New Extension of $q$-Bernoulli Polynomials

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Abstract

In this paper, we construct new $q$-extension of Bernoulli polynomials and Bernoulli polynomials of order $k$. These $q$-Bernoulli polynomials are useful to study various identities of Carlitz’s $q$-Bernoulli numbers.

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1. Introduction

Throughout this paper $\mathbb{Z}_p, \mathbb{Q}_p$ and $\mathbb{C}_p$ will respectively denote the ring of $p$-adic integers, the field of $p$-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$. Let $\nu_p$ be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = \frac{1}{p}$.

When one speaks of $q$-extension, $q$ is considered in many ways such as indeterminate, a complex number $q \in \mathbb{C}$ or $p$-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we assume that $|1 - q|_p < 1$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. The $q$-number of $x$ is denoted by

$$[x]_q = \frac{1 - q^x}{1 - q}.$$


Note that \(\lim_{q \to 1} [x]_q = x\). Let \(d\) be a fixed positive integer and let \(p\) be a fixed prime number. We set
\[
X_d = \lim_{N \to \infty} \mathbb{Z}/dp^N \mathbb{Z}, \quad X^* = \bigcup_{0 < a < dp} (a + dp\mathbb{Z}_p),
\]
\[a + dp^N \mathbb{Z}_p = \{ x \in X | x \equiv a \pmod{dp^N} \},\]
where \(a \in \mathbb{Z}\) lies in \(0 \leq a < dp^N\).

Let \(UD(\mathbb{Z}_p)\) be the space of uniformly differentiable functions on \(\mathbb{Z}_p\). For \(f \in UD(\mathbb{Z}_p)\), the \(p\)-adic \(q\)-integral on \(\mathbb{Z}_p\) is defined by Kim as follows:
\[
I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \to \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x) q^x \quad (\text{see } [10, 11]). \quad (1.1)
\]

As is well known, Bernoulli polynomials are defined by the generating function to be
\[
\frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n \quad \text{for } |t| < 2\pi \quad \text{with } t \in \mathbb{C}. \quad (1.2)
\]
In the special case, \(x = 0\), \(B_n = B_n(0)\) are called the \(n\)-th Bernoulli numbers. From (1.2), we note that
\[
B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k} = \sum_{k=0}^{n} \binom{n}{k} B_{n-k} x^k. \quad (1.3)
\]

In [4, 5], L. Carlitz defined the \(q\)-extension of Bernoulli numbers as follows:
\[
\beta_{0,q} = 1, q(q\beta + 1)^k - \beta_{k,q} = \delta_{k,1} \quad (1.4)
\]
with the usual convention of replacing \(\beta^l\) by \(\beta_{l,q}\) where \(\delta_{k,1}\) is the Kronecker’s symbol.

Note that, from (1.4), we note that
\[
\beta_{n,q} = \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{l + 1}{[l+1]_q}, \quad (n \geq 0). \quad (1.5)
\]
L. Carlitz have defined \(q\)-extension of Bernoulli polynomials as follows:
\[
\beta_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} [x]_q^{n-l} q^{lx} \beta_{l,q}, \quad (\text{see } [3,4]). \quad (1.6)
\]
In [10], Kim showed that Carlitz’s \(q\)-Bernoulli numbers and polynomials can be expressed as an \(p\)-adic \(q\)-integral on \(\mathbb{Z}_p\) as follows:
\[
\beta_{n,q} = \int_{\mathbb{Z}_p} [x]_q^n d\mu_q(x), \quad \beta_{n,q}(x) = \int_{\mathbb{Z}_p} [x + y]_q^n d\mu_q(x). \quad (1.6)
\]
New Extension of $q$-Bernoulli polynomials

$q$-Bernoulli numbers and polynomials have been studied by many mathematicians, and possess many interesting properties (see [1-17]). As new extension of Bernoulli polynomials, T. Kim et. al., Seo et. al., Y. Simsek gave new $q$-extensions of Bernoulli polynomials (see [13, 16, 17]). In [16], authors defined modified $q$-Bernoulli polynomials $\tilde{\beta}_{n,q}(x)$ by generating function, and represent $\tilde{\beta}_{n,q}(x)$ as a $p$-adic $q$-integral on $\mathbb{Z}_p$.

In this paper, we construct new $q$-extension of Bernoulli polynomials, and Bernoulli polynomials of order $k$. Finally, we investigate several properties of those polynomials.

2. A new extension of $q$-Bernoulli polynomials

As a new $q$-extension of Bernoulli polynomials, we define the modified $q$-Bernoulli polynomials which are defined by the generation function to be

$$\sum_{n=0}^{\infty} \tilde{\beta}_{n,q}(x)(1-q)^n \frac{t^n}{n!} = \int_{\mathbb{Z}_p} q^{-y}e^{(x+y)(1-q)t}d\mu_q(y).$$  \hspace{1cm} (2.1)

Note that, by (1.1), the following equations are obtained easily:

$$\tilde{B}_{n,q} = \int_{\mathbb{Z}_p} q^{-x}q^n d\mu_q(x) = \lim_{N \to \infty} \frac{1}{p^{N-1}} \sum_{x=0}^{[x]_q} [x]^n_q$$

$$= \frac{1}{(1-q)^n} \sum_{l=0}^{n} \left( \begin{array}{c} n \\ l \end{array} \right) (-1)^l \frac{l^t}{[l]_q},$$  \hspace{1cm} (2.2)

and $\tilde{\beta}_{n,q}$ is called $n$-th modified $q$-Bernoulli numbers which are defined in [13]. From (2.2), we obtain the following equation:

$$\int_{\mathbb{Z}_p} q^{-y}e^{(x+y)(1-q)t}d\mu_q(y)$$

$$= e^{x(1-q)t} \sum_{n=0}^{\infty} (1-q)^n \int_{\mathbb{Z}_p} q^{-y}[y]_q^n d\mu_q(y) \frac{t^n}{n!}$$

$$= e^{x(1-q)t} \sum_{n=0}^{\infty} \sum_{l=0}^{n} \left( \begin{array}{c} n \\ l \end{array} \right) (-1)^l \frac{l^t}{[l]_q} \frac{t^n}{n!}$$

$$= e^{x(1-q)+1-1)l} \sum_{l=0}^{\infty} (-1)^l \frac{l^t}{[l]_q} \frac{t^n}{n!}$$

$$= \left( \sum_{m=0}^{\infty} ((1-q)x+1)^m \frac{t^m}{m!} \right) \left( \sum_{l=0}^{\infty} (-1)^l \frac{l^t}{[l]_q} \frac{t^n}{n!} \right)$$

$$= \sum_{n=0}^{\infty} \sum_{l=0}^{n} (-1)^l \left( \begin{array}{c} n \\ l \end{array} \right) \frac{l^t}{[l]_q} ((1-q)x+1)^{n-l} \frac{t^n}{n!}. \hspace{1cm} (2.3)
Thus, by (2.1) and (2.3), we get
\[
\bar{\beta}_{n,q}(x) = \frac{1}{(1-q)^n} \sum_{l=0}^{n} (-1)^l \binom{n}{l} \frac{l}{[l]_q} ((1-q)x + 1)^{n-l}
\]
\[
= \frac{1}{(1-q)^n} \sum_{l=0}^{n} \sum_{j=0}^{n-l} (-1)^l \binom{n}{l} \frac{l}{[l]_q} (1-q)^j x^j.
\]
(2.4)

Therefore, by (2.4), we obtain the following theorem.

**Theorem 2.1.** For \( n \geq 0 \),
\[
\bar{\beta}_{n,q}(x) = \frac{1}{(1-q)^n} \sum_{l=0}^{n} (-1)^l \binom{n}{l} \frac{l}{[l]_q} ((1-q)x + 1)^{n-l}
\]
\[
= \sum_{l=0}^{n} \sum_{j=0}^{n-l} (-1)^l \binom{n}{l} \frac{l}{[l]_q} (1-q)^j x^j.
\]
(2.5)

Note that, by (2.1), we get
\[
\sum_{n=0}^{\infty} \bar{\beta}_{n,q}(x)(1-q)^n \frac{t^n}{n!} = \int_{\mathbb{Z}_p} q^{-y} e^{(x+[y]_q)(1-q)t} d\mu_q(y)
\]
\[
= \sum_{n=0}^{\infty} (1-q)^n \int_{\mathbb{Z}_p} q^{-y} (x+[y]_q)^n d\mu_q(y) \frac{t^n}{n!}.
\]
(2.6)

Thus, by (2.6), we obtain the following theorem.

**Theorem 2.2.** For \( n \geq 0 \),
\[
\bar{\beta}_{n,q}(x) = \int_{\mathbb{Z}_p} q^{-y} (x+[y]_q)^n d\mu_q(y).
\]

When \( x = 0 \), \( \bar{\beta}_{n,q}(0) = \bar{B}_{n,q} \) are \( n \)-th modified \( q \)-Bernoulli numbers. Note that
\[
\int_{\mathbb{Z}_p} q^{-y} (x+[y]_q)^n d\mu_q(y) = \sum_{l=0}^{n} \binom{n}{l} x^l \int_{\mathbb{Z}_p} q^{-y} [y]_q^{n-l} d\mu_q(y)
\]
\[
= \sum_{l=0}^{n} \binom{n}{l} x^l \bar{B}_{n-l,q}.
\]
(2.7)

Therefore, by (2.7), we obtain the following corollary.

**Corollary 2.3.** For \( n \geq 0 \), we have
\[
\bar{\beta}_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} \bar{B}_{n-l,q} x^l = \sum_{l=0}^{n} \binom{n}{l} \bar{B}_{l,q} x^{n-l},
\]
where \( \bar{B}_{n,q} \) are the \( n \)-th modified \( q \)-Bernoulli numbers.
3. A new approach to modified $q$-Bernoulli polynomials of order $k$

From now on, we consider the modified $q$-Bernoulli numbers of order $k$ as follows:

\[
\tilde{b}_{n,q} = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{-x_1-\cdots-x_k} [x_1 + \cdots + x_k]^n d\mu_q(x_1) \cdots d\mu_q(x_k).
\]

Note that, by (1.1),

\[
\tilde{b}_{n,q} = \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \left( \frac{l}{[l]_q} \right)^k.
\]  

Let us consider the modified $q$-Bernoulli polynomials of order $k$ as follows:

\[
\tilde{b}_{n,q}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x + [ x_1 + \cdots + x_k ]_q)^n d\mu_q(x_1) \cdots d\mu_q(x_k)
\]

\[
= \sum_{l=0}^{n} \binom{n}{l} \tilde{b}_{n,q} x^{n-l}
\]

where \( \tilde{b}_{n,q}(0) = \tilde{b}_{n,q} \).

By (3.1),

\[
(1-q)^n \tilde{b}_{n,q} = \sum_{l=0}^{n} \binom{n}{l} (-1)^l \left( \frac{l}{[l]_q} \right)^k.
\]

Consider the equation

\[
\sum_{n=0}^{\infty} (1-q)^n \tilde{b}_{n,q} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \left( \frac{l}{[l]_q} \right)^k \frac{t^l}{l!}
\]

\[
= \left( \sum_{m=0}^{\infty} \frac{t^m}{m!} \right) \left( \sum_{l=0}^{\infty} (-1)^l \left( \frac{l}{[l]_q} \right)^k \frac{t^l}{l!} \right)
\]

\[
= e^t \left( \sum_{l=0}^{\infty} (-1)^l \left( \frac{l}{[l]_q} \right)^k \frac{t^l}{l!} \right).
\]
Since
\[ e^{(1-q)x} \sum_{n=0}^{\infty} (1-q)^n \tilde{b}_{n,q} \frac{t^n}{n!} = \left( \sum_{l=0}^{\infty} (1-q)^l \frac{t^l}{l!} \right) \left( \sum_{n=0}^{\infty} (1-q)^n \tilde{b}_{n,q} \frac{t^n}{n!} \right) = \sum_{m=0}^{\infty} (1-q)^m \sum_{n=0}^{m} \binom{m}{n} \tilde{b}_{n,q} x^{m-n} \frac{t^m}{m!} = \sum_{m=0}^{\infty} (1-q)^m \tilde{b}_{m,q}(x) \frac{t^m}{m!} \tag{3.3} \]

and
\[ e^{(1-q)x} \left( e^{t} \sum_{l=0}^{\infty} (1-q)^l \frac{t^l}{l!} \right) = e^{((1-q)x+1)t} \left( \sum_{l=0}^{\infty} (1-q)^l \frac{t^l}{l!} \right) = \sum_{m=0}^{\infty} (1-q)x+1)^m \frac{t^m}{m!} \left( \sum_{l=0}^{\infty} (1-q)^l \frac{t^l}{l!} \right) = \sum_{n=0}^{\infty} \sum_{l=0}^{n} (1-q)^l \binom{n}{l} \frac{t^l}{l!} ((1-q)x+1)^{n-l} \frac{t^n}{n!}. \tag{3.4} \]

By (3.3) and (3.4),

\[ (1-q)^n \tilde{b}_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} (1-q)^l \frac{t^l}{l!} ((1-q)x+1)^{n-l} \]
\[ = \sum_{l=0}^{n} \sum_{j=0}^{n-l} \binom{n}{l} \binom{n-l}{j} (1-q)^l \frac{t^l}{l!} (1-q)^j x^j. \]

Thus, we have the following result.

**Theorem 3.1.** For \( n \geq 1 \),

\[ \tilde{b}_{n,q}(x) = \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (1-q)^l \frac{t^l}{l!} ((1-q)x+1)^{n-l} \]
\[ = \sum_{l=0}^{n} \sum_{j=0}^{n-l} \binom{n}{l} \binom{n-l}{j} (1-q)^l \frac{t^l}{l!} (1-q)^j x^j. \]
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Note that, by (3.2) and (3.3), the generating function of $\tilde{b}_{n,q}$ is represented as follows:

$$
\sum_{n=0}^{\infty} (1 - q)^n \tilde{b}_{n,q}(x) \frac{t^n}{n!} = e^{(1-q)x} \sum_{n=0}^{\infty} (1 - q)^n \tilde{b}_{n,q} \frac{t^n}{n!} = e^{(1-q)x+1} t \sum_{l=0}^{\infty} (-1)^l \left( \frac{l}{[l]_q} \right) \frac{t^l}{l!}.
$$

So, we have the following corollary.

**Corollary 3.2.** For $n \geq 0$,

$$
\sum_{n=0}^{\infty} (1 - q)^n \tilde{b}_{n,q}(x) \frac{t^n}{n!} = e^{(1-q)x+1} t \sum_{l=0}^{\infty} (-1)^l \left( \frac{l}{[l]_q} \right) \frac{t^l}{l!}
$$

where $\tilde{b}_{n,q}$ are the $n$-th modified $q$-Bernoulli numbers of order $k$.

**References**


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