The Concentration Spectrum of the Poles in a Given Region at the Compensating Approach
to the Synthesis of the Feedback Matrix

E. G. Kreventsov

Research Institute of Measuring Instruments
Novosibirsk, Russia

Abstract

The problem of finding an algorithm for matrix feedback, providing the specified zeros individual transfer functions, until finally solved in control theory of linear stationary systems. The siting of individual zeros of transfer functions has been widely discussed. As for a multidimensional system matrix system provides a predetermined range of the poles is not unique, then, generally speaking, this is not the only can be used to organize the above mentioned zeros, followed by the possibility of compensation. However, procedures that would allow arbitrarily select all the zeros of the transfer functions of individual multidimensional system does not yet exist. Attempt to solve this problem and the subject of this article.

Keywords: Shift zeros of the transfer functions, The calculation of the matrix the feedback in the original basis, Providing a full range of system poles, Compensation zeros of the transfer matrix, Modification methods for calculating matrix feedback.

1. Introduction

The modern control theory of linear systems based on state-space methods [1-3, 6, 7, 11, 12], has an extensive array of feedback systems [3, 5, 12, 15].
However, the matrix of feedback (SF) satisfies only one requirement - placing the roots of the characteristic equation of the system. The rest of the requirements for the control system (such as decoupling the control channels of the system, the compensation of the transfer matrix of zeros) are ignored in the calculation of the matrix. The theory of linear systems gives rise explicit and implicit algorithms for computing the coefficients of the matrix [2, 5, 6, 7, 14]. Under the explicit algorithms are understood such calculation algorithms, which can be made nonlinear equations for the unknown coefficients of the matrix SF, and their number is equal to the number of unknowns [2, 11, 14]. Among such algorithms can also be attributed, and such algorithms in which the transition from the original basis for a new, more appropriate to calculate the coefficients of the matrix [2, 11]. By the implicit algorithms should be attributed such algorithms, which formed the system of nonlinear equations with a number greater than the number of unknown coefficients compiled equations. By the implicit algorithms should include such algorithms, which formed the system of nonlinear equations with a number greater than the number of unknown coefficients compiled equations. In such cases, part of the "free" matrix coefficients given arbitrarily, or from which, or the requirements for the system [1], and thus, the original is not redefining the system of nonlinear equations becomes a system to be equal to the number of unknowns and equations. However, in all known systems is not fully resolved the issue of compensation zeros of the transfer matrix [1, 6, 7, 11, 14]. In domestic and foreign literature payment methods such zeros and management is not fully developed, since when the spectrum of individual zeros of the transfer matrix is changed and range poles. Not compensated zeros strongly distort the resulting transient response of the system [14]. Such systems generally provide only a shift of the pole, leaving unchanged the number of dominant zeroes. For example, by placing all of the eigenvalues of the system on the real axis of the complex plane, the resulting transient response overshoot can be 60 percent or more. Similar problems in the modern control theory of linear systems are solved by selecting such an arbitrary spectrum of poles at which the transient quality output meets the specified requirements. Avoid such drawbacks modifications allow various kinds of methods of calculation of the matrix feedback, in which the result set of zeros of the transfer matrix offset without degrading the quality of the remaining indicators.

2. Statement of the problem

Consider a multidimensional, fully control and observe a linear stationary system \((S)\) which has the dimension of the vectors of input and output match. Let \(S\) be a state-space description:

\[
\begin{aligned}
\dot{x} &= (A + BK)x + Bu, \\
y &= Cx,
\end{aligned}
\]  

\(1\)
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where $x$ – $n$ - dimensional vector of state coordinates, $v$ – $m$ - dimensional vector of control impacts, $y$ – $m$ - dimensional vector of output variables, $n \geq m$, $A$ – own matrix system, $B$ – matrix controls, $C$ – matrix output, $K$ – feedback matrix vector coordinates state. Dimension of these vectors and matrices as follows:

$x \in \mathbb{R}^n$, $v \in \mathbb{R}^m$, $y \in \mathbb{R}^m$, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C, K \in \mathbb{R}^{m \times m}$. It is assumed that the vector of the state coordinates $x$ is fully accessible to direct measurement.

Require that it was possible to calculate the feedback matrix $K$, through the matrix $A$, $B$, $C$, i.e., in the original basis. Require that the algorithm for calculating the matrix feedback provides a set of full range of poles of the system (1) and the full range of individual zeros of transfer functions. Require that the poles and zeros of the transfer functions for individual $S$ were coincident as to be able to put all the poles of the system in strictly defined points of the complex plane.

3. The method of solution

We represent the output equation in (1), as

$$\begin{bmatrix} y_i \\ y_{i+1} \end{bmatrix}_{i=1}^{m} = \begin{bmatrix} c_i (A + BK)^0 \\ c_i (A + BK)^1 \end{bmatrix}_{i=1}^{m} x, \quad \text{where} \quad c_i \quad \text{i-th row}, \quad i = 1, m, \quad \text{matrix} \quad C \quad \text{in model (1).}$$

Us express $y_i$ through first equation of system (1), we have:

$$\begin{bmatrix} y_i \\ y_{i+1} \end{bmatrix}_{i=1}^{m} = \begin{bmatrix} c_i A^{i/2} (A + BK) \end{bmatrix}_{i=1}^{m} x + \begin{bmatrix} c_i A^{i/2} B \end{bmatrix}_{i=1}^{m} v. \quad (2)$$

Differentiation will continue as long as the block matrix $[c_i A^{i/2} B]_{i=1}^{m}$ will give the first non-zero block row. Let there be given positive integers number $j_1, j_2, \ldots, j_m$, which are the minimal indices of the matrix product $c_i A^{i/2} B$, $i = 1, m$, $j = 0, n - 1$:

$$j_i = \min_j \left\{ \begin{array}{l} c_i A^{i/2} \neq [0]^{k \times m} \\
 j = 0, n - 1, i = 1, m,
 \end{array} \right. \quad \text{and} \quad j_i = (n - 1) \quad \text{when} \quad j_i \neq (n - 1).

(3)

Then the computation of the matrix product $c_i A^{i/2} B$, gives:

$$c_i (A + BK)^k = c_i A^k, \quad k = 0, j_i,$n
\begin{align*}
\begin{aligned}
(c_i A^{i/2} (A + BK))^{k - j_i} & = c_i A^{i/2} (A + BK)^{k - j_i}, \quad k = (j_i + 1), n.
\end{aligned}
\end{align*}

(4)

Differentiation of (2), taking into account (4) provides:
Given Kelly Hamilton theorem and expression (4) we have:

\[
\[ y \]_{i=1}^{m} = [ c_{i} (A + BK) ]_{i=1}^{m} x,
\]

\[
\[ y^{(j_{i})} \]_{i=1}^{m} = [ c_{i} (A + BK)^{j_{i}} ]_{i=1}^{m} x,
\]

\[
\[ y^{(j_{i}+1)} \]_{i=1}^{m} = [ c_{i} (A + BK)^{j_{i}+1} ]_{i=1}^{m} x + [ c_{i} (A + BK)^{j_{i}} B ]_{i=1}^{m} v,
\]

\[
\[ y^{(n)} \]_{i=1}^{m} = [ c_{i} (A + BK)^{n} ]_{i=1}^{m} x + [ c_{i} (A + BK)^{n-1} B ]_{i=1}^{m} v + \ldots
\]

\[
\ldots + [ c_{i} (A + BK)^{j_{i}} B ]_{i=1}^{m} v^{(n-j_{i}-1)}.
\]

Given Kelly Hamilton theorem and expression (4) we have:

\[
[ c_{i} (A + BK)^{n} ]_{i=1}^{m} = - \left[ c_{i} \sum_{k=0}^{n-1} \alpha_{k} (A + BK)^{k} \right]_{i=1}^{m},
\]

where \( \alpha_{k} \) - constants. As a result of (2) may be represented as

\[
[ y^{(n)} ]_{i=1}^{m} - \left[ \sum_{k=0}^{n-1} \alpha_{k} y^{(k)} \right]_{i=1}^{m} = Sp(L(K) \Theta),
\]

where \( Sp(\bullet) \) - denotes the trace of \( \Theta \), \( \Theta \subseteq R^{mxn} \). Matrix \( \Theta \) in (7) has the form:

\[
\Theta = \begin{pmatrix}
\cdots & \cdots & \cdots & \cdots \\
\end{pmatrix},
\]

Matrix \( L(K) \) - has the form, \( L(K) \in R^{nxn} \):

\[
L(K) = \begin{pmatrix}
\cdots & \cdots & \cdots & \cdots \\
\end{pmatrix}.
\]

Next we consider the \( i \) - fifth non-zero block row of the matrix (8) -

\[
L_{i}(K) = c_{i} (A + BK)^{j_{i}} B, \quad i = 1, m, \quad \text{then rank} \ L_{i}(K) = 1.
\]

Using the expression (4), we have:

\[
\left( c_{1} A^{j_{1}+k} \cdots c_{m} A^{j_{m}+k} \right)^{T} = [0]^{b \times n}.
\]
The resulting expression is valid for any positive integer \( k \), \( k \in \mathbb{Z} \). Write (2) as follows:

\[
\begin{bmatrix}
  y_{i}^{(j_{i+1})}
\end{bmatrix}_{j_{i}=1,m} = \begin{bmatrix}
  c_{j}A^{j_{i}+1} + c_{j}A^{j_{i}}BK
\end{bmatrix}_{j_{i}=1,m} x + \begin{bmatrix}
  c_{j}A^{j_{i}}B
\end{bmatrix}_{j_{i}=1,m} v. \tag{9}
\]

Rewrite (9), using (6) [9]:

\[
\begin{bmatrix}
  \sum_{k=0}^{j_{i}+1} \alpha_{k}y_{i}^{(k)}
\end{bmatrix}_{j_{i}=1,m} = \begin{bmatrix}
  c_{j}A^{j_{i}+1} + c_{j} \sum_{k=0}^{j_{i}} \alpha_{k}A^{k} + c_{j}A^{j_{i}}BK
\end{bmatrix}_{j_{i}=1,m} x + \begin{bmatrix}
  c_{j}A^{j_{i}}B
\end{bmatrix}_{j_{i}=1,m} v, \tag{10}
\]

where \( \alpha_{k} \) - factors, the magnitude of which is unknown. These factors define the poles of the spectrum \( S \). Given the differential equation (10), we introduce the following notation [8] - matrix \( D \)- nonsingular \( \det D \neq 0 \) and \( \exists D^{-1}, D \in \mathbb{R}^{m \times m} \), \( \text{rank } D = m \):

\[
D = \begin{bmatrix}
  \begin{bmatrix}
  c_{1}A^{j_{1}}B \\
  c_{2}A^{j_{2}}B \\
  \vdots \\
  c_{m}A^{j_{m}}B
\end{bmatrix}
\end{bmatrix}^{T}, \tag{11}
\]

matrix \( A^{*}, A^{*} \in \mathbb{R}^{m \times m} \):

\[
A^{*} = \begin{bmatrix}
  \begin{bmatrix}
  c_{1}A^{j_{1}+1} \\
  c_{2}A^{j_{2}+1} \\
  \vdots \\
  c_{m}A^{j_{m}+1}
\end{bmatrix}
\end{bmatrix}^{T}. \tag{12}
\]

Considering (6) and (10) define a matrix of poles \( K_{p}, K_{p} \in \mathbb{R}^{m \times m} \):

\[
K_{p} = \begin{bmatrix}
  \begin{bmatrix}
  \sum_{k=0}^{j_{1}} \alpha_{k}A^{k_{1}}
  \end{bmatrix} \\
  \begin{bmatrix}
  \sum_{k=2}^{j_{2}} \alpha_{k}A^{k_{2}}
  \end{bmatrix} \\
  \vdots \\
  \begin{bmatrix}
  \sum_{km=0}^{jm} \alpha_{km}A^{km}
  \end{bmatrix}
\end{bmatrix}^{T}, \tag{13}
\]

where \( \alpha_{k}, k = 0, j_{i} \) - coefficients, the amount of which is determined by the spectrum and \( S \) poles of the system can be specified in the synthesis of control systems. Since \( (A^{*} + K_{p} + D \cdot K_{p}) = [0] \), from the last expression, we can find a matrix \( K \), given the notation (11), (12), (13):

\[
K = -D^{-1} \cdot [A^{*} + K_{p}] = -\begin{bmatrix}
  \begin{bmatrix}
  c_{j}A^{j_{i}+1}
  \end{bmatrix}_{j_{i}=1,m}^{-1} \cdot \begin{bmatrix}
  c_{j}A^{j_{i}+1} + \sum_{k=0}^{j_{i}} \alpha_{k}c_{j}A^{k}
  \end{bmatrix}_{j_{i}=1,m}
\end{bmatrix}. \tag{14}
\]

Coefficients \( \alpha_{k} \), in the resulting expression, we ask, for example, the binomial expansion:

\[
\alpha_{k} = C_{j_{i}+1}^{k} \cdot \Omega_{k}^{k} = \frac{\Omega_{k}^{k} \cdot (j_{i}+1)!}{k! \cdot (j_{i}+1-k)!}, \quad k = 1, (j_{i}+1),
\]

where \( \Omega_{k}^{k} \) - coefficients, the amount of which is determined by the spectrum and \( S \) poles of the system can be specified in the synthesis of control systems.
where $\binom{k}{j+1}$ - binomial coefficients (can be determined by "Pascal's triangle scheme"), $\Omega$ - geometric average root. Considering equation (5) can be written to the transfer matrix $S$, $W(p) \in \mathbb{R}[p]^{m \times m}$:

$$W(p) = \frac{1}{\sum_{k=0}^{n} \alpha_k p^k} \left[ e_i A^{ji} \left( \sum_{k=0}^{n-j_i-1} \alpha_k p^k \right) B \right]_{j_i=1,m}. \quad (15)$$

In the expression (15) $p$ - Laplace operator.

**Definition 1.**

Set of complex numbers $z_i$ is called trailing zeros of the system $S$ of the form (1), if the rank of the transfer matrix (15) decreases with $p = z_i$, $i = 1, \mu$, $\mu \leq (n-m)$, $\mu \in \mathbb{Z}_{>0}$:

$$\text{rank} \left[ \frac{1}{\sum_{k=0}^{n} \alpha_k p^k} \left[ e_i A^{ji} \left( \sum_{k=0}^{n-j_i-1} \alpha_k p^k \right) B \right]_{j_i=1,m} \right] = p = z_i < m. \quad (16)$$

Since we consider a system with the same number of inputs and outputs $m$, then the above mentioned definition may be amended as follows: if the system is controllable and observable, then its transfer matrix has one minor $m$-th order, which can be represented as:

$$\det \left[ \frac{1}{\sum_{k=0}^{n} \alpha_k p^k} \left[ e_i A^{ji} \left( \sum_{k=0}^{n-j_i-1} \alpha_k p^k \right) B \right]_{j_i=1,m} \right] = \psi(p) \cdot \left( \sum_{k=0}^{n} \alpha_k p^k \right). \quad (17)$$

In this case, the zero polynomial $\psi(p) \in \mathbb{R}[p]^\mu$ - “numerator” minor (17):

$$\psi(p) = \det \left( \frac{pI_n - (A + BK) \ C}{\text{B}} \right) \left[ \begin{array}{c} -B \end{array} \right] = \det \left[ e_i A^{ji} \left( \sum_{k=0}^{n-j_i-1} \alpha_k p^k \right) B \right]_{j_i=1,m} \left( \sum_{k=0}^{n} \alpha_k p^k \right)^{m-1}. \quad (18)$$

Expression (18) is valid for any $K$, and in particular for the $K=[0]$. This follows directly from the principle of invariance of the set of trailing zeros for the matrix.
Concentration spectrum of poles

Block matrix, standing under the sign of the determinant in (18) is called the "Rosenbrock matrix" or "matrix system" [14], for which we introduce the following notation - $P(p), \ P(p) \in \mathbb{R}[p]^{(n+m) \times (n+m)}$. 

**Definition 2.**

Unimodal (unimodular) [4], we call such a polynomial a square matrix, elementary operation - "taking the determinant" of which gives the actual number of - $\det P(p) = c, \ c \in \mathbb{R}$. Otherwise, the operation "taking the determinant" of a polynomial of a square matrix give a polynomial.

Rewrite the expression (9), using (6):

$$\det \left[ c_i A_i^{j-i} \left( \sum_{k=0}^{n-j-1} \alpha_k \cdot p^k \right) B \right] = \psi(p) \left( \sum_{k=0}^{n} \alpha_k \cdot p^k \right)^{m-1}. \quad (19)$$

We introduce the notation: $\Omega_n$ - set of roots $\psi(p) = 0$, $\Omega_2$ - the set of roots of a polynomial $\left[ \sum_{k=0}^{n} \alpha_k \cdot p^k \right]^{m-1}$, $\Omega_K$ - set of the polynomial on the left side of expression (19). Then the set of $\Omega_K$ can be represented as the union of $\Omega_n$ and $\Omega_2$: $\Omega_K = \Omega_n \cup \Omega_2$, see Fig. 1.

![Fig.1. Set of roots $\Omega_K$ for the case of $\det P(p) = \psi(p)$](image)

Set $\Omega_K$ consists of the set of zeros of individual transfer functions. It is expedient to exclude from set $\Omega_K$ those parts which can not be affected by means of a matrix $K$, i.e. all set $\Omega_n$. As a result, we consider a class of systems with unimodal Rosenbrock matrix $P(p)$. We rewrite (19) in this case:

$$\det \left[ c_i A_i^{j-i} \left( \sum_{k=0}^{n-j-1} \alpha_k \cdot p^k \right) B \right] = c \left( \sum_{k=0}^{n} \alpha_k \cdot p^k \right)^{m-1}. \quad (20)$$
Expression (20) is called "an identity of individual zeros of the system". Consider the set of $\Omega_K$ in this case. A plurality of $\Omega_n$ is an empty set - $\Omega_n = \emptyset$, since in this case the zero polynomial $\psi(p)$, has no roots, by definition 2. Set $\Omega_K$ coincides with the set $\Omega_2$, $\Omega_K = \Omega_2$, see Fig. 2.

Fig.2. Set of roots $\Omega_K$ for the case of $\det P(p) = \psi(p) = cp^0 = c$

As a result, we can conclude that the set of $\Omega_K$ is directly determined by the spectrum of the system poles. That is, the range of individual transfer functions of zeros, in this case, for a multidimensional system, when $m > 1$ can be exposed by means of a shift of $K$. We now consider the individual transfer functions for the system $S$.

\[
\sum_{k=0}^{n} \alpha_k p^k \left[ c_i A^{i_1} \left( \sum_{k=0}^{n-j_i-1} \alpha_k \cdot p^k \right) b_i^T \right]_{i=1,m} = \frac{1}{j_i+1} \sum_{k=0}^{n} \alpha_k p^k \left[ c_i A^{j_i} b_i^T \right]_{i=1,m},
\]

(21)

where $b_i^T$ – column vector consisting of elements arranged in a matrix column $B$, $b_i^T \in \mathbb{R}^{n \times 1}$, $i = 1, m$. From expression (21) shows that for the transfer function $S$, will have a "compensated" zeros. Demonstrativeness of this result can be demonstrated following interpretation. Let us consider the expression (2). Perform calculations for matrices standing for the vector $x$, we have:

\[
\left[ c_i A^{i_j} (A + BK) \right]_{j=1,m} = \left[ c_i A^{i_j} + c_i A^{i_j} b_i^T K \right]_{j=1,m} = [0],
\]

(22)

so, as $\left[ c_i A^{i_j} b_i^T K \right]_{j=1,m} = -\left[ c_i A^{i_{j+1}} \right]_{j=1,m}$. Considering (22), expression (2) can be represented as follows:

\[
\sum_{k=0}^{j_i+1} \alpha_k y_i^{(k)} \left[ c_i A^{j_i} b_i^T \right]_{j=1,m} = \left[ c_i A^{j_i} b_i^T \right]_{j=1,m} v.
\]

(23)
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Considering equation (23) shows that the individual transfer functions are compensated. Since their numerators are no polynomials and the numerical coefficients are determined by the expression \[ c_i A^j b_i^T \]

4. Illustrative example

Consider the example of a system [8], in which the set of trailing zeros is empty (\( \Omega_n = \emptyset \)), and the set of \( \Omega_K \) is not empty. Suppose that the matrix \( A, B, C \) square, and \( A, B, C \in \mathbb{R}^{m \times m} \):

\[
A = \begin{pmatrix}
    a_{11} & \cdots & a_{1m} \\
    \vdots & \ddots & \vdots \\
    a_{m1} & \cdots & a_{mm}
\end{pmatrix}, \quad B = \begin{pmatrix}
    b_{11} & \cdots & b_{1m} \\
    \vdots & \ddots & \vdots \\
    b_{m1} & \cdots & b_{mm}
\end{pmatrix}, \quad C = \begin{pmatrix}
    c_{11} & \cdots & c_{1m} \\
    \vdots & \ddots & \vdots \\
    c_{m1} & \cdots & c_{mm}
\end{pmatrix}.
\]

(24)

Require that the matrix \( B, C \) were nonsingular: \( \det B \neq 0, \det C \neq 0 \). In such a system matrix \( P(p) \) is unimodal Rosenbrock matrix. Since all four blocks of the matrix \( P(p) \) square, order \( m \), then the determinant can be calculated from Schur [4]:

\[
\det P(p) = \det \left( p I_m - A \right) \cdot \det \left( C \right) \cdot \det \left( \left[ p I_m - A \right]^{-1} \right) \cdot \det \left( B \right) = \det \left( CB \right)
\]

(25)

In the resulting expression \( \det P(p) = \det \left( CB \right) \) no polynomials block. Matrix \( B, C \) are numeric and not contain a part of the Laplace operator, then Rosenbrock matrix \( P(p) \), matrix is unimodal. One can easily show that for a system with a square matrix \( S \) (24) all set individual zeros of transfer functions defined by the set of poles:

\[
\left( \sum_{k=0}^{m-1} \alpha_k \cdot p^k \right)^m \cdot \det \left( \left[ c_i, i=1,m \right] B \right) \equiv \det \left( CB \right) \cdot \left( \sum_{k=0}^{m-1} \alpha_k \cdot p^k \right)^{m-1}.
\]

(26)

Resulting characteristic polynomial to a normalized form (unitary) and setting poles spectrum focused at the origin of coordinates - \( \alpha_1 = 1, \alpha_0 = 0 \) we obtain:

\[
\left[ W_i(p) \right]_{i=1,m} = \frac{1}{\sum_{k=0}^{m-1} \alpha_k p^k} \cdot \left( \left[ c_i B \right]_{i=1,m} \right) = \frac{1}{p} \left( CB \right).
\]

(27)
Matrix \( K = -\left[ (CB)^{-1} \cdot (CA + K_P) \right] \) for the system (1), where the matrix poles \( K_P, K_P \in \mathbb{R}^{m \times n} \):

\[
K_P = \left( \begin{array}{cccc}
\alpha_{k1}e_1 & \alpha_{k2}e_2 & \cdots & \alpha_{km}e_m
\end{array} \right)^T.
\] (13a)

The resulting expression (26) and (27) clearly show that all the zeros of the individual transfer functions of the system \( S \) are fully compensated and the whole spectrum of poles can be arbitrarily set using (13) and (13a).

5. Conclusions

When placing the roots of the characteristic equation of the system transient response significantly depends on the actual poles and zeros of the set. The proposed algorithm for finding a feedback matrix completely inhibits the action of individual zeros of the system (in the particular case of pulling them in the point of origin of the complex plane). In calculating the coefficients is not necessary to form a system of nonlinear equations or the transition from the original basis for a new, more acceptable to the calculation (for example, the canonical form of control), as in the classical approach of calculation. The proposed algorithm is applicable if the following conditions. If a set of trailing zeros is empty, then the spectrum of individual zeros of the transfer functions can be arbitrarily set using the feedback matrix, and the spectrum of the system poles can be arbitrarily set using matrix SF. This fact can be used as the basis of the compensation algorithm for calculating the matrix state feedback.

Acknowledgements. I wish to express my gratitude for the assistance in the preparation of the manuscript, Siberian State University of Telecommunications and Informatics, Astrakhan State Technical University, Tomsk State University, as well as the Novosibirsk State Technical University.

References


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Received: January 11, 2014