Minimizing a Special Type of Functions

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Abstract

We solve a minimization problem of a particular class of functions. This problem appears in different aspects such as regression, $l_1$-norm minimization of curves in $\mathbb{R}^d$, $d$ is a positive integer, and approximation theory. A special case of this problem is directly related to quantile regression. Then we obtain a number of corollaries and discuss some applications of them.

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1 Introduction

Consider the following problem:
Problem 1:
\[
\begin{align*}
\text{Given that } f : \mathbb{R} \rightarrow \mathbb{R} \text{ is a strictly increasing continuous function.} \\
\text{Minimize } g(x) = \sum_{r=1}^{n} |f(x) - f(a_r)|, \\
\text{where } a_r < a_{r+1}, \text{ for all } 1 \leq r \leq n - 1.
\end{align*}
\]

In fact, special versions of this problem arise in different applications. For example, in quantile regression, if \( Y \) is any real-valued function and \( \{y_1, y_2, \ldots, y_n\} \) is a random sample of \( Y \)'s with \( y_1 < y_2 < \cdots < y_n \), then the median, known also as the half sample quantile, is nothing but the solution of the minimization problem
\[
\min_{x \in \mathbb{R}} \sum_{i=1}^{n} |f(x) - f(y_i)|, \text{ where } f \text{ is defined as } f(x) = x.
\]

In addition, different articles discussed the solution of Problem 1 under some constraints in the case where \( f \) is the identity map on \( \mathbb{R} \). See for example [1], [2] and [5]. However, Problem 1 is equivalent to the following constrained separable problem:

\[
\begin{align*}
\text{Minimize } g(x) &= \sum_{r=1}^{n} |f(x_r) - f(a_r)| \\
\text{Subject to } \\
(x_1 - x_2) + \sum_{i \neq 1,2} (x_1 - x_i)^2 &\leq 0 \\
(x_2 - x_3) + \sum_{i \neq 2,3} (x_2 - x_i)^2 &\leq 0 \\
(x_3 - x_4) + \sum_{i \neq 3,4} (x_3 - x_i)^2 &\leq 0 \\
&\vdots \\
(x_{n-1} - x_n) + \sum_{i \neq n-1,n} (x_{n-1} - x_i)^2 &\leq 0 \\
(x_n - x_1) + \sum_{i \neq 1,n} (x_n - x_i)^2 &\leq 0
\end{align*}
\]

where \( a_r < a_{r+1}, \) for all \( 1 \leq r \leq n - 1 \), and \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \).

Separable problems arise frequently in practice, particularly for time-dependent optimization. In addition, different techniques were obtained to solve such problems, see for example [3] and [6].
The importance of studying minimization problems is due to their applications in different aspects such as economics, engineering, physics and Mathematics. For example, in [4] the author considered the problem of minimizing the total costs in the multi-mode resource-constrained project scheduling which is known in industrial engineering.

In the present paper, we solve Problem 1, and obtain some corollaries. Then we introduce some applications related to the obtained results.

2 Main Result

**Theorem 2.1** Given that \( f : \mathbb{R} \rightarrow \mathbb{R} \) is a strictly increasing continuous function. Let \( \{a_r\}_{r=1}^n \) be a strictly increasing sequence of real numbers, where \( n > 1 \) is an integer. Define \( g : \mathbb{R} \rightarrow [0, \infty) \) as:

\[
g(x) = \sum_{r=1}^{n} |f(x) - f(a_r)|, \quad x \in \mathbb{R}.
\]

Then

1. If \( n \) is odd, \( g \) attains its smallest value at \( a_{\frac{n+1}{2}} \) with

\[
g_{\text{smallest}} = \sum_{i=\frac{n+1}{2}}^{n} f(a_i) - \sum_{i=1}^{\frac{n-1}{2}} f(a_i).
\]

2. If \( n \) is even, \( g \) attains its smallest value at any \( x \in [a_{\frac{n}{2}}, a_{\frac{n+2}{2}}] \) with

\[
g_{\text{smallest}} = \sum_{i=\frac{n}{2}+2}^{n} f(a_i) - \sum_{i=1}^{\frac{n-1}{2}} f(a_i).
\]

**Proof:** Using the fact that \( f \) is strictly increasing, we can rewrite the function \( g \) as follows:

\[
g(x) = \begin{cases} 
-\sum_{r=1}^{n} (f(x) - f(a_r)), & \text{if } x \leq a_1 \\
\sum_{r=1}^{i} (f(x) - f(a_r)) - \sum_{r=i+1}^{n} (f(x) - f(a_r)), & \text{if } a_i < x \leq a_{i+1}, i = 1, \ldots, n-1 \\
\sum_{r=1}^{n} (f(x) - f(a_r)), & \text{if } x > a_n
\end{cases}
\]

Now, we have two cases to consider:
1 $n$ is odd. Let $x$ be any number in the interval $(-\infty, a_{\frac{n+1}{2}}]$. Then $x \in (-\infty, a_1]$ or $x \in (a_i, a_{i+1}]$ for some $1 \leq i \leq \frac{n-1}{2}$. If $x \in (-\infty, a_1]$,

$$g \left( \frac{a_{n+1}}{2} \right) - g(x) = \left( \sum_{r=1}^{n-1} (f \left( \frac{a_{n+1}}{2} \right) - f(a_r)) - \sum_{r=\frac{n+1}{2}+1}^{n} (f \left( \frac{a_{n+1}}{2} \right) - f(x)) \right)$$

$$+ \sum_{r=1}^{\frac{n+1}{2}} (f(x) - f(a_r))$$

$$\leq \left( \sum_{r=1}^{n-1} (f \left( \frac{a_{n+1}}{2} \right) - f(x)) - \sum_{r=\frac{n+1}{2}+1}^{n} (f \left( \frac{a_{n+1}}{2} \right) - f(x)) \right)$$

$$+ \sum_{r=1}^{\frac{n+1}{2}} (f(x) - f(a_r))$$

$$= \sum_{r=1}^{\frac{n+1}{2}} (f(x) - f(a_r)) \leq 0.$$ 

Thus, $g \left( \frac{a_{n+1}}{2} \right) \leq g(x)$. If $x \in (a_i, a_{i+1}]$ for some $1 \leq i \leq \frac{n-1}{2}$, then

$$g \left( \frac{a_{n+1}}{2} \right) - g(x) = \sum_{r=1}^{i} (f \left( \frac{a_{n+1}}{2} \right) - f(a_r)) - \sum_{r=\frac{n+1}{2}+1}^{n} (f \left( \frac{a_{n+1}}{2} \right) - f(a_r))$$

$$- \sum_{r=1}^{i} (f(x) - f(a_r)) + \sum_{r=i+1}^{n} (f(x) - f(a_r))$$

$$\leq \sum_{r=1}^{i} (f \left( \frac{a_{n+1}}{2} \right) - f(x)) - \sum_{r=\frac{n+1}{2}+1}^{n} (f \left( \frac{a_{n+1}}{2} \right) - f(x))$$

$$+ \sum_{r=i+1}^{\frac{n+1}{2}} (f(x) - f(a_r))$$

$$\leq \sum_{r=i+1}^{\frac{n+1}{2}} (f(x) - f(a_r)) \leq 0.$$ 

Thus, $g \left( \frac{a_{n+1}}{2} \right) \leq g(x)$, which implies that $g \left( \frac{a_{n+1}}{2} \right) \leq g(x)$ for each $x \in (-\infty, a_{\frac{n+1}{2}}]$. Using a similar argument, we can show that $g \left( \frac{a_{n+1}}{2} \right) \leq$
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\( g(x) \) for each \( x \in \left[ a_{n+1}, \infty \right) \). Therefore,

\[
g_{\text{smallest}} = g \left( a_{\frac{n+1}{2}} \right)
\]

\[
= \sum_{j=1}^{\frac{n}{2}} \left( f \left( a_{\frac{n+1}{2}} \right) - f \left( a_j \right) \right) - \sum_{i=\frac{n+1}{2}+1}^{n} \left( f \left( a_{\frac{n+1}{2}} \right) - f \left( a_i \right) \right)
\]

\[
= \sum_{j=1}^{\frac{n}{2}} \left( f \left( a_{\frac{n+1}{2}} \right) - f \left( a_j \right) \right) + \sum_{i=\frac{n+3}{2}}^{n} \left( f \left( a_i \right) - f \left( a_{\frac{n+1}{2}} \right) \right)
\]

\[
= \sum_{i=\frac{n+3}{2}}^{n} f \left( a_i \right) - \sum_{i=1}^{\frac{n}{2}} f \left( a_i \right).
\]

2 \( n \) is even.

Using similar arguments as in Case 1, we can show that, \( g \left( a_{\frac{n}{2}} \right) \leq g(x) \) for each \( x \in (-\infty, a_{\frac{n}{2}}] \), and \( g \left( a_{\frac{n+2}{2}} \right) \leq g(x) \) for each \( x \in \left[ a_{\frac{n+2}{2}}, \infty \right) \).

On the other hand, if \( x \in \left( a_{\frac{n}{2}}, a_{\frac{n+2}{2}} \right] \) then

\[
g \left( a_{\frac{n}{2}} \right) - g(x) = \sum_{r=1}^{\frac{n}{2}} \left( f \left( a_{\frac{n}{2}} \right) - f \left( a_r \right) \right) - \sum_{r=\frac{n+1}{2}}^{n} \left( f \left( a_{\frac{n}{2}} \right) - f \left( a_r \right) \right)
\]

\[
- \sum_{r=1}^{\frac{n}{2}} \left( f \left( x \right) - f \left( a_r \right) \right) + \sum_{r=\frac{n+2}{2}}^{n} \left( f \left( x \right) - f \left( a_r \right) \right)
\]

\[
= \sum_{r=1}^{\frac{n}{2}} \left( f \left( a_{\frac{n}{2}} \right) - f \left( x \right) \right) - \sum_{r=\frac{n+2}{2}}^{n} \left( f \left( a_{\frac{n}{2}} \right) - f \left( x \right) \right)
\]

\[
= (f \left( a_{\frac{n}{2}} \right) - f \left( x \right)) \frac{n}{2} - (f \left( a_{\frac{n}{2}} \right) - f \left( x \right)) \frac{n}{2} = 0.
\]

Thus, \( g \) is constant on \( \left[ a_{\frac{n}{2}}, a_{\frac{n+2}{2}} \right] \), which implies that \( g \) attains its smallest value at any \( x \in \left[ a_{\frac{n}{2}}, a_{\frac{n+2}{2}} \right] \), and

\[
g_{\text{smallest}} = \sum_{i=\frac{n}{2}+2}^{n} f \left( a_i \right) - \sum_{i=1}^{\frac{n-1}{2}} f \left( a_i \right).
\]
**Corollary 2.2** Given that \( f : \mathbb{R} \rightarrow \mathbb{R} \) is a strictly decreasing continuous function. Let \( \{a_r\}_{r=1}^n \) be a strictly increasing sequence of real numbers, where \( n > 1 \) is an integer. Define \( g : \mathbb{R} \rightarrow [0, \infty) \) as:

\[
g(x) = \sum_{r=1}^{n} |f(x) - f(a_r)|, \quad x \in \mathbb{R}.
\]

Then

1. If \( n \) is odd, \( g \) attains its smallest value at \( a_{\frac{n+1}{2}} \).

2. If \( n \) is even, \( g \) attains its smallest value at any \( x \in \left[ a_{\frac{n}{2}}, a_{\frac{n+2}{2}} \right] \).

**Proof:** By rewriting \( g(x) \) as \( g(x) = \sum_{r=1}^{n} \left| (-f)(x) - (-f)(a_r) \right| \) and applying Theorem 2.1.

**Corollary 2.3** Given that \( f : \mathbb{R} \rightarrow \mathbb{R} \) is a strictly increasing surjective function. Let \( \{a_r\}_{r=1}^n \) be a strictly increasing sequence of real numbers, where \( n > 1 \) is an integer. Define \( g : \mathbb{R} \rightarrow [0, \infty) \) as:

\[
g(x) = \sum_{r=1}^{n} |f(x) - a_r|, \quad x \in \mathbb{R}.
\]

Then \( g \) attains its smallest value at \( f^{-1}\left( a_{\frac{n+1}{2}} \right) \) when \( n \) is odd, and at any \( x \in f^{-1}\left( \left[ a_{\frac{n}{2}}, a_{\frac{n+2}{2}} \right] \right) \) when \( n \) is even.

**Corollary 2.4** Let \( A \) be any topological space, and let \( f : A \rightarrow \mathbb{R} \) be any continuous function. Suppose that \( \{b_r\}_{r=1}^n \subseteq \text{Range}(f) \) is a strictly increasing sequence of real numbers, where \( n > 1 \) is an integer. Define \( g : \mathbb{R} \rightarrow [0, \infty) \) as:

\[
g(x) = \sum_{r=1}^{n} |f(x) - b_r|, \quad x \in \mathbb{R}.
\]

Then

1. If \( n \) is odd, \( g \) attains its smallest value, call it \( m_0 \), at any \( x \in \left\{ t \in A : f(t) = b_{\frac{n+1}{2}} \right\} \), and \( m_0 = \sum_{r=1}^{n} \left| b_{\frac{n+1}{2}} - b_r \right| \).

2. If \( n \) is even, \( g \) attains its smallest value, call it \( m_0 \), at any \( x \in \left\{ t \in A : f(t) \in \left[ b_{\frac{n}{2}}, b_{\frac{n+2}{2}} \right] \right\} \), and \( m_0 = \sum_{r=1}^{n} \left| b_{\frac{n}{2}} - b_r \right| \).
3 Examples

In this section, we introduce some examples about using the main results in different contexts.

**Example 3.1** Let \( f(x) = \sqrt[3]{x} \). Let \( \{a_r\}_{r=1}^n \) be a strictly increasing sequence of real numbers, where \( n > 1 \) is an integer. Using Corollary 2.2, \( \sum_{r=1}^n |\sqrt[3]{x} - a_r| \) attains its smallest value at \( a_{\frac{n+1}{2}}^3 \) when \( n \) is odd, and at any \( x \in \left[a_{\frac{n}{2}}, a_{\frac{n+2}{2}}\right] \) when \( n \) is even.

**Example 3.2** Let \( f(x) = \tan^{-1} x \). Let \( \{a_r\}_{r=1}^n \) be a strictly increasing sequence of real numbers, where \( n > 1 \) is an integer. Using Theorem 2.1, \( \sum_{r=1}^n |\tan^{-1} x - \tan^{-1} a_r| \) attains its smallest value at \( a_{\frac{n+1}{2}} \) when \( n \) is odd, and at any \( x \in \left[a_{\frac{n}{2}}, a_{\frac{n+2}{2}}\right] \) when \( n \) is even.

The next example shows an application of Corollary 2.3 in the context of real normed linear spaces.

**Example 3.3** Let \( n \) be any integer with \( n > 1 \). Let \( X \) be any real normed linear space. Let \( \{a_r\}_{r=1}^n \) be any sequence of elements in \( X \) such that \( \{\|a_r\|\}_{r=1}^n \) is strictly increasing. Using Corollary 2.3, we get that

1. If \( n \) is odd, \( \sum_{r=1}^n \|x - a_r\| \) attains its smallest value, call it \( m_0 \), at any \( x \in \left\{t \in X : \|t\| = \left\|a_{\frac{n+1}{2}}\right\|\right\} \), and \( m_0 = \sum_{r=1}^n \left|\left\|a_{\frac{n+1}{2}}\right\| - \|a_r\|\right| \). Using the reverse triangle inequality, we get that \( \min_{x \in X} \sum_{r=1}^n \|x - a_r\| \geq \sum_{r=1}^n \left|\left\|a_{\frac{n+1}{2}}\right\| - \|a_r\|\right| \).

2. If \( n \) is even, \( \sum_{r=1}^n \|x - a_r\| \) attains its smallest value, call it \( m_0 \), at any \( x \in \left\{t \in X : \|t\| \in \left[\|a_{\frac{n}{2}}\|, \|a_{\frac{n+2}{2}}\|\right]\right\} \), and \( m_0 = \sum_{r=1}^n \left|\left\|a_{\frac{n}{2}}\right\| - \|a_r\|\right| \). Using the reverse triangle inequality, we get that

\[
\min_{x \in X} \sum_{r=1}^n \|x - a_r\| \geq \sum_{r=1}^n \left|\left\|a_{\frac{n}{2}}\right\| - \|a_r\|\right| .
\]

The next example shows an application of Theorem 2.1 in the context of \( l_1 \)-norm of real vector-valued functions defined on \( \mathbb{R} \).
Example 3.4 Consider the curve $r : \mathbb{R} \rightarrow \mathbb{R}^4$ defined as

$$r(x) = (2^x - 8, 2^x - 4, 2^x - 16, 2^x - 2).$$

Using Theorem 2.1, $\|r(x)\|_1$ attains its smallest value at any $x \in [2, 3]$, and it is equal to 18.

We close the section by the following Corollary which is an application of Theorem 2.1 in the context of integrals.

Corollary 3.5 Let $h : \mathbb{R} \rightarrow \mathbb{R}$ be a non-negative continuous function. Let $\{a_r\}_{r=1}^n$ be a strictly increasing sequence of real numbers, where $n > 1$ is an integer. Define $g : \mathbb{R} \rightarrow [0, \infty)$ as:

$$g(x) = \sum_{r=1}^n \left| \int_{a_r}^x h(t) \, dt \right|, \quad x \in \mathbb{R}.$$

Then

1. If $n$ is odd, $g$ attains its minimum at $a_{\frac{n+1}{2}}$.

2. If $n$ is even, $g$ attains its minimum at any $x \in \left[a_{\frac{n}{2}}, a_{\frac{n+2}{2}}\right]$.

Proof: Note that

$$g(x) = \sum_{r=1}^n \left| \int_0^x h(t) \, dt - \int_0^{a_r} h(t) \, dt \right|.$$

Since $\int_0^x h(t) \, dt$ is a strictly increasing function over $\mathbb{R}$, the result follows by using Theorem 2.1.

4 Conclusion

Although most optimization problems involve constraints which bound the design space, unconstrained minimization strategies are important for many structural analysis problems. The importance of studying Problem 1 is due to its applications in different aspects. It arises in different contexts such as integration and approximation theory. In addition, it is related to quantile analysis, and is equivalent to a constrained separable problem over the whole space $\mathbb{R}^n$. The results obtained in the present paper can be generalized to include more optimization problems in different contexts. We will consider some of these generalizations in future projects.

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