Doubly Connected Domination in the Corona and Lexicographic Product of Graphs

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Abstract

Let $G$ be a simple connected graph. A connected dominating set $S \subseteq V(G)$ is called a doubly connected dominating set of $G$ if the subgraph $(V(G)\setminus S)$ induced by $V(G)\setminus S$ is connected. In this paper, we show that any three positive integers $a$, $b$, and $c$ with $4 \leq a \leq b \leq c$, where $b \leq 2a - 3$ if $a$ is odd and $b \leq 2a - 2$ if $a$ is even, are realizable as the total domination, connected domination, and doubly connected domination numbers. Also, we characterize the doubly connected dominating sets in the corona and lexicographic product of graphs and determine the corresponding doubly connected domination numbers.

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1 Introduction

Cyman et al. in [6] introduced the concept of doubly connected domination and obtained characterizations for the lower and upper bounds of the doubly connected domination number. Relationships of the doubly connected domination number to other invariants such as connected domination (see [6]) and outer-connected domination (see [5]) have also been established. In [6], the authors further characterized those graphs for which the doubly connected domination number and the connected domination number are equal.

Four years later, Akhbari et al. in [1] investigated the concept and obtained a series of results which are themselves Nordhaus-Gaddum type inequalities to prove the upperbound of the sum of the doubly connected domination number of some graph and its complement. Karami et al. [9] further established the upper bounds of the doubly connected domination subdivision number in terms of the order of the given graph or its edge connectivity number.

Let $G$ be a simple graph. A set $S \subseteq V(G)$ is a dominating set of $G$ if for every $v \in V(G) \setminus S$, there exists $u \in S$ such that $uv \in E(G)$. If $S$ is a dominating set and every $u \in S$ has at least one neighbor in $S$, then $S$ is called a total dominating set. The total domination number is the minimum cardinality of a total dominating set of $G$, and is denoted by $\gamma_t(G)$. A total dominating set $S$ of $G$ with $|S| = \gamma_t(G)$ is called a $\gamma_t$-set.

A dominating set $S \subseteq V(G)$ is called a connected dominating set of $G$ if the subgraph $\langle S \rangle$ induced by $S$ is connected. The connected domination number of $G$, denoted by $\gamma_c(G)$, is the smallest cardinality of a connected dominating set of $G$. A connected dominating set $S$ of $G$ with $|S| = \gamma_c(G)$ is called a $\gamma_c$-set.

A connected dominating set $S \subseteq V(G)$ is called a doubly connected dominating set of $G$ if the subgraph $\langle V(G) \setminus S \rangle$ induced by $V(G) \setminus S$ is connected. The doubly connected domination number of $G$, denoted by $\gamma_{cc}(G)$, is the smallest cardinality of a doubly connected dominating set of $G$. A doubly connected dominating set $S$ of $G$ with $|S| = \gamma_{cc}(G)$ is called a $\gamma_{cc}$-set. Other variants of the standard concept of domination can be found in the books by Hedetniemi et al. [7, 8]. For undefined terms, see [2, 3, 4].

2 Realization Problem

We begin this section by considering a remark.

**Remark 2.1** For any connected graph $G$, $\gamma_t(G) \leq \gamma_c(G) \leq \gamma_{cc}(G)$.

The above remark is clear since every doubly connected dominating set is a connected dominating set and every connected dominating set is a total dominating set.
**Theorem 2.2** Let $a$, $b$, and $c$ be positive integers such that $4 \leq a \leq b \leq c$, where $b \leq 2a - 3$ if $a$ is odd and $b \leq 2a - 2$ if $a$ is even. Then there exists a connected graph $G$ such that $\gamma_t(G) = a$, $\gamma_c(G) = b$ and $\gamma_{cc}(G) = c$.

**Proof:** Consider the following cases:

**Case 1.** $a = b = c$

Let $G = G_1$ be the graph shown in Figure 1. Then the set $\{x_i : i = 1, 2, ..., a\}$ is a $\gamma_t$-set, $\gamma_c$-set, and $\gamma_{cc}$-set of $G$. Thus, $\gamma_t(G) = a$, $\gamma_c(G) = b = a$, and $\gamma_{cc}(G) = c = b = a$.

![Figure 1: Graph $G_1$](image)

**Case 2.** $a = b < c$

Let $G = G_2$ be the graph shown in Figure 2. Then the set $A = \{x_i : i = 1, 2, ..., a\}$ is a $\gamma_t$-set and $\gamma_c$-set of $G$, and $B = A \cup \{y_j : j = 1, 2, ..., c - a\}$ is a $\gamma_{cc}$-set of $G$. Thus, $\gamma_t(G) = |A| = a$, $\gamma_c(G) = |A| = b = a$, and $b < \gamma_{cc}(G) = |B| = c$.

![Figure 2: Graph $G_2$](image)

**Case 3.** $a < b \leq c$

Consider the following subcases:

**Subcase 1.** $a$ is even

Suppose $b$ is even and $b = 2a - 2$. Consider the graphs $G_3$ and $G_4$ shown in Figure 3. The set $A = \{x_i : i = 1, 2, ..., \frac{b-a+2}{2}\} \cup \{z_i : i = 1, 2, ..., \frac{b-a+2}{2}\}$ is a $\gamma_t$-set of $G_3$ and $G_4$, the set $B = A \cup \{q_r : r = 1, 2, ..., b-a\}$ is a $\gamma_c$-set of $G_3$ and $G_4$, the set $B$ is a $\gamma_{cc}$-set of $G_3$, and the set $C = B \cup \{y_j : j = 1, 2, ..., c-b\}$ is a $\gamma_{cc}$-set of $G_4$. Thus $\gamma_t(G_3) = \gamma_t(G_4) = |A| = 2 \left(\frac{b-a+2}{2}\right) = b - a + 2 = \ldots$
\[a - 2 + 2 = a, \quad \gamma_c(G_3) = \gamma_{cc}(G_3) = \gamma_c(G_4) = |B| = a + b - a = b, \text{ and} \]
\[\gamma_{cc}(G_4) = |B| + c - b = b + c - b = c. \text{ Therefore, if } b = c, \text{ then we take } G = G_3;\]

otherwise choose \(G = G_4.\)

\[G_3: z_1 \quad z_2 \quad z_3 \quad z_4 \quad \cdots \quad z_{\frac{b-a+2}{2}} \quad q_1 q_2 q_3 q_4 q_5 q_6 \quad \cdots \quad q_{b-a-1} q_{b-a} \]

\[G_4: z_1 \quad z_2 \quad z_3 \quad z_4 \quad \cdots \quad z_{\frac{b-a+2}{2}} \quad y_1 y_2 y_3 \quad \cdots \quad y_{c-b} \]

**Figure 3:** Graphs \(G_3\) and \(G_4\)

Suppose \(b < 2a - 2\). Let \(m = 2a - b - 2\) and \(n = c - b\). Consider the graphs \(G_5\) and \(G_6\) shown in Figure 4. Let \(A = \{x_i : i = 1, 2, \ldots, \frac{b-a+2}{2}\} \cup \{z_i : i = 1, 2, \ldots, \frac{b-a+2}{2}\} \cup \{v_s : s = 1, 2, \ldots, m\}, \quad B = A \cup \{q_r : r = 1, 2, \ldots, b-a\},\]

and \(C = B \cup \{y_j : j = 1, 2, \ldots, n\}. \) Then \(A\) is a \(\gamma_t\)-set of \(G_5\) and \(G_6\), \(B\) is a \(\gamma_c\)-set of \(G_5\) and \(G_6\), \(B\) is a \(\gamma_{cc}\)-set of \(G_5\), and \(C\) is a \(\gamma_{cc}\)-sets of \(G_6\). Thus, \(\gamma_t(G_5) = \gamma_t(G_6) = |A| = 2 \left(\frac{b-a+2}{2}\right) + m = b-a+2+m = b-a+2+2a-b-2 = a,\)
\[\gamma_c(G_5) = \gamma_c(G_6) = \gamma_{cc}(G_5) = |B| = a + b - a = b, \text{ and } \gamma_{cc}(G_6) = |C| = b + n = c. \]

If \(b = c\), then we take \(G = G_5.\) If \(b < c\), then we choose \(G = G_6.\)

\[G_5: z_1 \quad z_2 \quad z_3 \quad \cdots \quad z_{\frac{b-a+2}{2}} \quad w_1 w_2 \quad \cdots \quad w_m \]\n
\[G_6: z_1 \quad z_2 \quad z_3 \quad \cdots \quad z_{\frac{b-a+2}{2}} \quad y_1 y_2 \quad \cdots \quad y_n \]

**Figure 4:** Graphs \(G_5\) and \(G_6\)

Suppose now that \(b\) is odd. Then \(b \leq 2a - 3\). Again, let \(m = 2a - b - 3\) and \(n = c - b.\) Consider the graphs \(G_7, G_8,\) and \(G_9\) shown in Figure 5.
Let $A_1 = \{x_i : i = 1, 2, ..., \frac{b-a+3}{2}\} \cup \{z_i : i = 1, 2, ..., \frac{b-a+3}{2}\}$, $A_2 = A_1 \cup \{q_r : r = 1, 2, ..., b-a\}$, $A_3 = A_1 \cup \{y_j : j = 1, 2, ..., n\}$, $A_4 = A_1 \cup \{v_s : s = 1, 2, ..., m\}$, $A_5 = A_1 \cup \{q_r : r = 1, 2, ..., b-a\}$, $A_6 = A_5 \cup \{y_j : j = 1, 2, ..., n\}$. Then the sets $A_1$, $A_2$, and $A_3$ are respectively, $\gamma_t$, $\gamma_c$, and $\gamma_{cc}$-sets of $G_7$. The sets $A_4$ is a $\gamma_t$-set of $G_8$ and $A_5$ is both a $\gamma_c$-set and a $\gamma_{cc}$-set of $G_8$. The sets $A_4$, $A_5$, and $A_6$ are respectively, $\gamma_t$, $\gamma_c$, and $\gamma_{cc}$-sets of $G_9$. Thus, $\gamma_t(G_7) = |A_1| = 2\left(\frac{b-a+3}{2}\right) = b-a+3 = 2a-3-a+3 = a$, $\gamma_c(G_7) = |A_2| = |A_1|+b-a = a+b-a = b$, $\gamma_{cc}(G_7) = |A_3| = |A_2|+n = b+c-b = c$, $\gamma_t(G_8) = \gamma_t(G_9) = |A_4| = |A_1|+m = a+2a-b-3 = 3a-2a+3-3 = a$, $\gamma_c(G_8) = \gamma_c(G_9) = \gamma_{cc}(G_8) = |A_5| = |A_4|+b-a = a+b-a = b$, and $\gamma_{cc}(G_9) = |A_6| = |A_5|+n = b+c-b = c$. If $b < 2a-3$ and $b < c$, then we choose $G = G_9$. If $b = 2a-3$, then we take $G = G_7$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{graphs.png}
\caption{Graphs $G_7$, $G_8$ and $G_9$}
\end{figure}

**Subcase 2.** $a$ is odd

Suppose $b$ is odd. If $b = 2a-3$, then let $G = G_{10}$ be the graph shown in Figure 6. Let $A = \{x_i : i = 1, 2, ..., \frac{a-1}{2}\} \cup \{z_i : i = 1, 2, ..., \frac{a-1}{2}\} \cup \{p_1\}$, $B = A \cup \{q_j : j = 1, 2, ..., b-a\}$. Then $A$ is a $\gamma_t$-set of $G_{10}$ and $B$ is both a $\gamma_c$-set and a $\gamma_{cc}$-set of $G_{10}$. Thus, $\gamma_t(G_{10}) = 2\left(\frac{a-1}{2}\right) + 1 = a - 1 + 1 = a$ and $\gamma_c(G_{10}) = \gamma_{cc}(G_{10}) = |A|+b-a = a+b-a = b$. If $b = c$, then we take $G = G_{10}$. 

Suppose \( b \) is odd and \( b < 2a - 3 \). Then \( b - a \) is even. Let \( m = 2a - b - 2 \) and \( n = c - b \). Consider the graphs \( G_{11} \) and \( G_{12} \) shown in Figure 7. Let \( A = \{x_i : i = 1, 2, \ldots, \frac{b-a+2}{2}\} \cup \{z_i : i = 1, 2, \ldots, \frac{b-a+2}{2}\}, B = A \cup \{q_j : j = 1, 2, \ldots, b-a\}, C = A \cup \{v_s : s = 1, 2, \ldots, m\}, D = B \cup \{v_s := 1, 2, \ldots, m\}, E = C \cup \{q_j : j = 1, 2, \ldots, b-a\}, \text{ and } F = E \cup \{y_k : k = 1, 2, \ldots, n\}. \) Then \( C \) is a \( \gamma_t \)-set of \( G_{11} \) and \( G_{12}, \) \( D \) is a \( \gamma_c \)-set of \( G_{12}, \) \( E \) is a \( \gamma_c \)-set and \( \gamma_{cc} \)-set of \( G_{11}, \) and \( F \) is a \( \gamma_{cc} \)-set of \( G_{12}. \) Thus, \( \gamma_t(G_{11}) = \gamma_t(G_{12}) = |C| = |A| + m = 2 \left( \frac{b-a+2}{2} \right) + m = b-a+2+2a-b-2 = a, \gamma_c(G_{11}) = \gamma_{cc}(G_{11}) = |E| = |C| + b-a = a+b-a = b, \gamma_c(G_{12}) = |D| = |B| + m = |A| + b-a = m = 2 \left( \frac{b-a+2}{2} \right) + b-a+2a-b-2 = b-a+2+a-2 = b, \text{ and } \gamma_{cc}(G_{12}) = |F| = |E| + n = b+n = b + c - b = c. \) If \( b = c, \) then we take \( G = G_{11}. \) If \( b < c, \) then we choose \( G = G_{12}. \)

Suppose \( b \) is even and \( b = 2a-4. \) Let \( n = c-b. \) Consider the graphs \( G_{13} \) and \( G_{14} \) shown in Figure 8. Let \( A = \{x_i : i = 1, 2, \ldots, \frac{b-a+3}{2}\} \cup \{z_i : i = 1, 2, \ldots, \frac{b-a+3}{2}\} \cup \{p_1\}, B = A \cup \{q_j : j = 1, 2, \ldots, b-a\}, C = B \cup \{y_k : k = 1, 2, \ldots, n\}. \) Then \( A \) is a \( \gamma_t \)-set of \( G_{13} \) and \( G_{14}, \) \( B \) is a \( \gamma_c \)-set of \( G_{13} \) and \( G_{14}, \) \( B \) is \( \gamma_{cc} \)-set of \( G_{13}, \) and \( C \) is a \( \gamma_{cc} \)-set of \( G_{14}. \) Thus, \( \gamma_t(G_{13}) = \gamma_t(G_{14}) = |A| = 2 \left( \frac{b-a+3}{2} \right) + 1 = b-a+4 = 2a-4-a+4 = a, \gamma_c(G_{13}) = \gamma_c(G_{14}) = \gamma_{cc}(G_{13}) = \gamma_{cc}(G_{14}) = |B| = |A| + b-a = a+b-a = b, \) and \( \gamma_c(G_{12}) = |C| = |B| + n = b+n = b+c-b = c. \) If \( b = c, \) then we take \( G = G_{13}. \) If \( b < c, \) then we choose \( G = G_{14}. \)
Thus, difference $\gamma$ $\gamma$ $\gamma$ $\gamma$ $\gamma$ $\gamma$

\begin{align*}
&\{b\} = \text{set of } G
&\text{is even and } G
&x \in C,
&\text{and } (z_i = 1, 2, \ldots, n).
&\gamma_i(G_15) = \gamma_i(G_16) = |A| = 2\left(\frac{b-a+3}{2}\right) + m = b - a + 3 + 2a - b - 3 = a,
&\gamma_c(G_15) = \gamma_c(G_16) = \gamma_{cc}(G_15) = |B| = |A| + b - a = a + b - a = b,
&\gamma_{cc}(G_16) = |C| = |B| + n = b + n = b + c - b = c. \text{ If } b < c, \text{ then we take } G = G_15. \text{ If } b = c, \text{ then we choose } G = G_16. \square
\end{align*}

As pointed out by Cyman et al. in [6], Theorem 2.2 also shows that the difference $\gamma_{cc} - \gamma_c$ can be made arbitrarily large.

Figure 8: Graphs $G_{13}$ and $G_{14}$

Figure 9: Graphs $G_{15}$ and $G_{16}$
3 Corona of Graphs

The corona of two graphs $G$ and $H$, denoted by $G \circ H$, is the graph obtained by taking one copy of $G$ of order $n$ and $n$ copies of $H$, and then joining the $i$-th vertex of $G$ to every vertex in the $i$-th copy of $H$. For every $v \in V(G)$, we denote by $H^v$ the copy of $H$ whose vertices are joined or attached to the vertex $v$. The following remark can be found in [6].

**Remark 3.1** If $S$ is a $\gamma_{cc}$-set of a connected graph $G$ of order $n \geq 3$, then every cut-vertex of $G$ is in $S$.

**Theorem 3.2** Let $G$ be connected graph and $H$ be any graph. Then a nonempty set $C \subset V(G \circ H)$ is $dcd$-set of $G \circ H$ if and only if there exists a vertex $v$ of $G$ such that

$$C = V(G) \cup \left( \bigcup_{u \in V(G) \setminus \{v\}} V(H^u) \right) \cup (V(H^v) \setminus T^v),$$

where $\langle T^v \rangle$ is a connected subgraph of $H^v$.

**Proof**: Let $C \subset V(G \circ H)$ be a $dcd$-set of $G \circ H$. Since every vertex $x$ of $G$ is a cut-vertex of $G \circ H$, $V(G) \subset C$ by Remark 3.1. Let $T = V(G \circ H) \setminus C$. Suppose there exist $y, z \in V(G)$, where $y \neq z$, such that $T \cap V(H^y) \neq \emptyset$ and $T \cap V(H^z) \neq \emptyset$. Since $y, z \in C$, it follows that $\langle T \rangle$ is disconnected, contrary to the fact that $C$ is a doubly connected dominating set. Hence, $T \subset V(H^v)$ for some $v \in V(G)$. Let $T^v = T$. Since $\langle T \rangle$ is a connected subgraph of $G \circ H$, it follows that $\langle T^v \rangle$ is a connected subgraph of $H^v$. Moreover,

$$C = V(G) \cup \left( \bigcup_{u \in V(G) \setminus \{v\}} V(H^u) \right) \cup (V(H^v) \setminus T^v).$$

The converse is clear. □

**Corollary 3.3** Let $G$ be a connected graph of order $n \geq 1$ and $H$ be any graph of order $m$. Then $\gamma_{cc}(G \circ H) = mn + n - k$, where

$$k = \max\{|K| : K \text{ is a component of } H\}.$$ 

In particular, if $H$ is connected, then $\gamma_{cc}(G \circ H) = mn + n - m$.

**Proof**: Let $K$ be a component of $H$ with $|V(K)| = k$. Choose $v \in V(G)$ and let $T^v \subseteq V(H^v)$ such that $\langle T^v \rangle \cong K$. Then

$$C = V(G) \cup \left( \bigcup_{u \in V(G) \setminus \{v\}} V(H^u) \right) \cup (V(H^v) \setminus T^v).$$
is a doubly connected dominating set of \( G \circ H \) by Theorem 3.2. Hence, 
\[ \gamma_{cc}(G \circ H) \leq |C| = mn + n - k. \] Next let \( C' \) be a minimum doubly connected dominating set of \( G \circ H \). Then

\[
C' = V(G) \bigcup \left( \bigcup_{u \in V(G) \setminus \{v\}} V(H^u) \right) \bigcup (V(H^v) \setminus T^x) \]

for some \( x \in V(G) \), where \( \langle T^x \rangle \) is a connected subgraph of \( H^v \). Hence, 
\[ \gamma_{cc}(G \circ H) = |C'| = mn + n - |T^x| \geq mn + n - k. \] Therefore, 
\[ \gamma_{cc}(G \circ H) = mn + n - k. \]

\[ \square \]

4 Lexicographic Product of Graphs

The **lexicographic product** of two graphs \( G \) and \( H \), denoted by \( G[H] \), is the graph with \( V(G[H]) = V(G) \times V(H) \) and \( (u_1, u_2)(v_1, v_2) \in E(G[H]) \) if either \( u_1v_1 \in E(G) \) or \( u_1 = v_1 \) and \( u_2v_2 \in E(H) \).

Now, let \( C \) be a non-empty subset of \( V(G[H]) \). Let 
\[ S = C_G = \{ x \in V(G) : (x, a) \in C \text{ for some } a \in V(H) \} \] (the \( G \)-projection of \( C \)). For each \( x \in S \), let \( T_x = \{ a \in V(H) : (x, a) \in C \} \). Then \( C = \bigcup_{x \in S} \{ x \} \times T_x \).

Recall that the **distance** \( d_G(u, v) \) in \( G \) of two vertices \( u, v \) is the length of a shortest \( u-v \) path in \( G \). A \( u-v \) path of length \( d_G(u, v) \) is called a **\( u-v \) geodesic**.

The next result characterizes the doubly connected dominating sets in the lexicographic product of two non-trivial connected graphs.

**Theorem 4.1** Let \( G \) and \( H \) be any non-trivial connected graphs. A nonempty proper subset \( C = \bigcup_{x \in S} \{ (x) \times T_x \} \) of \( V(G[H]) \), where \( S \subseteq V(G) \) and \( T_x \subseteq V(H) \) for every \( x \in S \), is a doubly connected dominating set of \( G[H] \) if and only if \( S \) is a connected dominating set in \( G \) and the following hold:

i. \( \langle (V(G) \setminus S) \cup S_1 \rangle \) is connected in \( G \) whenever \( (V(G) \setminus S) \cup S_1 \neq \emptyset \);

ii. \( \langle V(H) \setminus T_x \rangle \) is connected whenever \( S_1 = \{ x \} \) and \( S = V(G) \); and

iii. \( T_x \) is a connected dominating set of \( H \) whenever \( S = \{ x \} \), where \( S_1 = \{ x \in S : T_x \neq V(H) \} \)

**Proof:** Let \( S_1 = \{ x \in S : T_x \neq V(H) \} \). Suppose that \( C = \bigcup_{x \in S} \{ (x) \times T_x \} \), where \( S \subseteq V(G) \) and \( T_x \subseteq V(H) \) for every \( x \in S \), is a doubly connected dominating set of \( G[H] \). Then \( S \) is a connected dominating set of \( G \). Let \( x, y \in (V(G) \setminus S) \cup S_1 \), where \( x \neq y \) and \( xy \notin E(G) \). Consider the following cases:

**Case 1.** \( x, y \in V(G) \setminus S \)
Let \( a \in V(H) \). Since \((x, a), (y, a) \in V(G[H]) \setminus C\) and \(\langle V(G[H]) \setminus C \rangle\) is connected, there exists a \((x, a) - (y, a)\) geodesic \([x_1, a_1], (x_2, a_2), \ldots, (x_k, a_k)\) in \(\langle V(G[H]) \setminus C \rangle\), where \((x_1, a_1) = (x, a), (x_k, a_k) = (y, a)\), and \(k \geq 3\). It follows that \([x_1, x_2, \ldots, x_k]\) is a \(x - y\) geodesic in \(\langle (V(G[H]) \setminus S) \cup S_1 \rangle\).

**Case 2.** \(x, y \in S_1\)

Let \(b \in V(H) \setminus T_x\) and \(c \in V(H) \setminus T_y\). Then \((x, b), (y, c) \notin C\). Since \(\langle V(G[H]) \setminus C \rangle\) is connected, there exists a \((x, b) - (y, c)\) geodesic \([x_1, b_1], (x_2, b_2), \ldots, (x_k, b_k)\) in \(\langle V(G[H]) \setminus C \rangle\), where \((x_1, b_1) = (x, b), (x_k, b_k) = (y, c)\), and \(k \geq 3\). Consequently, \([x_1, x_2, \ldots, x_k]\) is a \(x - y\) geodesic in \(\langle (V(G) \setminus S) \cup S_1 \rangle\).

**Case 3.** \(x \in V(G) \setminus S\) and \(y \in S_1\)

Let \(b \in T_y\). Then \((x, b), (y, b) \notin C\). Using the same argument as in Case 1, there exists a \(x - y\) geodesic in \(\langle (V(G) \setminus S) \cup S_1 \rangle\).

Accordingly, \(\langle (V(G) \setminus S) \cup S_1 \rangle\) is connected in \(G\).

Suppose \(S_1 = \{x\}\) and \(S = V(G)\). Then \(V(G[H]) \setminus C = \{x\} \times (V(H) \setminus T_x)\). Since \(\langle V(G[H]) \setminus C \rangle\) is connected in \(G[H]\), it follows that \(\langle V(H) \setminus T_x \rangle\) is connected in \(H\). Next, suppose \(S = \{x\}\). If \(T_x = V(H)\), then \(\langle T_x \rangle\) is a connected dominating in \(H\). So suppose \(T_x \neq V(H)\), i.e., \(x \in S_1\). Let \(d \in V(H) \setminus T_x\). Since \((x, d) \notin C\) and \(C\) is a dominating set in \(G[H]\), there exists \((x, e) \in C\) such that \((x, d)(x, e) \in E(G[H])\). It follows that \(e \in T_x\) and \(de \in E(H)\). Thus \(T_x\) is a dominating set in \(H\). Moreover, since \(\langle C \rangle = \langle \{x\} \times T_x \rangle \cong \langle T_x \rangle\) is connected in \(G[H]\), it follows that \(\langle T_x \rangle\) is connected in \(H\).

For the converse, suppose that \(S\) is a connected dominating set of \(G\) satisfying condition (i), (ii), and (iii). Let \((x, a) \notin C\) and consider the following cases:

**Case 1.** \(x \in S\) (hence, \(x \in S_1\))

If \(S = \{x\}\), then \(S_1 = \{x\}\). By (iii), there exists \(b \in T_x\) such that \(ab \in E(H)\). Hence \((x, b) \in C\) and \((x, b)(x, a) \in E(G[H])\).

**Case 2.** \(x \notin S\)

Since \(S\) is a dominating set, there exists \(y \in S\) such that \(xy \in E(G)\). Let \(c \in T_y\). Then \((y, c) \in C\) and \((x, a)(y, c) \in E(G[H])\).

Therefore \(C\) is a dominating set in \(G[H]\).

Next, let \((u, p), (v, q) \in C\), where \((u, p) \neq (v, q)\) and \((u, p)(v, q) \notin E(G[H])\). Consider the following cases:

**Case 1.** \(u = v\)

Then \(p \neq q\) and \(pq \notin E(H)\). If \(S = \{u\}\), then \(\langle T_u \rangle\) is connected by (iii). Hence there exists a \(p - q\) geodesic \([p_1, p_2, \ldots, p_k]\) in \(\langle T_u \rangle\), where \(p = p_1, p_k = q\), and
\[ k \geq 3. \text{ Hence } [(u, p_1), (u, p_2), \ldots, (u, p_k)] \text{ is a } (u, p)-(u, q) \text{ geodesic in } C. \]

**Case 2.** \( u \neq v \)

Then \( uv \notin E(G) \). Since \( (S) \) is connected, there exists a \( u - v \) geodesic \([u_1, u_2, \ldots, u_r]\) in \( (S) \), where \( u_1 = u, u_r = v \), and \( r \geq 3 \). Set \( p_1 = p, p_r = q \) and choose \( p_i \in T_u \) for each \( i = 2, 3, 4, \ldots, r - 1 \). Then \([(u_1, p_1), (u_2, p_2), \ldots, (u_r, p_r)]\) is a \((u, p)-(v, q)\) geodesic in \( (C) \).

Therefore \( (C) \) is connected in \( G[H] \).

Next, let \((u, a), (v, b) \in V(G[H]) \setminus C\), where \((u, a) \neq (v, b)\) and \((u, a)(v, b) \notin E(G[H])\). Consider the following cases:

**Case 1.** \( u, v \in S \) (hence, \( u, v \in S_1 \))

If \( u = v \), then \( T_u = T_v \notin V(H) \). Also, \( a, b \in V(H) \setminus T_u \), \( a \neq b \) and \( ab \notin E(H) \). If \(|(V(G) \setminus S) \cup S_1| = 1\), then \( S_1 = \{u\} \) and \( S = V(G) \). Hence, by (ii), \( (V(H) \setminus T_u) \) is connected. Thus there exists an \( a - b \) geodesic \([a_1, a_2, \ldots, a_t]\) in \((V(H) \setminus T_u), a = a_1, b = a_t, \text{ and } t \geq 3 \). It follows that \([(u, a_1), (u, a_2), \ldots, (u, a_t)]\) is a \((u, a)-(v, b)\) geodesic in \((V(G[H]) \setminus C)\). If \(|(V(G) \setminus S) \cup S_1| \geq 2\), then by (i), there exists \( w \in (V(G) \setminus S) \cup S_1\) such that \( uw \in E(G) \). Let \( c \in V(H) \) such that \((w, c) \notin C\). Then \([(u, a), (w, c), (v, b)]\) is a \((u, a)-(v, b)\) geodesic in \((V(G[H]) \setminus C)\).

If \( u \neq v \), then there exists a \( u - v \) geodesic \([u_1, u_2, \ldots, u_k]\) in \((V(G) \setminus S) \cup S_1\) by (i), where \( u = u_1, v = u_k\), and \( k \geq 3 \). Set \( b_1 = a, b_k = b \), and choose \( b_i \in V(H) \) such that \((u_i, b_i) \notin C\) for \( i = 2, 3, \ldots, k - 1 \). Then \([(u_1, b_1), (u_2, b_2), \ldots, (u', b_k)]\) is a \((u, a)-(v, b)\) geodesic in \((V(G[H]) \setminus C)\).

**Case 2.** \( u, v \in V(G) \setminus S \)

If \( u = v \), then \( a \neq b \) and \( ab \notin E(H) \). Since \( H \) is connected, there exists an \( a - b \) geodesic \([a_1, a_2, \ldots, a_t]\) in \( H \), where \( a = a_1, b = a_r \), and \( t \geq 3 \). If \(|(V(G) \setminus S) \cup S_1| = 1\), then \( S_1 = \emptyset \) and \( V(G) \setminus S = \{u\} \). Hence, \([(u, a_1), (u, a_2), \ldots, (u, a_r)]\) is a \((u, a)-(v, b)\) geodesic in \((V(G[H]) \setminus C)\). If \(|(V(G) \setminus S) \cup S_1| \geq 2\), then by (i), there exists \( w \in (V(G[H]) \setminus S) \cup S_1\) such that \( uw \in E(G) \). Let \( c \in V(H) \) such that \((w, c) \notin C\). Then \([(u, a), (w, c), (v, b)]\) is a \((u, a)-(v, b)\) geodesic in \((V(G[H]) \setminus C)\).

If \( u \neq v \), then there exists a \( u - v \) geodesic \([u_1, u_2, \ldots, u_k]\) in \((V(G) \setminus S) \cup S_1\) by (i), where \( u_1 = u, u_k = v\), and \( k \geq 3 \). Set \( b_1 = a, b_k = b \) and choose \( b_i \in V(H) \) such that \((u_i, b_i) \notin C\) for \( i = 2, 3, \ldots, k - 1 \). Then \([(u_1, b_1), (u_2, b_2), \ldots, (u'_k, b_k)]\) is a \((u, a)-(v, b)\) geodesic in \((V(G[H]) \setminus C)\).

**Case 3.** \( u \in V(G) \setminus S \) and \( v \in S_1 \).

By (i), there exists a \( u - v \) geodesic \([u_1, u_2, \ldots, u_k]\) in \((V(G) \setminus S) \cup S_1\), where \( u = u_1, v = u_k\), and \( k \geq 3 \). Then using the same argument as in the preceding case, a \((u, a)-(v, b)\) geodesic in \((V(G[H]) \setminus C)\) exists.
Therefore, \( \langle V(G[H]) \rangle \setminus C \) is connected. Hence, \( C \) is a doubly connected dominating set of \( G[H] \). \( \square \)

**Corollary 4.2** Let \( G \) and \( H \) be non-trivial connected graphs. Then

\[
\gamma_{cc}(G[H]) = \begin{cases} 
1 & \text{if } \gamma(G) = 1 \text{ and } \gamma(H) = 1 \\
2 & \text{if } \gamma(G) = 1 \text{ and } \gamma(H) \neq 1 \\
\gamma_c(G) & \text{if } \gamma(G) \neq 1.
\end{cases}
\]

**Proof:** Consider the following cases:

**Case 1.** \( \gamma(G) = 1 \)

Suppose \( \gamma(H) = 1 \). Let \( S = \{x\} \) and \( D = \{y\} \) be dominating sets of \( G \) and \( H \), respectively. Let \( T_x = \{y\} \). Then \( C = \{(x, y)\} \) is a doubly connected dominating set of \( G[H] \) by Theorem 4.1. Thus, \( \gamma_{cc}(G[H]) \leq |C| = 1 \). This implies that \( \gamma_{cc}(G[H]) = 1 \).

Suppose \( \gamma(H) \neq 1 \). Let \( S' = \{x_1\} \) be a dominating set of \( G \). Since \( G \) is non-trivial, there exists \( x_2 \in V(G) \) such that \( x_1x_2 \in E(G) \). Let \( C = \{(x_1, a), (x_2, a)\} \) where \( a \in V(H) \). Since \( H \) is non-trivial, \( T_{x_1} \neq V(H) \) and \( T_{x_2} \neq V(H) \), i.e., \( S_1 = \{x \in S : T_x \neq V(H)\} = S = \{x_1, x_2\} \). By Theorem 4.1, \( C \) is a doubly connected dominating set of \( G[H] \). Thus, \( \gamma_{cc}(G[H]) \leq |C| = 2 \). Next, let \( C' \) be a doubly connected dominating set of \( G[H] \) such that \( |C'| < |C| \). Then \( |C'| = 1 \). Let \( C' = \{(x, c)\} \). Since \( \gamma(H) \neq 1 \), there exists \( b \in V(H) \) such that \( bc \in E(H) \), i.e., \( (x, b)(x, c) \notin E(G[H]) \). This implies that \( C' \) is not a dominating set of \( G[H] \), contrary to our assumption. Thus, \( \gamma_{cc}(G[H]) \neq 1 \). Therefore, \( \gamma_{cc}(G[H]) = 2 \).

**Case 2.** \( \gamma(G) \neq 1 \)

Let \( S \) be a \( \gamma_{cc} \)-set of \( G \). Since \( \gamma(G) \neq 1 \), it follows that \( |S| \neq 1 \). Let \( a \in V(H) \) and let \( T_a = \{a\} \forall x \in S \). Since \( H \neq K_1 \), \( S_1 = \{x \in S : T_x \neq V(H)\} = S \). By Theorem 4.1, \( C = \bigcup_{x \in S} \{x\} \times T_x \) is a doubly connected dominating set of \( G[H] \). Hence, \( \gamma_{cc}(G[H]) \leq |C| = |S| = \gamma_c(G) \). On the other hand, if \( C^* = \bigcup_{x \in S^*} \{x\} \times T_x \) is a \( \gamma_{cc} \)-set of \( G[H] \), then \( S^* \) is a connected dominating set of \( G \) by Theorem 4.1. Hence, \( \gamma_c(G) \leq |S^*| \leq |C^*| = \gamma_{cc}(G[H]) \). Therefore, \( \gamma_{cc}(G[H]) = \gamma_c(G) \). \( \square \)

**References**


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