2} > -\infty, \]
\[ 0 \leq E\left(\frac{(Y_i - X_i)^2}{1 + I_i}\right)|X_o = x_o, Y_o = y_r\right) \leq 2(C_2^2 + C_1^2)e^{c_2} < +\infty. \]
This implies that \( g_o(t) \) has \( n \)-th derivative function on \((0, +\infty)\) with \( n = 1, 2. \)
Thus, \( f_o(t) \) has \( n \)-th derivative function on \((0, +\infty)\) with \( n = 1, 2 \) and
\[ f_o'(t) = E\left((Y_i - X_i)(1 + I_i)^{-1} e^{(Y_i - X_i)(1 + I_i)^{-1}}|X_o = x_o, I_o = y_r\right), \]
\[ f_o''(t) = E\left((Y_i - X_i)(1 + I_i)^{-1} e^{(Y_i - X_i)(1 + I_i)^{-1}}|X_o = x_o, I_o = y_r\right). \]
Hence \( f_o'(t) \geq 0 \).
This implies that
\[ f_o(t) \] is a convex function with \( f_o'(0) = 0 \) \quad (11) \] and
\[ f_o''(0) = E\left((Y_i - X_i)(1 + I_i)^{-1}|X_o = x_o, Y_o = y_r\right) \leq E(Y)|Y_o = y_r\right) - E(X)|X_o = x_o \right) < 0. \quad (12) \]
By \( P\left( (Y_i - X_i)(1 + I_i)^{-1} > 0 \middle| X_o = x_o, Y_o = y_o \right) > 0 \), we can find some constant \( \delta_u > 0 \) such that
\[
P\left( (Y_i - X_i)(1 + I_i)^{-1} > \delta_u > 0 \middle| X_o = x_o, Y_o = y_o \right) > 0.
\]
Then, we get
\[
f_o(t) = E \left\{ e^{(Y_i - X_i)(1 + I_i)^{-1}} \middle| X_o = x_o, Y_o = y_o \right\} - 1
\]
\[
\geq E \left\{ e^{(Y_i - X_i)(1 + I_i)^{-1}} \middle| X_o = x_o, Y_o = y_o, I_o > 0 \right\} - 1
\]
\[
\geq e^{\delta_u \cdot P\left( (Y_i - X_i)(1 + I_i)^{-1} > \delta_u \middle| X_o = x_o, Y_o = y_o \right)} - 1.
\]
This implies that
\[
\lim_{t \to +\infty} f_o(t) = +\infty.
\]  
(13)
From (11), (12) and (13) there exists a unique positive constant \( R_o \) satisfying (10). This completes the proof.

Let: \( R_o = \min\left\{ R_o > 0 : E \left\{ e^{R_o \cdot (Y_i - X_i)(1 + I_i)^{-1}} \middle| X_o = x_o, Y_o = y_o \right\} = 1, x_i \in E_X, y_i \in E_Y \right\} \).

Thus, \( R_o > 0 \).

Using Lemma 2.1, we obtain a probability inequality for \( \psi^{(i)}(u, x_i, y_i) \) by the martingale approach.

**Theorem 2.1.** If model (1) satisfies assumptions 1.1 to 1.5 and (9) then, for any \( u > 0, x_i \in E_X, y_i \in E_Y \),
\[
\psi^{(i)}(u, x_i, y_i) \leq e^{-R_o u}.
\]  
(14)

**Proof.**
Consider the process \( \{U^{(1)}_{n}\} \) is given by (2), we let
\[
V^{(1)}_{n} = U^{(1)}_{n} \prod_{j=1}^{n} (1 + I_j)^{-1} = u + \sum_{j=1}^{n} (X_j - Y_j) \prod_{i=1}^{j} (1 + I_i)^{-1},
\]  
(15)
and \( S^{(1)}_{n} = e^{-R_o \cdot V^{(1)}_{n}} \). Thus, we have: \( S^{(1)}_{n+1} = S^{(1)}_{n} e^{-R_o \cdot (X_{n+1} - Y_{n+1}) \prod_{i=1}^{n} (1 + I_i)^{-1}} \).

With any \( n \geq 1 \), we have
\[
E \left( S^{(1)}_{n+1} \middle| X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_n, I_1, I_2, \ldots, I_n \right)
\]
\[
= S^{(1)}_{n} E \left( e^{-R_o \cdot (X_{n+1} - Y_{n+1}) \prod_{i=1}^{n} (1 + I_i)^{-1}} \middle| X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_n, I_1, I_2, \ldots, I_n \right)
\]
\[
= S^{(1)}_{n} E \left( e^{-R_o \cdot (X_{n+1} - Y_{n+1}) \prod_{i=1}^{n} (1 + I_i)^{-1}} \middle| X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_n, I_1, I_2, \ldots, I_n \right).
\]
From $0 \leq \prod_{i=1}^{n} (1 + I_i)^{-1} \leq 1$ and Jensen’s inequality implies

$$S_n^{(1)} E \left( e^{-R_n (X_{n0} - Y_{n0}) (1 + I_{n0})^{-1} \prod_{i=1}^{n} (1 + I_i)^{-1}} \bigg| X_1, X_2, ..., X_n, Y_1, Y_2, ..., Y_n, I_1, I_2, ..., I_n \right)$$

$$\leq S_n^{(1)} E \left( e^{-R_n (X_{n0} - Y_{n0}) (1 + I_{n0})^{-1} \prod_{i=1}^{n} (1 + I_i)^{-1}} \bigg| X_1, X_2, ..., X_n, Y_1, Y_2, ..., Y_n, I_1, I_2, ..., I_n \right).$$

In addition,

$$E \left( e^{-R_n (X_{n0} - Y_{n0}) (1 + I_{n0})^{-1} \prod_{i=1}^{n} (1 + I_i)^{-1}} \bigg| X_1, X_2, ..., X_n, Y_1, Y_2, ..., Y_n, I_1, I_2, ..., I_n \right)$$

$$= E \left( e^{-R_n (X_{n0} - Y_{n0}) (1 + I_{n0})^{-1} \prod_{i=1}^{n} (1 + I_i)^{-1}} \bigg| X_n, Y_n \right) = E \left( e^{-R_n (X_{n0} - Y_{n0}) (1 + I_{n0})^{-1} \prod_{i=1}^{n} (1 + I_i)^{-1}} \bigg| X_n, Y_n \right) \leq 1.$$

Thus, we have

$$E \left( S_n^{(1)} \bigg| X_1, X_2, ..., X_n, Y_1, Y_2, ..., Y_n, I_1, I_2, ..., I_n \right) \leq S_n^{(1)}.$$

Hence, \{S_n^{(1)}, n = 1, 2, ...\} is a supermartingale with respect to the $\sigma$ - filtration $S_n^{(1)} = \sigma\{X_1, ..., X_n, Y_1, ..., Y_n, I_1, ..., I_n\}.$

Define $T_{\nu}^{(1)} = \min \left\{ n : V_n^{(1)} < 0 \big| U_n^{(1)} = u, X_0 = x_0, Y_0 = y_0 \right\},$ with $V_n^{(1)}$ is given by (15).

Hence, $T_{\nu}^{(1)}$ is a stopping time and $n \wedge T_{\nu}^{(1)} = \min(n, T_{\nu}^{(1)})$ is a finite stopping time.

Therefore, from the optional stopping theorem for supermartingales, we have

$$E \left( S_n^{(1)} \bigg| n \wedge T_{\nu}^{(1)} \right) \leq E \left( S_{n \wedge T_{\nu}^{(1)}}^{(1)} \right) = e^{-R_{\nu} a}.$$

This implies that

$$e^{-R_{\nu} a} \geq E \left( S_n^{(1)} \bigg| n \wedge T_{\nu}^{(1)} \right) \geq E \left( S_n^{(1)} \bigg| n \wedge T_{\nu}^{(1)} \right) \cdot 1_{V_n^{(1)} \leq a} = E \left( S_n^{(1)} \cdot 1_{V_n^{(1)} \leq a} \right) = E \left( e^{-R_{\nu} \gamma^{(1)}} \cdot 1_{V_n^{(1)} \leq a} \right).$$

(16)

From $V_n^{(1)} < 0$ and $R_{\nu} > 0$ then (16) becomes

$$e^{-R_{\nu} a} \geq E \left( 1_{V_n^{(1)} \leq a} \right) = P(T_{\nu}^{(1)} \leq n).$$

(17)

In addition,

$$\psi_n^{(1)} (u, x, y) = P \left( \bigcup_{k=1}^{n} (U_{\nu}^{(1)} < 0) \bigg| U_{\nu}^{(1)} = u, X_0 = x_0, Y_0 = y_0 \right)$$

$$= P \left( \bigcup_{k=1}^{n} (V_n^{(1)} < 0) \bigg| U_n^{(1)} = u, X_0 = x_0, Y_0 = y_0 \right) = P(T_{\nu}^{(1)} \leq n).$$

(18)

Combining (17) and (18) imply that

$$\psi_n^{(1)} (u, x, y) \leq e^{-R_{\nu} a}.$$

(19)

Thus, (14) follows by letting $n \to \infty$ in (19).

Similarly, we have Lemma 2.2.

**Lemma 2.2.** Let model (3) satisfy assumptions 1.1 to 1.5. If any $x_i \in E_X, y_i \in E_Y,$
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\[ E\left[ Y_t (1 + I_t)^{-1} \mid Y_o = y_r \right] < E(X_t \mid X_o = x_t) \] and
\[ P\left( Y_t (1 + I_t)^{-1} - X_t > 0 \mid X_o = x_t, Y_o = y_r \right) > 0, \]
then, there exists a unique positive constant \( R_o \) satisfying:
\[ E\left( e^{R_o [Y_t (1 + I_t)^{-1} - X_t]} \right) \mid X_o = x_t, Y_o = y_r = 1. \]

**Proof.**
Define
\[ f_o(t) = E\left( e^{\frac{Y_t (1 + I_t)^{-1} - X_t}{Y_t (1 + I_t)^{-1} - X_t}} \right) \mid X_o = x_t, Y_o = y_r \]
\[ = h_o(t) - 1, \text{ with } h_o(t) = E\left( e^{\frac{Y_t (1 + I_t)^{-1} - X_t}{Y_t (1 + I_t)^{-1} - X_t}} \right) \mid X_o = x_t, Y_o = y_r. \]

Let \( C_1 = \max \{ x_t : x_t \in E_{Y_t} \} \geq 0, C_2 = \max \{ y_r : y_r \in E_{Y_t} \} \geq 0. \)

With any \( t \in (0; +\infty), \) from \( X_t \geq 0, I_t \geq 0, Y_t \geq 0, \) we have
\[ 0 \leq h_o(t) = E\left( e^{\frac{X_t (1 + I_t)^{-1}}{X_t (1 + I_t)^{-1} - X_t}} \right) \mid X_o = x_t, Y_o = y_r \leq e^{C_2} < +\infty, \]
\[ E\left( \frac{Y_t}{1 + I_t} - X_t \right) e^{\frac{X_t (1 + I_t)^{-1}}{X_t (1 + I_t)^{-1} - X_t}} \mid X_o = x_t, Y_o = y_r \leq C_2 e^{C_2} < +\infty. \]

In addition, we also have
\[ E\left( \frac{Y_t}{1 + I_t} - X_t \right) e^{\frac{X_t (1 + I_t)^{-1}}{X_t (1 + I_t)^{-1} - X_t}} \mid X_o = x_t, Y_o = y_r \geq -C_2 e^{C_2} > -\infty, \]
\[ 0 \leq E\left( \frac{Y_t}{1 + I_t} - X_t \right)^2 e^{\frac{X_t (1 + I_t)^{-1}}{X_t (1 + I_t)^{-1} - X_t}} \mid X_o = x_t, Y_o = y_r \leq 2(C_2^2 + C_1^2) e^{C_2} < +\infty. \]

This implies that \( h_o(t) \) has \( n \)-th derivative function on \((0, +\infty)\) with \( n = 1, 2. \)

Thus, \( f_o(t) \) has \( n \)-th derivative function on \((0, +\infty)\) with \( n = 1, 2 \) and
\[ f_o(t) = E\left( Y_t (1 + I_t)^{-1} - X_t \right) e^{\frac{Y_t (1 + I_t)^{-1} - X_t}{Y_t (1 + I_t)^{-1} - X_t}} \mid X_o = x_t, Y_o = y_r \]
\[ = f_o(t) - 1, \text{ with } f_o(t) = E\left( Y_t (1 + I_t)^{-1} - X_t \right) e^{\frac{Y_t (1 + I_t)^{-1} - X_t}{Y_t (1 + I_t)^{-1} - X_t}} \mid X_o = x_t, Y_o = y_r. \]

Hence \( f_o(t) \geq 0. \) Which implies that
\[ f_o(t) \text{ is a convex function with } f_o(0) = 0 \]
and
\[ f_o(0) = E\left( Y_t (1 + I_t)^{-1} - X_t \right) \mid X_o = x_t, Y_o = y_r < 0. \]
By \( P\left((Y(1+I)^{-1} - X_i) > 0 \right| X_o = x_i, Y_o = y_r) > 0 \), we can find some constant \( \delta_u > 0 \) such that
\[
P\left((Y(1+I)^{-1} - X_i) > \delta_u \right| X_o = x_i, Y_o = y_r) > 0.
\]
Then, we get
\[
f_u(t) = E \left(e^{t(Y(1+I)^{-1} - X_i)} \middle| X_o = x_i, Y_o = y_r\right) - 1
\geq E \left(e^{t(Y(1+I)^{-1} - X_i)} \middle| X_o = x_i, Y_o = y_r\right) 1_{\{[X(1+I)^{-1} - X_i] > \delta_u \}} - 1
\geq e^{\delta_u}.P\left((Y(1+I)^{-1} - X_i) > \delta_u \right| X_o = x_i, Y_o = y_r\right) - 1.
\]
This implies that
\[
\lim_{t \to +\infty} f_u(t) = +\infty. \quad (24)
\]
From (22), (23) and (24) there exists a unique positive constant \( R_u \) satisfying (21).
This completes the proof.

Let: \( \overline{R_u} = \min \left\{ R_u \mid 0 < E \left(e^{R_u(Y(1+I)^{-1} - X_i)} \middle| X_o = x_i, Y_o = y_r\right) = 1, x_i \in E_x, y_r \in E_y \right\} \).
Thus, \( \overline{R_u} > 0 \).
Use Lemma 3.2, we obtain a probability inequality for \( \psi^{(2)}(u,x_i,y_r) \) by the martingale approach.

**Theorem 2.2.** If model (3) satisfies assumptions 1.1 to 1.5 and (20) then, for any \( u > 0, x_i \in E_x, y_r \in E_y, \)
\[
\psi^{(2)}(u,x_i,y_r) \leq e^{-\overline{R_u}}. \quad (25)
\]

**Proof.**
Consider the process \( \{U_n^{(2)}\} \) given by (4), we let
\[
V_n^{(2)} = U_n^{(2)} \prod_{j=1}^n (1+I_j)^{-1} = u + \sum_{j=1}^n (X_j(1+I_j) - Y_j) \prod_{j=1}^n (1+I_j)^{-1}, \quad (26)
\]
and \( S_n^{(2)} = e^{-\overline{R_u}V_n^{(2)}} \). Thus, we have
\[
S_n^{(2)} = S_1^{(2)} e^{-\overline{R_u}V_n^{(2)}} = e^{-\overline{R_u}V_n^{(2)}} = e^{-\overline{R_u}V_n^{(2)}} \prod_{m=1}^n (1+I_m)^{-1}.
\]
With any \( n \geq 1 \), we have
\[
E \left(S_n^{(2)} \mid X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_n, I_1, I_2, \ldots, I_n\right)
= S_1^{(2)} E \left(e^{-\overline{R_u}V_n^{(2)}} \prod_{m=1}^n (1+I_m)^{-1} \mid X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_n, I_1, I_2, \ldots, I_n\right)
\]

\[= S_1^{(2)} \left(e^{-\overline{R_u}V_n^{(2)}} \prod_{m=1}^n (1+I_m)^{-1} \right) \left(X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_n, I_1, I_2, \ldots, I_n\right) \]
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\[ S_n^{(2)} = \mathbb{E} \left( e^{-\mathcal{R}_n(X_{n+1},Y_{n+1}(I_{n+1}^{-1}))} \prod_{i=1}^{n} X_i, X_{n+1}, Y_i, Y_{n+1}, I_i, I_{n+1}^{-1} \right) \]

From \( 0 \leq \prod_{i=1}^{n} (1 + I_i)^{-1} \leq 1 \) and Jensen’s inequality implies
\[ E \left( S_n^{(2)} \right) \geq S_n^{(2)} \mathbb{E} \left( e^{-\mathcal{R}_n(X_{n+1},Y_{n+1}(I_{n+1}^{-1}))} \prod_{i=1}^{n} X_i, X_{n+1}, Y_i, Y_{n+1}, I_i, I_{n+1}^{-1} \right) \]
\leq S_n^{(2)} \mathbb{E} \left( e^{-\mathcal{R}_n(X_{n+1},Y_{n+1}(I_{n+1}^{-1}))} \prod_{i=1}^{n} X_i, X_{n+1}, Y_i, Y_{n+1}, I_i, I_{n+1}^{-1} \right) .

In addition,
\[ E \left( e^{-\mathcal{R}_n(X_{n+1},Y_{n+1}(I_{n+1}^{-1}))} \prod_{i=1}^{n} X_i, Y_i \right) = E \left( e^{-\mathcal{R}_n(X_{n+1},Y_{n+1}(I_{n+1}^{-1}))} \prod_{i=1}^{n} X_i, Y_i \right) \leq 1 .

Thus, we have
\[ E \left( S_n^{(2)} \right) \leq S_n^{(2)} .
\]

Hence, \( \{ S_n^{(2)}, n = 1, 2, \ldots \} \) is a supermartingale with respect to the \( \sigma \)-filtration
\[ \mathcal{F}_n^{(2)} = \sigma \{ X_1, \ldots, X_n, Y_1, \ldots, Y_n, I_1, \ldots, I_n \} .
\]

Define \( T_n^{(2)} = \min \left\{ n : V_n^{(2)} < 0 \left( U_n^{(2)} = u, X_o = x_r, Y_o = y_r \right) \right\} \), with \( V_n^{(2)} \) is given by (26).

Hence, \( T_n^{(2)} \) is a stopping time and \( n \wedge T_n^{(2)} = \min(n, T_n^{(2)}) \) is a finite stopping time.

Therefore, from the optional stopping theorem for supermartingales, we have
\[ E \left( S_n^{(2)} \right) \leq E \left( S_n^{(2)} \right) = e^{-\mathcal{R}_n} .
\]

This implies that
\[ e^{-\mathcal{R}_n} \geq E \left( S_n^{(2)} \right) = E \left( S_n^{(2)} \right) \leq n .
\]

From \( V_n^{(2)} < 0 \) and \( \mathcal{R}_n > 0 \) then (27) becomes
\[ e^{-\mathcal{R}_n} \geq E \left( 1_{(V_n^{(2)} < 0)} \right) = P(T_n^{(2)} \leq n) . \tag{28} \]

In addition,
\[ \psi_n^{(2)}(u, x_r, y_r) = P \left( \bigcup_{k=1}^{n} \left( U_k^{(2)} < 0 \right) \bigg| U_0^{(2)} = u, X_o = x_r, Y_o = y_r \right) \]
\[ = P \left( \bigcup_{k=1}^{n} \left( V_k^{(2)} < 0 \right) \bigg| U_0^{(2)} = u, X_o = x_r, Y_o = y_r \right) = P(T_n^{(2)} \leq n) . \tag{29} \]

Combining (28) and (29) imply that
\[ \psi_n^{(2)}(u, x_r, y_r) \leq e^{-\mathcal{R}_n} . \tag{30} \]

Thus, (30) follows by letting \( n \to \infty \) in (25).

This completes the proof.
3. Conclusion

Our main results in this paper, Theorem 2.1 and Theorem 2.2 give upper bounds for $\psi^{(1)}(u,x,y)$ and $\psi^{(2)}(u,x,y)$ by the martingale approach.

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