An Analytically Tractable Multi-asset Stochastic Volatility Model

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Abstract

A multi-asset stochastic volatility model is presented. This is based on the Heston stochastic volatility model. It has the advantage of being sufficiently flexible to interpret the individual asset dynamics as well as the interdependencies among them, while still being analytically tractable. A formula for the joint transition probability density function of the stochastic process implicitly defined by the model is deduced. This formula is expressed as a one dimensional integral of an explicitly known function and allows us to derive elementary formulae for the conditional mean and variance of the asset price log-returns. A calibration procedure is proposed and tested on simulated data.

Keywords: Stochastic volatility model, Kolmogorov backward equation, maximum likelihood function, calibration procedure
1 Introduction

This paper proposes a multi-asset stochastic volatility model (GH model for short) based on the Heston model [14]. This model is analytically tractable and describes satisfactorily the dynamics of n correlated asset prices. The GM model was developed in the aftermath of some recent studies on the Heston stochastic volatility model [3], [11], [16] and some generalizations of the Heston model in the multiscale framework [12], [9], [7], [8]. The GH model is obtained from the Heston stochastic volatility model by adding an extra term to the stochastic differential equation satisfied by each asset price. Roughly speaking, this term expresses the correlation among the asset prices. The assumption made on the correlation structure enables us to derive a semi-explicit formula for the transition probability density function associated with the state variables. This formula is expressed by a one dimensional integral of an elementary function and it can be evaluated using simple quadrature rules such as the midpoint rule. Moreover, elementary formulae for the conditional mean of the asset log-returns and their variance are deduced (see, formulae (26) and (27)). The formula for the transition probability density function is used to calibrate the GH model applying the maximum likelihood approach. This approach is commonly used to calibrate stochastic volatility models (see, for example, [2], [20], [10]) and its alternative is the mean least square approach. The latter is usually applied when formulae for option prices are available (see, for example, [4], [5], [13]). A numerical experiment on simulated data is proposed. This experiment shows the performance of the calibration procedure and the GH model’s ability to forecast asset prices. The paper is organized as follows. In Section 2 we present a generalization of the Heston stochastic volatility model and a formula for the transition probability density function associated with the GH model. Furthermore, formulae for the conditional expected values of the asset log-returns and of their variances are shown. In Section 3 the formulae proposed in Section 2 are proven. In Section 4 we propose a calibration procedure based on the maximum likelihood approach to estimate the parameters of the GH model. Some experiments on simulated data are presented to show the GH model’s ability to describe the multi asset dynamics.

2 The asset stochastic volatility model

We now describe the details of the proposed model. Letting \( t \) be a real variable that denotes time, we consider the (vector valued real) stochastic process \( (S_{it}, v_{it}), \ i = 1, 2, \ldots, n, \ t > 0 \), solution of the following system of stochastic differential equations:

\[
\begin{align*}
    dS_{it} & = S_{it} \mu dt + S_{it} \sqrt{v_{it}} dW_{0,t} + S_{it} a_i dQ_{i,t}^0, \\
    dv_{it} & = \chi_i (\theta_i - v_{it}) dt + \varepsilon_i \sqrt{v_{it}} dW_{1,t}^1,
\end{align*}
\]

(1) (2)
where $\mu_i, a_i, \chi_i, \theta_i, \varepsilon_i$, for $i = 1, 2, \ldots, n$, are suitable real constants, satisfying the following conditions:

\begin{align}
    a_i, \chi_i, \theta_i, \varepsilon_i &> 0 \\
    \frac{2\chi_i\theta_i}{\varepsilon_i^2} &> 1,
\end{align}

and $W_{i,t}^0, Q_{i,t}^0, W_{i,t}^1$ are standard Wiener processes such that $W_{i,0}^0 = W_{i,0}^1 = Q_{i,0}^0 = 0$, $i = 1, 2, \ldots, n$. Let us denote with $dW_{i,t}^0, dQ_{i,t}^0, dW_{i,t}^1$ the stochastic differentials of the Wiener processes just mentioned, we assume that the following relations hold for $i, j = 1, 2, \ldots, n$:

\begin{align}
    E(dW_{i,t}^0 dQ_{j,t}^0) &= 0 \quad \forall i, j \\
    E(dW_{i,t}^0 dQ_{i,t}^0) &= 0 \quad \forall i, j \\
    E(dW_{i,t}^0 dW_{j,t}^1) &= 0 \quad i \neq j, \quad E(dW_{i,t}^0 dW_{i,t}^1) = \rho_{i,j} dt, \\
    E(dW_{i,t}^1 dW_{j,t}^1) &= 0 \quad i \neq j, \quad E(dW_{i,t}^1 dW_{i,t}^1) = dt \\
    E(dQ_{i,t}^0 dQ_{j,t}^0) &= \rho_{i,j} dt \quad i \neq j, \quad E(dQ_{i,t}^0 dQ_{i,t}^0) = dt,
\end{align}

where $E(\cdot)$ denotes the expected value of $\cdot$, $\rho_i, \rho_{i,j} \in (-1, 1)$ are constants known as correlation coefficients. Note that the condition (4) guarantees that when $v_{i,t}$ is positive with probability one at time $t = 0$, $v_{i,t}$, solution of (2), remains positive with probability one for $t > 0$, $i = 1, 2, \ldots, n$. Finally we introduce the following two matrices:

\begin{align}
    \Gamma_{i,j} &= \begin{cases} 
        \rho_{i,j} & i \neq j \\
        1 & i = j 
    \end{cases}, i, j = 1, 2, \ldots, n, \\
    D_{i,j} &= \begin{cases} 
        0 & i \neq j \\
        a_i & i = j 
    \end{cases}, i, j = 1, 2, \ldots, n.
\end{align}

Note that model (1)-(2) is a generalization of the Heston model. In fact it is obtained adding the term $a_i S_{i,t} dQ_{i,t}^0$ to the Heston process that describes the dynamic of the asset price $S_i$, $i = 1, 2, \ldots, n$. Specifically, when we choose $a_i = 0$, $i = 1, 2, \ldots, n$ the GH model reduces to the Heston model for $n$ uncorrelated assets (see equations (5), (7), (7)). When $a_i \neq 0$, $i = 1, 2, \ldots, n$ a correlation structure among the assets is introduced and this correlation does not depend explicitly on the asset stochastic volatility. This particular correlation structure allows us to derive a semi-explicit formula for the transition probability density function associated with the process $(S_{i,t}, v_{i,t})$, $t > 0$, $i = 1, 2, \ldots, n$. 
To derive this semi-explicit formula we consider the asset log-return $x_{i,t}$, $t \geq 0$, $i = 1, 2, \ldots, n$, given by:

$$ x_{i,t} = \ln \frac{S_{i,t}}{S_{i,0}}. \quad (13) $$

The dynamical system (1)–(2) can be rewritten as follows:

$$ dx_{i,t} = (\mu_i - 1/2 a_i^2 - 1/2 v_{i,t}) dt + \sqrt{v_{i,t}} dW_{i,t}^0 + a_i dQ_{i,t}^0, \quad t > 0, \quad i = 1, 2, \ldots, n, \quad (14) $$

$$ dv_{i,t} = \chi_i(\theta_i - v_{i,t}) dt + \varepsilon_i \sqrt{v_{i,t}} dW_{i,t}^1, \quad t > 0, \quad i = 1, 2, \ldots, n. \quad (15) $$

Equations (14)–(15) must be equipped with initial conditions as follows:

$$ x_{i,0} = \tilde{x}_{i,0}, \quad i = 1, 2, \ldots, n, \quad (16) $$

$$ v_{i,0} = \tilde{v}_{i,0}, \quad i = 1, 2, \ldots, n, \quad (17) $$

where $\tilde{x}_{i,0}, \tilde{v}_{i,0}$, $i = 1, 2, \ldots, n$, are random variables that we assume to be concentrated in a point with probability one. For simplicity, we identify the random variables $\tilde{x}_{i,0}, \tilde{v}_{i,0}$, $i = 1, 2, \ldots, n$, with the points where they are concentrated. Equation (13) implies $\tilde{x}_{i,0} = 0$, $i = 1, 2, \ldots, n$.

Let $R^+$ be the set of the positive real number, $m$ be a positive integer, $\mathbb{R}^m$ be the $m$-dimensional real Euclidean space and $\mathbb{R}^{m^+}$ be the cartesian product of $m$ copies of $\mathbb{R}^+$. Let $(x_1, x_2, \ldots, x_n, v_1, \ldots, v_n, t, x_1', \ldots, x_n', v_1', \ldots, v_n', t') \in \mathbb{R}^n \times \mathbb{R}^{n^+} \times \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^{n^+} \times \mathbb{R}^+$, $t' > t$, and $p_f(x_1, \ldots, x_n, v_1, \ldots, v_n, t, x_1', \ldots, x_n', v_1', \ldots, v_n', t')$ be the transition probability density function associated with the stochastic process implicitly defined by (14)–(15), that is, the probability density function of having $x_{i,t'} = x_i', v_{i,t'} = v_i'$, given the fact that we have $x_{i,t} = x_i, v_{i,t} = v_i$, $i = 1, 2, \ldots, n$, when $t' > t$. To derive a formula for $p_f$, in analogy with Lipton (2001) (pages 602–605) [15], we consider the Kolmogorov backward equation satisfied by $p_f$ as a function of the “past” variables $(x_1, \ldots, x_n, v_1, \ldots, v_n, t) \in \mathbb{R}^n \times \mathbb{R}^{n^+} \times \mathbb{R}^+$, that is:

$$ -\frac{\partial p_f}{\partial t} = \frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2 p_f}{\partial x_i^2} (v_i + a_i^2) + \frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2 p_f}{\partial v_i^2} \varepsilon_i^2 v_i + \sum_{i=1}^{n} \frac{\partial^2 p_f}{\partial x_i \partial v_i} \varepsilon_i v_i \rho_i $$

$$ + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \frac{\partial^2 p_f}{\partial x_i \partial x_j} \rho_{i,j} a_i a_j + \sum_{i=1}^{n} \frac{\partial p_f}{\partial x_i} \left( \mu_i - \frac{a_i^2}{2} - \frac{v_i}{2} \right) + \sum_{i=1}^{n} \frac{\partial p_f}{\partial v_i} \chi_i (\theta_i - v_i), \quad (18) $$

with the final condition:

$$ p_f(x_1, \ldots, x_n, v_1, \ldots, v_n, t', x_1', \ldots, x_n', v_1', \ldots, v_n', t') = \prod_{i=1}^{n} \delta(x_i - x_i') \delta(v_i - v_i'), \quad (19) $$
where \( \delta(\cdot) \) denotes Dirac’s delta and suitable boundary conditions. The transition probability density function \( p_f \) is the fundamental solution of the Kolmogorov backward equation (18) with final condition (19) and suitable boundary conditions.

As shown in Section 3 the following integral representation formula for \( p_f \) holds:

\[
p_f(x_1, \ldots, x_n, v_1, \ldots, v_n, t, x'_1, \ldots, x'_n, v'_1, \ldots, v'_n, t') = \frac{1}{(2\pi)^n} \prod_{j=1}^{n} \int_{\mathbb{R}} dk_j 2\tilde{M}_j \cdot \]

\[
e^{ik_j \left[(x'_j-x_j)-(t'-t) \left(\mu_j - \frac{a_j^2}{2} \right)\right]} e^{-\frac{2x_jk_j(\nu_j+\zeta_j)t}{\varepsilon_j^2}} e^{-2x_jk_j(\nu_j+\zeta_j)t} e^{-2v_j(\zeta_j^2 - \nu_j^2) s_{j,\gamma}/(\varepsilon_j^2 s_{j,\beta})}.
\]

\[
e^{-\tilde{M}_j(v_j+v'_j) \left(\nu_j \varepsilon_j^2 \right)^{1/2}} \left( 2\tilde{M}_j(v_j+v'_j) \right)^{1/2} \cdot \]

\[
e^{-\frac{(\nu_j - \nu'_j)^2}{2\varepsilon_j^2}} \] \( \cdot \) \( D_{\Gamma D}k \), \hspace{1cm} (20)

where \( I_a \) is the modified Bessel function of order \( a \) (see, for example, Abramowitz and Stegun, 1970 [1]) and the quantities \( \nu_j, \zeta_j, s_{j,\beta}, s_{j,\gamma} \) and \( \tilde{v}_j, M_j, j = 1, 2, \ldots, n \), are given by:

\[
\nu_j = -\frac{1}{2} \left( \chi_j + ik_j \varepsilon_j \rho_j \right), \hspace{1cm} (21)
\]

\[
\zeta_j = \frac{1}{2} \left( 4v_j^2 + \varepsilon_j^2 (k_j^2 - ik_j) \right)^{1/2}, \hspace{1cm} (22)
\]

\[
s_{j,\gamma} = 1 - e^{-2\zeta_j \tau}, \hspace{1cm} s_{j,\beta} = (\zeta_j - \nu_j) + (\zeta_j + \nu_j) e^{-2\zeta_j \tau}, \hspace{1cm} \tau > 0, \hspace{1cm} (23)
\]

\[
\tilde{v}_j = \frac{4v_j s_{j,\beta}^2}{(s_{j,\beta})^2}, \hspace{1cm} M_j = \frac{2s_{j,\beta}}{\varepsilon_j^2 s_{j,\gamma}}, \hspace{1cm} \tau > 0. \hspace{1cm} (24)
\]

Integrating equation (20) with respect to the “future” variances \( (v'_1, v'_2, \ldots, v'_n) \) we obtain:

\[
p_{M,f}(x_1, x_2, \ldots, x_n, v_1, \ldots, v_n, t, x'_1, x'_2, \ldots, x'_n, t') = \]

\[
\frac{1}{(2\pi)^n} \prod_{j=1}^{n} \int_{\mathbb{R}} dk_j e^{ik_j \left[(x'_j-x_j)-(t'-t) \left(\mu_j - \frac{a_j^2}{2} \right)\right]} e^{-\frac{2x_jk_j(\nu_j+\zeta_j)t}{\varepsilon_j^2}} e^{-2x_jk_j(\nu_j+\zeta_j)t} e^{-2v_j(\zeta_j^2 - \nu_j^2) s_{j,\gamma}/(\varepsilon_j^2 s_{j,\beta})} \cdot \]

\[
e^{-2v_j(\zeta_j^2 - \nu_j^2) s_{j,\gamma}/(\varepsilon_j^2 s_{j,\beta})} e^{-\frac{(\nu_j - \nu'_j)^2}{2\varepsilon_j^2}} \cdot \hspace{1cm} (25)
\]

where the matrices \( \Gamma \) and \( D \) are given in (11) and (12). Note that the function \( p_{M,f} \) depends on the “past” variances at time \( t \). The quantities \( v_1, v_2, \ldots, v_n \) are not really observable in the market so that we consider them as model parameters in the calibration procedure. Finally, using formula (20) we derive
the expected value and the variance of the variable $x_{i,t'}$ conditioned to the observation made at $t$, $t' > t$, of the variables $x_{i,t}$ and $v_{i,t}$, $i = 1, 2, \ldots, n$. Let us denote with $\tilde{x}_i$, $i = 1, 2, \ldots, n$, the observation made of the log-returns $x_{i,t}$, $i = 1, 2, \ldots, n$ and with $\tilde{v}_i$, $i = 1, 2, \ldots, n$, the variances at time $t$, we obtain the following formulae for the conditional mean and variance of the asset log-returns:

$$\hat{x}_{i,t,t'} = E(x_{i,t'} | x_{i,t} = \tilde{x}_i, v_{i,t} = \tilde{v}_i) = \tilde{x}_i + (t' - t)(\mu_i - \frac{\theta_i}{2} - a_i^2) + \frac{(\theta_i - \tilde{v}_i)}{2\chi_i}(1 - e^{-\chi_i(t' - t)}), \quad t' > t, \quad i = 1, 2, \ldots, n. \quad (26)$$

$$\hat{v}_{i,t,t'} = E((x_{i,t'} - \hat{x}_{i,t'})^2 | x_{i,t} = \tilde{x}_i, v_{i,t} = \tilde{v}_i) = (t' - t)a_i^2 + \frac{\theta_i}{8\chi_i} \left[ 1 - e^{-\chi_i(t' - t)} \right]^2$$

$$+ \theta_i \left( t' - t - \frac{1 - e^{-\chi_i(t' - t)}}{\chi_i} \right) \left[ \left( \rho_i - \frac{\varepsilon_i}{2\chi_i} \right)^2 \rho_i - \frac{3\varepsilon_i}{2\chi_i} + (1 - \rho_i^2) \right]$$

$$+ \frac{\tilde{v}_i}{\chi_i} (1 - e^{-\chi_i(t' - t)}) \left[ \left( \rho_i - \frac{\varepsilon_i}{2\chi_i} \right)^2 + (1 - \rho_i^2) \right]$$

$$+ \frac{\tilde{v}_i (t' - t)\varepsilon_i e^{-\chi_i(t' - t)}}{\chi_i} \left( -\rho_i + \frac{\varepsilon_i}{2\chi_i} \right),$$

$$t' > t, \quad i = 1, 2, \ldots, n, \quad (27)$$

and finally:

$$\hat{\rho}_{i,j,t,t'} = E((x_{i,t'} - \hat{x}_{i,t'})(x_{j,t'} - \hat{x}_{j,t'}) | x_{i,t} = \tilde{x}_i, v_{i,t} = \tilde{v}_i, x_{j,t} = \tilde{x}_j, v_{j,t} = \tilde{v}_j)$$

$$= a_i a_j \rho_{i,j} (t' - t), \quad t' > t, \quad i \neq j, \quad i, j = 1, 2, \ldots, n. \quad (28)$$

### 3 Derivation of the integral representation formula for the transition probability density function

Let us derive the integral representation formula for $p_f$ (see formula (20)). Letting $\tau = t' - t$, we introduce the function $p_b$ defined as follows:

$$p_b(\tau, x_1, \ldots, x_n, v_1, \ldots, v_n, x'_1, \ldots, x'_n, v'_1, \ldots, v'_n) = p_f(x_1, \ldots, x_n, v_1, \ldots, v_n, t, x'_1, \ldots, x'_n, v'_1, \ldots, v'_n, t'), \quad t' = t + \tau, \quad \tau > 0 \quad (29)$$

In fact it is easy to see that the function $p_f$ is only a function of $\tau = t' - t$ instead of being a function of $t$ and $t'$ separately. Using the fact that $\tau = t' - t$
and (29), equation (18) can be rewritten as an equation for $p_b$, that is:

$$
\frac{\partial p_b}{\partial \tau}(\tau, x_1, \ldots, x_n, v_1, \ldots, v_n, x'_1, \ldots, x'_n, v'_1, \ldots, v'_n) = \frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2 p_b}{\partial x_i^2} (v_i + a_i^2) + \frac{n}{2} \sum_{i=1}^{n} \frac{\partial^2 p_b}{\partial v_i^2} \varepsilon^2_i v_i + \sum_{i=1}^{n} \frac{\partial^2 p_b}{\partial x_i \partial v_i} \varepsilon_i v_i \rho_i + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \frac{\partial^2 p_b}{\partial x_i \partial x_j} \rho_{i,j} a_i a_j
$$

$$
+ \sum_{i=1}^{n} \frac{\partial p_b}{\partial x_i} \left( \mu_i - \frac{a_i^2}{2} - \frac{v_i}{2} \right) + \sum_{i=1}^{n} \frac{\partial p_b}{\partial v_i} \chi_i (\theta_i - v_i),
$$

with initial condition:

$$
p_b(0, x_1, \ldots, x_n, v_1, \ldots, v_n, x'_1, \ldots, x'_n, v'_1, \ldots, v'_n) = \prod_{i=1}^{n} \delta(x_i - x'_i) \delta(v_i - v'_i),
$$

and with the appropriate boundary conditions. Now we consider the following representation formula for $p_b$:

$$
p_b(\tau, x_1, \ldots, x_n, v_1, \ldots, v_n, x'_1, \ldots, x'_n, v'_1, \ldots, v'_n) =
\frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}} dk_1 e^{ik_1 x'_1} \ldots \int_{\mathbb{R}} dk_n e^{ik_n x'_n} \int_{\mathbb{R}} dl'_1 e^{il'_1 v'_1} \ldots \int_{\mathbb{R}} dl'_n e^{il'_n v'_n} \cdot
\hat{f}(\tau, x_1, \ldots, x_n, v_1, \ldots, v_n, k_1, \ldots, k_n, l'_1, \ldots, l'_n),
$$

where $\hat{f}$ is the Fourier transform with respect to the “future” variables $(x'_1, \ldots, x'_n, v'_1, \ldots, v'_n) \in \mathbb{R}^n \times \mathbb{R}^n$ of the function obtained by extending $p_b$, as a function of the variables $(x'_1, \ldots, x'_n, v'_1, \ldots, v'_n) \in \mathbb{R}^n \times \mathbb{R}^n$, with zero when $v'_j \notin \mathbb{R}^+$, $j = 1, 2, \ldots, n$, moreover $k_j, l_j$, $j = 1, 2, \ldots, n$, are the conjugate variables in the Fourier transform of $x'_j$ and $v'_j$, $j = 1, 2, \ldots, n$. Letting $l'_j = \frac{2}{\varepsilon_j} l_j$, $j = 1, 2, \ldots, n$, from formula (32) we obtain:

$$
p_b(\tau, x_1, \ldots, x_n, v_1, \ldots, v_n, x'_1, \ldots, x'_n, v'_1, \ldots, v'_n) =
\frac{1}{(2\pi)^{2n}} \left( \prod_{j=1}^{n} \frac{2}{\varepsilon_j^2} \right) \int_{\mathbb{R}} dk_1 e^{ik_1 x'_1} \ldots \int_{\mathbb{R}} dk_n e^{ik_n x'_n} \int_{\mathbb{R}} dl_1 e^{\frac{2}{\varepsilon_1} l_1 v_1} \ldots \int_{\mathbb{R}} dl_n e^{\frac{2}{\varepsilon_n} l_n v_n} \cdot
f(\tau, x_1, \ldots, x_n, v_1, \ldots, v_n, k_1, \ldots, k_n, l_1, \ldots, l_n)
$$

where

$$
f(\tau, x_1, \ldots, x_n, v_1, \ldots, v_n, k_1, \ldots, k_n, l_1, \ldots, l_n) =
\hat{f}(\tau, x_1, \ldots, x_n, v_1, \ldots, v_n, k_1, \ldots, k_n, l'_1, \ldots, l'_n(l_n)).
$$

Since the Fourier transform has been taken with respect to the “future” variables $(x'_1, \ldots, x'_n, v'_1, \ldots, v'_n)$, it is easy to see that $f$ satisfies equation (30), so
problem (30) is translated into the problem of solving the following partial differential equation satisfied by the function \( f \) as a function of the “past” variables \((x_1, \ldots, x_n, v_1, \ldots, v_n)\):

\[
\begin{align*}
\frac{\partial f}{\partial \tau}(\tau, x_1, \ldots, x_n, v_1, \ldots, v_n, k_1, \ldots, k_n, l_1, \ldots, l_n) &= \frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2} (v_i + a_i^2) \\
&+ \frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2 f}{\partial v_i^2} \varepsilon_i^2 v_i + \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i \partial v_i} \varepsilon_i v_i \rho_i + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1 \neq i}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j} \rho_{i,j} a_i a_j \\
&+ \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \left( \mu_i - \frac{a_i^2}{2} - \frac{v_i}{2} \right) + \sum_{i=1}^{n} \frac{\partial f}{\partial v_i} \chi_i (\theta_i - v_i)
\end{align*}
\]

(35)

with initial condition:

\[
f(0, x_1, \ldots, x_n, v_1, \ldots, v_n, k_1, \ldots, k_n, l_1, \ldots, l_n) = \prod_{j=1}^{n} e^{-i k_j x_j} e^{-\frac{\varepsilon_j}{2} l_j v_j}
\]

(36)

Since the coefficients of the partial differential operator appearing on the right hand side of (35) are first degree polynomials in \( v_j, j = 1, 2, \ldots, n \), we seek a solution of problem (35), (36) of the following form (see [15]):

\[
f(\tau, x_1, \ldots, x_n, v_1, \ldots, v_n, k_1, \ldots, k_n, l_1, \ldots, l_n) = \prod_{j=1}^{n} e^{-i k_j x_j} e^{-\frac{\varepsilon_j}{2} l_j v_j} B_j(\tau, k_1, \ldots, k_n, l_1, \ldots, l_n)
\]

\[
e^A(\tau, k_1, \ldots, k_n, l_1, \ldots, l_n)
\]

(37)

where \( A \) and \( B_j, j = 1, 2, \ldots, n \), are functions to be determined. Substituting formula (37) into equation (35), after some elementary algebra one notices that, in order to satisfy (35), it is sufficient for the functions \( A \) and \( B_j, j = 1, 2, \ldots, n \) to satisfy the following ordinary differential equations:

\[
\frac{dA}{d\tau}(\tau, k_1, \ldots, k_n, l_1, \ldots, l_n) = -\frac{1}{2} \varepsilon \Gamma \theta D_k - i \sum_{j=1}^{n} k_j \left( \mu_j - \frac{a_j^2}{2} \right) - \sum_{j=1}^{n} B_j \chi_j \theta_j
\]

(38)

\[
\frac{dB_j}{d\tau}(\tau, k_1, \ldots, k_n, l_1, \ldots, l_n) = \frac{\varepsilon_j^2}{2} B_j^2 + (i k_j \varepsilon_j \rho_j + \chi_j) B_j - \frac{1}{2} (k_j^2 - i k_j),
\]

\[
j = 1, 2, \ldots, n,
\]

(39)

with initial condition:

\[
A(0, k_1, \ldots, k_n, l_1, \ldots, l_n) = 0, \quad B_j(0, k_1, \ldots, k_n, l_1, \ldots, l_n) = i l_j, \quad j = 1, 2, \ldots, n,
\]

(40)
where \( \underline{k} = (k_1, k_2, \ldots, k_n)^T \in \mathbb{R}^n \) and the superscript \( T \) means transposed.

Note that the initial condition (40) has been obtained by imposing (36) on the function \( f \) given by (37). Equation (39) is a Riccati equation that can be solved elementarily, then integrating (38) with respect to \( \tau \) we obtain:

\[
A(\tau, k_1, \ldots, k_n, l_1, \ldots, l_n) = -\frac{1}{2} k^T D \Gamma D k \tau - \frac{1}{4} \sum_{j=1}^{n} k_j \left( \mu_j - \frac{a_j^2}{2} \right)
- \sum_{j=1}^{n} \frac{2 \chi_j \theta j}{\varepsilon_j^2} \ln \left( \frac{(v_j + \zeta_j - \nu l_j) e^{-2 \zeta_j \tau} + (\nu l_j - v_j + \zeta_j)}{2 \zeta_j} \right)
- \sum_{j=1}^{n} \frac{2 \chi_j \theta j (v_j + \zeta_j)}{\tau_j}
\]

(41)

\[
B_j(\tau, k_j, l_j) = \frac{(v_j - \zeta_j)(v_j + \zeta_j - \nu l_j) e^{-2 \zeta_j \tau} + (v_j + \zeta_j)(\nu l_j - v_j + \zeta_j)}{(v_j + \zeta_j - \nu l_j) e^{-2 \zeta_j \tau} + (\nu l_j - v_j + \zeta_j)}
\]

\[ j = 1, 2, \ldots, n, \]

(42)

where

\[
\nu_j = -\frac{1}{2} (\chi_j + \nu k_j \varepsilon_j \rho_j), \quad j = 1, 2, \ldots, n,
\]

(43)

\[
\zeta_j = \frac{1}{2} \left( 4 \nu_j^2 + \varepsilon_j^2 (k_j^2 - \nu k_j) \right)^{1/2}, \quad j = 1, 2, \ldots, n.
\]

(44)

Substituting equations (41) and (42) into equation (33) and integrating with respect to the variables \( l_j, j = 1, 2, \ldots, n \) (see, for example, Oberhettinger, 1973 [18]), an elementary computation gives formula (20).

4 A calibration problem for the GH model and simulated results

In this section we use the maximum likelihood approach to calibrate the parameters of the stochastic model (14) and (15). This model is parameterized by \( 6n + n(n - 1)/2 \) real quantities: \( \mu_i, \chi_i, \theta_i, \varepsilon_i, a_i, i = 1, 2, \ldots, n \) and the correlation coefficients \( \rho_i, i = 1, 2, \ldots, n \), \( \rho_{i,j}, i = 1, 2, \ldots, n-1, j = i+1, i+2, \ldots, n \). Moreover, we consider the initial stochastic volatilities \( \tilde{v}_i, i = 1, 2, \ldots, n \), as parameters that must be determined since they are not observable in the financial markets. In this way we must assign \( 7n + n(n - 1)/2 \) parameters. To
this end we introduce the set of the feasible parameters:

$$\mathcal{M} = \left\{ \Theta \in \mathbb{R}^{7n+n(n-1)/2}, \Theta = (\epsilon_i, \theta_i, \rho_i, \chi_i, \mu_i, a_i, \bar{v}_i), \quad |\epsilon_i, \chi_i, \theta_i, a_i| \geq 0, \right\}$$

\[
\frac{2\chi_i\theta_i}{\epsilon_i^2} > 1,\ -1 < \rho_i < 1,\ i = 1, 2, \ldots, n, \]
\[
-1 < \rho_{i,j} < 1,\ i = 1, 2, \ldots, n-1,\ j = i+1, i+2, \ldots, n \}
\]

(45)

and we choose \(m_{ob}\) asset log-return observations. We denote with \(\bar{x}_{j,i}\) the \(j-th\) asset log-return observed at \(t = t_i, j = 1, 2, \ldots, n, i = 1, 2, \ldots, m_{ob}\) and we choose \(t_i < t_{i+1}, i = 1, 2, \ldots, m_{ob}\) with \(t_1 = 0\) and \(t_{m_{ob}+1} = +\infty\).

We consider the (log-likelihood) function defined by:

\[
F(\Theta) = \sum_{i=1}^{m_{ob}^{-1}} \ln p_{M,f}^*(\bar{x}_{1,i}, \bar{x}_{2,i}, \ldots, \bar{x}_{n,i}, t_i, \bar{x}_{1,i+1}, \bar{x}_{2,i+1}, \ldots, \bar{x}_{n,i+1}, t_{i+1} )
\]

(46)

where \(p_{M,f}^*(x_1, x_2, \ldots, x_n, t, x'_1, x'_2, \ldots, x'_n, t'| \Theta) = p_{M,f}(x_1, x_2, \ldots, x_n, \bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n, t, x'_1, x'_2, \ldots, x'_n, t')\) We note that \(p_{M,f}^*\) can be rewritten as follows:

\[
p_{M,f}^*(x_1, x_2, \ldots, x_n, t, x'_1, x'_2, \ldots, x'_n, t'| \Theta) = \int_{\mathbb{R}^n} dk e^{-\frac{(t-t_i)^T D(\Gamma-I) Dk}{2}} \prod_{j=1}^{n} e^{-\frac{(t'_j-t_j)^2}{2}} g_j(k_j, x_j, x'_j, t'-t), \Theta \in \mathcal{M},
\]

(47)

where \(k = (k_1, k_2, \ldots, k_n) \in \mathbb{R}^n\), and for \(j = 1, 2, \ldots, n,\)

\[
g_j(k_j, x_j, x'_j, \tau) = \frac{1}{2\pi} e^{\frac{1}{2} k_j^T [x'_j - x_j - \tau (\mu_j + \frac{1}{2} \epsilon_j^2)]} e^{-2 x_j \theta_j (\zeta_j + \nu_j) \tau / \epsilon_j^2} e^{-2 x_j \theta_j \ln (s_{j,\alpha} / (2 \zeta_j)) / \epsilon_j^2} e^{-2 \zeta_j (\zeta_j - \nu_j) s_{j,\alpha} / (\epsilon_j^2 s_{j,\alpha})}, k_j, x_j, x'_j \in \mathbb{R}, \tau \in \mathbb{R}^+.\]

(48)

The \(n\)-dimensional integral appearing in (47) can be reduced to the sum of the products of one dimensional integrals when \(\tau = t' - t\) goes to zero. In fact, we have:

\[
e^{-\frac{1}{2} \tau k^T D(\Gamma-I) Dk} = 1 - \frac{1}{2} \tau k^T D(\Gamma-I) Dk + o(\tau),\ \tau \to 0.
\]

(49)

Using (49) in (47) we approximate the function \(p_{M,f}^*\) with the following func-
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\[ p_{M,f}(x_1, x_2, \ldots, x_n, t, x'_1, x'_2, \ldots, x'_n, t'|\Theta) = \prod_{h=1}^{n} \int_{\mathbb{R}^+} dk_h e^{-\frac{(\alpha' - \alpha_h)}{2} k_h^2 a_h^2} g_h(k_h, x_h, x'_h, t' - t) + \]

\[ -\frac{1}{2} \tau \sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{i,j} a_i a_j \left( \prod_{h=1, h \neq i, h \neq j}^{n} \int_{\mathbb{R}^+} dk_h e^{-\frac{(\alpha' - \alpha_h)}{2} k_h^2 a_h^2} g_h(k_h, x_h, x'_h, t' - t) \right) \cdot \left( \int_{\mathbb{R}^+} dk_i k_i e^{-\frac{(\alpha' - \alpha_i)}{2} k_i^2 a_i^2} g_i(k_i, x_i, x'_i, t' - t) \right) \cdot \left( \int_{\mathbb{R}^+} dk_j k_j e^{-\frac{(\alpha' - \alpha_j)}{2} k_j^2 a_j^2} g_j(k_j, x_j, x'_j, t' - t) \right), \quad \Theta \in \mathcal{M}. \] (50)

where \( g_i(k_i, x_i, x'_i, \tau), k_i, x_i, x'_i \in \mathbb{R}, \tau \in \mathbb{R}^+, i = 1, 2, \ldots, n, \) are given in (48).

Using (50) we approximate the log-likelihood function (46) with the following function:

\[ F^a(\Theta) = \sum_{i=1}^{m_{ob}-1} \ln p_{M,f}^a(\tilde{x}_{1,i}, \tilde{x}_{2,i}, \ldots, \tilde{x}_{n,i}, t_i, \tilde{x}_{1,i+1}, \tilde{x}_{2,i+1}, \ldots, \tilde{x}_{n,i+1}, t_{i+1}|\Theta), \]

\[ \Theta \in \mathcal{M}, \] (51)

and we propose the following calibration problem:

\[ \max_{\Theta \in \mathcal{M}} F^a(\Theta). \] (52)

We note that problem (52) is solvable with an affordable computer time also when the number of the assets in portfolios are few tens and that the solution \( \Theta^a \) of problem (52) is an approximation of the solution of the following problem:

\[ \max_{\Theta \in \mathcal{M}} F(\Theta), \] (53)

that can be stated: given the observations \((\tilde{x}_{1,i}, \tilde{x}_{2,i}, \ldots, \tilde{x}_{n,i})\) at time \( t = t_i, i = 1, 2, \ldots, m_{ob}, \) determine the vector \( \Theta^* \in \mathcal{M} \subset \mathbb{R}^{7n+n(n-1)/2} \) that makes “more likely” the observations made.

Note that problem (52) is one of the simplest way of formulating the calibration problem, it can be improved as suggested, for example, in [6].

The calibration procedure solves problem (52) using a technique based on a variable metric steepest ascent method. Specifically, starting from an initial guess \( \Theta^0 \in \mathcal{M}, \) at every iteration, the current approximation of the solution of (52) is updated with a step in the direction of the gradient of the (log-)likelihood function (51) computed in a suitable metric (see [19] and [17]).
<table>
<thead>
<tr>
<th>$i$</th>
<th>$\epsilon_i$</th>
<th>$\theta_i$</th>
<th>$\rho_i$</th>
<th>$X_i$</th>
<th>$\mu_i$</th>
<th>$a_i$</th>
<th>$\tilde{v}_{0,i}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.8</td>
<td>0.1</td>
<td>-0.9</td>
<td>20.94</td>
<td>0.026</td>
<td>0.05</td>
<td>0.7</td>
</tr>
<tr>
<td>2</td>
<td>0.05</td>
<td>0.025</td>
<td>-0.55</td>
<td>0.94</td>
<td>0.01</td>
<td>0.05</td>
<td>0.3</td>
</tr>
<tr>
<td>3</td>
<td>0.1</td>
<td>0.15</td>
<td>-0.25</td>
<td>3.0</td>
<td>-0.015</td>
<td>0.05</td>
<td>0.1</td>
</tr>
<tr>
<td>4</td>
<td>0.12</td>
<td>0.2</td>
<td>-0.25</td>
<td>13.0</td>
<td>-0.03</td>
<td>0.05</td>
<td>0.01</td>
</tr>
</tbody>
</table>

Table 1: Value of the model parameters used to generate the synthetic data

Let us go into the details of the numerical experiments. The first experiment tests the reliability of the aforementioned calibration procedure. We choose $n = 4$ assets. The first components of the vector $\Theta^*$ used to generate the data are shown in Table 1 and the remaining components which are the correlation coefficients $\rho_{i,j}$ are chosen as follows: $\rho_{i,j} = -0.5$, $i = 1, 2, \ldots, n-1$, $j = i+1, i+2, \ldots, n$. We simulate data integrating numerically one trajectory of the stochastic differential equations (14), (15) (whose parameters are assigned by $\Theta^*$) with the initial conditions (16), (17) using the explicit Euler’s method. Specifically, we generate daily data for a period of thirteen months. We consider a “year” made of 252 working days and a “month” of 21 working days. As a consequence, we generate $21*13$ daily log-return observations $\tilde{x}_{i,t}$, $i = 1, 2, \ldots, 4$, $t = 1, 2, \ldots, 21*13$.

To proceed with the calibration we consider four time windows of daily data: twelve-months, six-months, three-months and one-month. We denote with $\Theta_{12}, \Theta_6, \Theta_3, \Theta_1$ the vectors obtained by the calibration procedure applied to these four windows. As previously mentioned, the calibration procedure consists in solving problem (52) and the vectors $\Theta_{12}, \Theta_6, \Theta_3, \Theta_1$ are the numerical solutions of this problem for the different time windows.

In the calibration procedure of the four time windows we use the simulated data relative to the first twelve months while we use the data of the last month to select the best vector from among $\Theta_{12}, \Theta_6, \Theta_3, \Theta_1$. Furthermore, all parameters are expressed in values per annum.

Table 2 and Table 3 show the first components of the vectors $\Theta_{12}, \Theta_6, \Theta_3, \Theta_1$. Looking at the tables we can see that the Euclidean distances of the vectors $\Theta_{12}, \Theta_6, \Theta_3, \Theta_1$ are sufficiently small. This finding is coherent with the fact that the data used in the calibration have been generated using only one vector, that is the vector $\Theta^*$ shown in Table 1. Hence, we must expect that $\Theta_{12} \approx \Theta_6 \approx \Theta_3 \approx \Theta_1 \approx \Theta^*$. In fact we have:

$$
\|\Theta_1 - \Theta^*\| = 0.0594, \|\Theta_3 - \Theta^*\| = 0.0542, \\
\|\Theta_6 - \Theta^*\| = 0.0279, \|\Theta_{12} - \Theta^*\| = 0.0246.
$$

(54)

Now we propose a criterion to choose one of the vectors $\Theta_{12}, \Theta_6, \Theta_3, \Theta_1$. This criterion can be useful when we work with real data where we do not
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We forecast the daily price of the four assets during the thirteenth month using formula (26). We select as the best vector the one that minimizes the mean of the relative errors between the observed and forecast prices from among $\Theta_1, \Theta_3, \Theta_6, \Theta_{12}$. We stress that the one day-ahead forecast values of the asset price $i$, $\hat{S}_{i,t,t+\Delta t}$, are computed using the following formula: $\hat{S}_{i,t,t+\Delta t} = e^{\hat{x}_{i,t,t+\Delta t}}$, where $\Delta t = 1/252$, and $\hat{x}_{i,t,t+\Delta t}$ is given by Eq. (26). In this equation we have chosen $\Theta$ equal to $\Theta_j$, $j = 1, 3, 6, 12$.

The mean relative errors on the forecast values obtained using $\Theta_1, \Theta_3, \Theta_6, \Theta_{12}$ are, respectively, 0.0540, 0.0469, 0.0504 and 0.0554.

By applying the criterion mentioned above we select $\Theta_j$ as the best vector. We conclude showing the quality of the forecast prices, $\hat{S}_{i,t,t+\Delta t}$, $i = 1, 2, 3, 4$. The quality is determined comparing the forecast and real (simulated) asset prices. Figure 2 shows the relative errors, $\epsilon_{i,t,t+\Delta t, i} = |\tilde{S}_{i,t,t+\Delta t} - \exp(\hat{x}_{i,t,t+\Delta t})|/|\exp(\tilde{x}_{i,t,t+\Delta t})|$, as a function of $t$ expressed in days. In Figure 2 the variable $t$ ranges from $t = 1$ to $t = 30$. That is, we consider the one day-ahead forecasts in the thirty days that follow the time period considered in the calibration procedure. The average relative error is about 5% and the largest error is 10%.

Figure 1: Simulated asset prices as a function of time

know the true values of the model parameters.
<table>
<thead>
<tr>
<th>$\Theta_1$</th>
<th>$\epsilon_i$</th>
<th>$\theta_i$</th>
<th>$\rho_i$</th>
<th>$\chi_i$</th>
<th>$\alpha_i$</th>
<th>$\tilde{v}_{i,0}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i = 1$</td>
<td>0.8123</td>
<td>0.0933</td>
<td>-0.9357</td>
<td>20.2483</td>
<td>0.0258</td>
<td>0.04584</td>
</tr>
<tr>
<td>$i = 2$</td>
<td>0.0463</td>
<td>0.0234</td>
<td>-0.5436</td>
<td>0.8530</td>
<td>0.010</td>
<td>0.04681</td>
</tr>
<tr>
<td>$i = 3$</td>
<td>0.1016</td>
<td>0.1552</td>
<td>-0.2297</td>
<td>2.740</td>
<td>-0.0160</td>
<td>0.0512</td>
</tr>
<tr>
<td>$i = 4$</td>
<td>0.1126</td>
<td>0.1866</td>
<td>-0.2729</td>
<td>14.2761</td>
<td>-0.0277</td>
<td>0.0508</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\Theta_3$</th>
<th>$\epsilon_i$</th>
<th>$\theta_i$</th>
<th>$\rho_i$</th>
<th>$\chi_i$</th>
<th>$\alpha_i$</th>
<th>$\tilde{v}_{i,0}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i = 1$</td>
<td>0.07343</td>
<td>0.1012</td>
<td>-0.9148</td>
<td>22.2365</td>
<td>0.0265</td>
<td>0.0450</td>
</tr>
<tr>
<td>$i = 2$</td>
<td>0.0538</td>
<td>0.0275</td>
<td>-0.5749</td>
<td>1.0277</td>
<td>0.0096</td>
<td>0.0459</td>
</tr>
<tr>
<td>$i = 3$</td>
<td>0.1080</td>
<td>0.1546</td>
<td>-0.2701</td>
<td>3.2769</td>
<td>-0.0140</td>
<td>0.0510</td>
</tr>
<tr>
<td>$i = 4$</td>
<td>0.1298</td>
<td>0.1918</td>
<td>-0.2718</td>
<td>12.7781</td>
<td>-0.0289</td>
<td>0.0530</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\Theta_6$</th>
<th>$\epsilon_i$</th>
<th>$\theta_i$</th>
<th>$\rho_i$</th>
<th>$\chi_i$</th>
<th>$\alpha_i$</th>
<th>$\tilde{v}_{i,0}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i = 1$</td>
<td>0.7675</td>
<td>0.0985</td>
<td>-0.9719</td>
<td>21.5808</td>
<td>0.0273</td>
<td>0.0459</td>
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<tr>
<td>$i = 2$</td>
<td>0.0466</td>
<td>0.0268</td>
<td>-0.5948</td>
<td>0.9013</td>
<td>0.0093</td>
<td>0.0586</td>
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<tr>
<td>$i = 3$</td>
<td>0.0962</td>
<td>0.1504</td>
<td>-0.2448</td>
<td>3.1739</td>
<td>-0.0148</td>
<td>0.0507</td>
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<tr>
<td>$i = 4$</td>
<td>0.1222</td>
<td>0.1837</td>
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<td>12.8239</td>
<td>-0.0306</td>
<td>0.0530</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\Theta_{12}$</th>
<th>$\epsilon_i$</th>
<th>$\theta_i$</th>
<th>$\rho_i$</th>
<th>$\chi_i$</th>
<th>$\alpha_i$</th>
<th>$\tilde{v}_{i,0}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i = 1$</td>
<td>0.7703</td>
<td>0.1080</td>
<td>-0.9500</td>
<td>21.3303</td>
<td>0.0265</td>
<td>0.0449</td>
</tr>
<tr>
<td>$i = 2$</td>
<td>0.0519</td>
<td>0.0233</td>
<td>-0.5447</td>
<td>0.9005</td>
<td>0.0095</td>
<td>0.0458</td>
</tr>
<tr>
<td>$i = 3$</td>
<td>0.1037</td>
<td>0.1628</td>
<td>-0.2491</td>
<td>3.2034</td>
<td>-0.0155</td>
<td>0.0510</td>
</tr>
<tr>
<td>$i = 4$</td>
<td>0.1226</td>
<td>0.1873</td>
<td>-0.2612</td>
<td>13.4085</td>
<td>-0.0289</td>
<td>0.0539</td>
</tr>
</tbody>
</table>

Table 2: All the components of the vectors except the correlation coefficients $\rho_{i,j}, i = 1, 2, \ldots, n-1, j = i+1, i+2, \ldots, n, n = 4$, obtained by the calibration procedure (see formula (45))

<table>
<thead>
<tr>
<th>$\Theta$</th>
<th>$\rho_{1,2}$</th>
<th>$\rho_{1,3}$</th>
<th>$\rho_{1,4}$</th>
<th>$\rho_{2,3}$</th>
<th>$\rho_{2,4}$</th>
<th>$\rho_{3,4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Theta_1$</td>
<td>-0.0524</td>
<td>-0.0481</td>
<td>-0.0494</td>
<td>-0.0491</td>
<td>-0.0541</td>
<td>-0.0549</td>
</tr>
<tr>
<td>$\Theta_3$</td>
<td>-0.0547</td>
<td>-0.0469</td>
<td>-0.0501</td>
<td>-0.0490</td>
<td>-0.0476</td>
<td>-0.0524</td>
</tr>
<tr>
<td>$\Theta_6$</td>
<td>-0.0509</td>
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<td>-0.0538</td>
<td>-0.0550</td>
<td>-0.0523</td>
<td>-0.0547</td>
</tr>
<tr>
<td>$\Theta_{12}$</td>
<td>-0.0547</td>
<td>-0.0469</td>
<td>-0.0501</td>
<td>-0.0490</td>
<td>-0.0476</td>
<td>-0.0524</td>
</tr>
</tbody>
</table>

Table 3: The correlation coefficients $\rho_{i,j}, i = 1, 2, 3, j = i+1, i+2, \ldots, 4$, contained in the vectors obtained by the calibration procedure (see formula (45))
Figure 2: Relative errors on the forecast asset prices

References


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