Inverse Boundary Value Problem for the Pseudoparabolic Equation of Third Order with Periodic and Integral Conditions

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Abstract

We analyze the solvability of the inverse problem with an unknown coefficient depended on time for the pseudoparabolic equation of third order with periodic and integral conditions. The initial problem is reduced to an equivalent problem. With the help of the Fourier method, the equivalent problem is reduced to a system of integral equations. The existence and uniqueness of the solution of the integral equations is proved. The obtained solution of the integral equations is also the only solution to the equivalent problem. Basing on the equivalence of the problems, the theorem of the existence and uniqueness of the classical solutions of the original problem is proved.

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1 Introduction

There are many cases where a practical problem creates a need to determine the coefficients or solve otherwise the right-hand side of a differential equation
using some known properties of the problem’s solution. Such problems are called inverse value problems in mathematical physics. Inverse value problems arise in a number of scientific fields such as seismology, mineral exploration, biology, medicine, quality control of industrial products, etc. All of these problems can be stated and solved by use of modern mathematics. The inverse value problems are well studied, nevertheless they are considered to be the frontier of modern mathematics.

Recently, the inverse value problems have been widely applied in various fields. Various types of inverse value problems for a number of partial differential equations have been studied in many papers. Among others, we would like to note the papers by A.N.Tikhonov [7], M.M.Lavrentyev [4,5], A.M.Denisov [2], M.I.Ivanchov [3] and their followers.

In searching of local and non-local boundary value problems for pseudoparabolic equations practical and theoretical interests assume great importance and is more actively studied now days.

In this paper, we prove the existence and uniqueness of the solution of an inverse boundary value problem for the equation of third order with non-local integral condition.

2 Problem statement and its reduction to equivalent problem.

Let’s consider for the equation
\[ u_t(x, t) - bu_{txx}(x, t) - a(t)u_{xx}(x, t) = p(t)u(x, t) + f(x, t), \]  
(1)
in the domain \( D_T = \{(x, t) : 0 \leq x \leq 1, \ 0 \leq t \leq T\} \) an inverse boundary problem with initial conditions
\[ u(x, 0) = \varphi(x) \quad (0 \leq x \leq 1), \]  
(2)
the periodic condition
\[ u(0, t) = u(1, t) \quad (0 \leq t \leq T), \]  
(3)
the non-local integral condition
\[ \int_0^1 u(x, t)dx = 0 \quad (0 \leq t \leq T) \]  
(4)
and with the additional condition
\[ u(x_0, t) = h(t) \quad (0 \leq t \leq T), \]  
(5)
where $x_0 \in (0, 1)$, $b > 0$ are the given numbers, $a(t) > 0$, $f(x, t)$, $\varphi(x)$, $h(t)$ are the given functions, and $\{u(x, t), p(t)\}$ are the desired functions.

The condition (4) is a non-local integral condition of first kind, i.e. the one not involving values of unknown functions at the domain’s boundary points.

**Definition.** The classic solution of problem (1)-(5) is the pair $\{u(x, t), p(t)\}$ of the functions $u(x, t)$ and $p(t)$ possessing the following properties:

1) the function $u(x, t)$ is continuous in $D_T$ together with all its derivatives contained in equation (1);
2) the function $p(t)$ is continuous on $[0, T]$;
3) all the conditions (1)-(5) are satisfied in the ordinary sense.

The following lemma is valid.

**Lemma1.** Let $b > 0$, $0 < a(t) \in C[0, T]$, $\varphi(x) \in C[0, 1]$, $f(x, t) \in C(D_T)$, $h(t) \in C^1[0, T]$, $h(t) \neq 0$ ($0 \leq t \leq T$), $\int_0^1 f(x, t)dx = 0$ ($0 \leq t \leq T$),

$$\int_0^1 \varphi(x)dx = 0, \varphi'(0) = \varphi'(1), \varphi(x_0) = h(0).$$

Then the problem on finding the classic solution of problem (1) – (5) is equivalent to the problem on defining of the function $u(x, t)$ and $p(t)$, possessing the properties 1) and 2) of definition of the classic solution of problem (1) – (5), from relations (1) – (3), and

$$u_x(0, t) = u_x(1, t) \quad (0 \leq t \leq T), \quad (6)$$

$$h'(t) - bu_{txx}(x_0, t) - a(t)u_xx(x_0, t) = p(t)h(t) + f(x_0, t) \quad (0 \leq t \leq T). \quad (7)$$

**Proof.** Let $\{u(x, t), p(t)\}$ be a solution of problem (1) – (5). Integrating equation (1) with respect to $x$ from 0 to 1, we have:

$$\frac{d}{dt} \int_0^1 u(x, t)dx - b(u_{tx}(1, t) - u_{tx}(0, t)) - a(t)(u_x(1, t) - u_x(0, t)) =$$

$$= p(t) \int_0^1 u(x, t)dx + \int_0^1 f(x, t)dx \quad (0 \leq t \leq T). \quad (8)$$

Taking into account that $\int_0^1 f(x, t)dx = 0$ ($0 \leq t \leq T$), allowing for (4), we find:

$$b \frac{d}{dt}(u_x(1, t) - u_x(0, t)) + a(t)(u_x(1, t) - u_x(0, t)) = 0. \quad (9)$$

By (2) and $\varphi'(0) = \varphi'(1)$ we get:

$$u_x(1, 0) - u_x(0, 0) = \varphi'(1) - \varphi'(0) = 0. \quad (10)$$
Since the problem (9), (10) has only a trivial solution, then \( u_x(1,t) = u_x(0,t) = 0 \), i.e. the condition (6) is fulfilled. Further assuming \( h(t) \in C^1[0,T] \), and differentiating (5), we obtain:

\[
u_t(x_0,t) = h'(t) \quad (0 \leq t \leq T).
\]

From (1) we have:

\[
u_t(x_0,t) - bu_{xx}(x_0,t) - a(t)u_{xx}(x_0,t) = p(t)u(x_0,t) + f(x_0,t) \quad (0 \leq t \leq T).
\]

Hence allowing for (5) and (11) we arrive at the fulfillment of (7). Now suppose that \{u(x,t), p(t)\} is a solution of problem (1) – (3), (6), (7). Then from (8), allowing for (6) and \( \int_0^1 f(x,t)dx = 0 \) \((0 \leq t \leq T)\), we find:

\[
\frac{d}{dt} \int_0^1 u(x,t)dx - p(t) \int_0^1 u(x,t)dx = 0 \quad (0 \leq t \leq T).
\]

By (2) and \( \int_0^1 \varphi(x)dx = 0 \), it is obvious that

\[
\int_0^1 u(x,0)dx = \int_0^1 \varphi(x)dx = 0 \quad (0 \leq t \leq T).
\]

Since of problem (13), (14) has only a trivial solution, then \( \int_0^1 u(x,t)dx = 0 \quad (0 \leq t \leq T) \) i.e. condition (4) is fulfilled.

Then from (7) and (12) we get:

\[
\frac{d}{dt}(u(x_0,t) - h(t)) = p(t)(u(x_0,t) - h(t)) \quad (0 \leq t \leq T).
\]

Further, by (2) and \( \varphi(x_0) = h(0) \), we obtain:

\[
u(x_0,0) - h(0) = \varphi(x_0) - h(0) = 0 \quad (0 \leq t \leq T).
\]

From (15) and (16), we conclude that condition (5) is fulfilled. The lemma is proved.

3 Investigation of the existence and uniqueness of the classic solution of the inverse boundary value problem

It is known [1] that the system

\[
1, \cos \lambda_1 x, \sin \lambda_1 x, \ldots, \cos \lambda_k x, \sin \lambda_k x, \ldots
\]
form is a basis in $L_2(0, 1)$, where $\lambda_k = 2k\pi$ \ (k = 1, 2, \ldots).

Since the system (17) form a basis in $L_2(0, 1)$, it is obvious that for each solution \{u(x, t), p(t)\} of the problems (1) – (3), (6), (7) its first component $u(x, t)$ has the form:

$$u(x, t) = \sum_{k=0}^{\infty} u_{1k}(t) \cos \lambda_k x + \sum_{k=1}^{\infty} u_{2k}(t) \sin \lambda_k x \quad (\lambda_k = 2\pi k), \quad (18)$$

where

$$u_{10}(t) = \int_0^1 u(x, t) dx,$$

$$u_{1k}(t) = 2 \int_0^1 u(x, t) \cos \lambda_k x dx, \quad u_{2k}(t) = 2 \int_0^1 u(x, t) \sin \lambda_k x dx \quad (k = 1, 2, \ldots).$$

Then applying the formal scheme of the Fourier method, from (1) and (2) we have:

$$u'_{10}(t) = F_{10}(t; u, p) \quad (0 \leq t \leq T), \quad (19)$$

$$= (1 + b\lambda_k^2)u_{1k}'(t) + a(t)\lambda_k^2 u_{1k}(t) = F_{ik}(t; u, p) \quad (0 \leq t \leq T; \quad i = 1, 2; \quad k = 1, 2, \ldots), \quad (20)$$

$$u_{10}(0) = \varphi_{10}, \quad u_{1k}(0) = \varphi_{ik} \quad (i = 1, 2; \quad k = 1, 2, \ldots), \quad (21)$$

where

$$F_{1k}(t; u, p) = p(t)u_{1k}(t) + f_{1k}(t) \quad (k = 0, 1, \ldots),$$

$$f_{10}(t) = \int_0^1 f(x, t) dx, \quad f_{1k}(t) = 2 \int_0^1 f(x, t) \cos \lambda_k x dx \quad (k = 1, 2, \ldots),$$

$$\varphi_{10} = \int_0^1 \varphi(x) dx, \quad \varphi_{ik} = 2 \int_0^1 \varphi(x) \cos \lambda_k x dx \quad (k = 1, 2, \ldots),$$

$$F_{2k}(t; u, a) = p(t)u_{2k}(t) + f_{2k}(t), \quad f_{2k}(t) = 2 \int_0^1 f(x, t) \sin \lambda_k x dx \quad (k = 1, 2, \ldots),$$

$$\varphi_{2k} = 2 \int_0^1 \varphi(x) \sin \lambda_k x dx \quad (k = 1, 2, \ldots).$$

Solving the problem (19)– (22), we find:

$$u_{10}(t) = \varphi_{10} + \int_0^t F_{10}(\tau; u, p) d\tau \quad (0 \leq t \leq T), \quad (22)$$

$$u_{ik}(t) = \varphi_{ik} e^{-\int_0^t \frac{a(\tau)^2}{1 + b\lambda_k^2} d\tau} + \frac{1}{1 + b\lambda_k^2} \int_0^t F_{ik}(\tau; u, p) e^{-\int_\tau^t \frac{a(\tau)^2}{1 + b\lambda_k^2} d\tau} d\tau \quad (0 \leq t \leq T; \quad i = 1, 2; \quad k = 1, 2, \ldots). \quad (23)$$
After substituting the expressions \( u_{1k}(t) \) \((k = 0, 1, \ldots)\) and \( u_{2k}(t) \) \((k = 1, 2, \ldots)\) into (18), for determining the component \( u(x, t) \) of the solution \( \{u(x, t), p(t)\} \) of problem (1) – (3), (6), (7) we get:

\[
\begin{align*}
  u(x, t) &= \varphi_{10} + \int_0^t F_{10}(\tau; u, p)d\tau + \sum_{k=1}^{\infty} \left\{ \varphi_{1k}e^{-\int_0^t \frac{a(s)\lambda_k^2}{1+b\lambda_k^2} ds} \right. \\
  &\quad + \frac{1}{1+b\lambda_k^2} \int_0^t F_{1k}(\tau; u, p)e^{-\int_0^\tau \frac{a(s)\lambda_k^2}{1+b\lambda_k^2} ds} d\tau \right\} \cos \lambda_k x + \\
  &\quad + \sum_{k=1}^{\infty} \left\{ \varphi_{2k}e^{-\int_0^t \frac{a(s)\lambda_k^2}{1+b\lambda_k^2} ds} + \frac{1}{1+b\lambda_k^2} \int_0^t F_{2k}(\tau; u, p)e^{-\int_0^\tau \frac{a(s)\lambda_k^2}{1+b\lambda_k^2} ds} d\tau \right\} \sin \lambda_k x. \\
\end{align*}
\]

(24)

Now, from (7), allowing for (18) we obtain:

\[
p(t) = [h(t)]^{-1} \left\{ h'(t) - f(x_0, t) + \sum_{k=1}^{\infty} \left[ b\lambda_k^2 u'_{1k}(t) + a(t)\lambda_k^2 u_{1k}(t) \right] \cos \lambda_k x_0 + \right. \\
  &\quad \left. + \sum_{k=1}^{\infty} \left[ b\lambda_k^2 u'_{2k}(t) + a(t)\lambda_k^2 u_{2k}(t) \right] \sin \lambda_k x_0 \right\}. \\
\]

(25)

By (20) and (23) we have:

\[
b\lambda_k^2 u'_{ik}(t) + a(t)\lambda_k^2 u_{ik}(t) = F_{ik}(t; u, p) - u'_{ik}(t) = \frac{b\lambda_k^2}{1+b\lambda_k^2} F_{ik}(t; u, p) + \\
  + \frac{a(t)\lambda_k^2}{1+b\lambda_k^2} \left[ \varphi_{ik}e^{-\int_0^t \frac{a(s)\lambda_k^2}{1+b\lambda_k^2} ds} + \frac{1}{1+b\lambda_k^2} \int_0^t F_{ik}(\tau; u, p)e^{-\int_0^\tau \frac{a(s)\lambda_k^2}{1+b\lambda_k^2} ds} d\tau \right] \\
  \quad (0 \leq t \leq T; \quad i = 1, 2; \quad k = 1, 2, \ldots). \\
\]

(26)

For finding an equation for the second component \( p(t) \) of the solution \( \{u(x, t), p(t)\} \) of problem (1) – (3), (6), (7), substitute expression (26) into (25):

\[
p(t) = [h(t)]^{-1} \left\{ h'(t) - f(x_0, t) + \right. \\
  &\quad \left. + \sum_{k=1}^{\infty} \left[ \frac{b\lambda_k^2}{1+b\lambda_k^2} F_{1k}(t; u, p) + \frac{a(t)\lambda_k^2}{1+b\lambda_k^2} \right] \times \right. \\
\]

(26)
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Thus, the solution of problem (1) – (3), (6), (7) is reduced to the solution of system (24), (27) with respect to the unknown functions $u(x, t)$ and $p(t)$.

**Lemma 2.** If $\{u(x, t), p(t)\}$ – is any solution of problem (1) – (3), (6), (7), then the functions

$$u_{10}(t) = \int_0^1 u(x, t) dx,$$

$$u_{1k}(t) = 2 \int_0^1 u(x, t) \cos \lambda_k x dx, \quad u_{2k}(t) = 2 \int_0^1 u(x, t) \sin \lambda_k x dx \quad (k = 1, 2, \ldots).$$

satisfies system (22), (23) in $[0, T]$.

**Remark.** From lemma 2 it follows that to prove the uniqueness of the solution of problem (1) – (3), (6), (7), it suffices to prove the uniqueness of the solution of system (24), (27).

In order to investigate the problem (1) – (3), (6), (7), consider the following spaces:

1. Denote by $B^3_{2, T}$ [6] the set of all the functions $u(x, t)$ of the form

$$u(x, t) = \sum_{k=0}^{\infty} u_{1k}(t) \cos \lambda_k x + \sum_{k=1}^{\infty} u_{2k}(t) \sin \lambda_k x \quad (\lambda_k = 2\pi k),$$

considered in $D_T$, where each of the functions $u_{1k}(t)$ ($k = 0, 1, \ldots$), $u_{2k}(t)$ ($k = 1, 2, \ldots$) is continuous on $[0, T]$ and

$$J_T(u) \equiv \|u_{10}(t)\|_{C[0, T]} + \left( \sum_{k=1}^{\infty} (\lambda_k^2 \|u_{1k}(t)\|_{C[0, T]}^2)^{\frac{1}{2}} \right)^2 +$$

$$+ \left( \sum_{k=1}^{\infty} (\lambda_k^3 \|u_{2k}(t)\|_{C[0, T]}^2)^{\frac{1}{2}} \right)^2 < +\infty.$$  

The norm in this set is determined as follows:

$$\|u(x, t)\|_{B^3_{2, T}} = J_T(u).$$

2. Denote by $E^3_T$ the space $B^3_{2, T} \times C[0, T]$ of the vector-functions $z(x, t) = \{u(x, t), a(t)\}$ with the norm

$$\|z(x, t)\|_{E^3_T} = J_T(z).$$
It is known that $B_{2,T}^3$ and $E_{T}^3$ are Banach spaces.

Now, in the space $E_{T}^3$, consider the operator

$$
\Phi(u, p) = \{P(u, p), H(u, p)\},
$$

where

$$
P(u, p) = \sum_{k=0}^{\infty} P_{ik}(t) \cos \lambda_k x + \sum_{k=1}^{\infty} P_{2k}(t) \sin \lambda_k x,
$$

$P_{10}(t)$, $P_{ik}(t)$, $(k = 1, 2, \ldots)$ and $H(u, p)$ equal to the right sides of (22),(23) and (27), respectively.

By means of simple transformations we find

$$
\|P_{10}(t)\|_{C[0,T]} \leq |\varphi_{10}| + \sqrt{T} \left( \int_{0}^{T} |f_{10}(\tau)|^2 d\tau \right)^{\frac{1}{2}} +
$$

$$
+ T \|p(t)\|_{C[0,T]} \|u_0(t)\|_{C[0,T]},
$$

(28)

$$
\left( \sum_{k=1}^{\infty} (\lambda_k^3 \|P_{ik}(t)\|_{C[0,T]}^2) \right)^{\frac{1}{2}} \leq \sqrt{3} \left( \sum_{k=1}^{\infty} (\lambda_k^3 |\varphi_{ik}|)^2 \right)^{\frac{1}{2}} +
$$

$$
+ \frac{\sqrt{3}}{b} \sqrt{T} \left( \int_{0}^{T} \sum_{k=1}^{\infty} (\lambda_k |f_{ik}(\tau)|)^2 d\tau \right)^{\frac{1}{2}} +
$$

$$
+ \frac{\sqrt{3}}{b} T \|p(t)\|_{C[0,T]} \left( \sum_{k=1}^{\infty} (\lambda_k^3 \|u_{ik}(t)\|_{C[0,T]}^2) \right)^{\frac{1}{2}},
$$

(29)

$$
\|H(u, p)\|_{C[0,T]} \leq \left\|\left[ h(t) \right]^{-1} \right\|_{C[0,T]} \left\{ \|h(t) - f(x_0, t)\|_{C[0,T]} +
$$

$$
+ \frac{1}{\sqrt{6}} \sum_{i=1}^{2} \left[ \left( \sum_{k=1}^{\infty} (\lambda_k |\varphi_{ik}|)^2 \right)^{\frac{1}{2}} + \left( \sum_{k=1}^{\infty} (\lambda_k \|f_{ik}(t)\|_{C[0,T]}^2) \right)^{\frac{1}{2}} +
$$

$$
+ \|p(t)\|_{C[0,T]} \left( \sum_{k=1}^{\infty} (\lambda_k^3 \|u_{ik}(t)\|_{C[0,T]}^2) \right)^{\frac{1}{2}} +
$$

$$
+ \frac{1}{b} \|a(t)\|_{C[0,T]} \left( \left( \sum_{k=1}^{\infty} (\lambda_k |\varphi_{ik}|)^2 \right)^{\frac{1}{2}} + \frac{\sqrt{T}}{b} \left( \int_{0}^{T} \sum_{k=1}^{\infty} (\lambda_k |f_{ik}(\tau)|)^2 d\tau \right)^{\frac{1}{2}} +
$$

$$
+ \frac{T}{b} \|p(t)\|_{C[0,T]} \left( \sum_{k=1}^{\infty} (\lambda_k^3 \|u_{ik}(t)\|_{C[0,T]}^2) \right)^{\frac{1}{2}} \right) \right\}.
$$

(30)
Suppose that the data of problem (1) – (3), (6), (7) satisfy the following conditions:

1. \( \varphi(x) \in C^2[0, 1], \varphi''(x) \in L_2(0, 1), \varphi(0) = \varphi(1), \varphi'(0) = \varphi'(1), \varphi''(0) = \varphi''(1). \)

2. \( f(x, t) \in C(D_T), f_x(x, t) \in L_2(D_T), f(0, t) = f(1, t). \)

3. \( b > 0, \quad 0 < a(t) \in C[0, T], \quad h(t) \in C^1[0, T], \quad h(t) \neq 0 \quad (0 \leq t \leq T). \)

Then from (28)–(30), we get:

\[
\|P(u, p)\|_{B^2_{2,T}} \leq A_1(T) + B_1(T) \|p(t)\|_{C[0,T]} \|u(x, t)\|_{B^2_{2,T}}, \tag{31}
\]

\[
\|H(u, p)\|_{C[0,T]} \leq A_2(T) + B_2(T) \|p(t)\|_{C[0,T]} \|u(x, t)\|_{B^3_{2,T}}, \tag{32}
\]

where

\[
A_1(T) = \|\varphi(x)\|_{L^2(0,1)} + \sqrt{T} \|f(x, t)\|_{L^2(D_T)} + 2\sqrt{3} \|\varphi''(x)\|_{L^2(0,1)} + \frac{2\sqrt{3}}{b} \sqrt{T} \|f_x(x, t)\|_{L^2(D_T)},
\]

\[
B_1(T) = \left( 1 + \frac{2\sqrt{3}}{b} \right) T,
\]

\[
A_2(T) = \|[h(t)]^{-1}\|_{C[0,T]} \left\{ \|h'(t) - f(x_0, t)\|_{C[0,T]} + \frac{2}{\sqrt{6}} \|f_x(x, t)\|_{C[0,T]} \|\varphi'(x)\|_{L^2(0,1)} + \frac{2}{b\sqrt{6}} \|a(t)\|_{C[0,T]} (\|\varphi'(x)\|_{L^2(0,1)} + \|f(x, t)\|_{L^2(D_T)}) \right\},
\]

\[
B_2(T) = \frac{2}{\sqrt{6}} \left( 1 + \frac{1}{b^2} T \|a(t)\|_{C[0,T]} \right) \|[h(t)]^{-1}\|_{C[0,T]}.
\]

From inequalities (31), (32) we obtain:

\[
\|P(u, p)\|_{B^2_{2,T}} + \|H(u, p)\|_{C[0,T]} \leq A(T) + B(T) \|p(t)\|_{C[0,T]} \|u(x, t)\|_{B^3_{2,T}}, \tag{33}
\]

where

\[
A(T) = A_1(T) + A_2(T), \quad B(T) = B_1(T) + B_2(T).
\]

So, we can prove the following theorem.

**Theorem 1.** Let conditions 1-3 be fulfilled, and

\[
(A(T) + 2)^2 B(T) < 1. \tag{34}
\]
Then problem (1) – (3), (6), (7) has a unique solution in the ball \( K = K_R(\|z\|_{E^3_T}) \leq R = A(T) + 2 \) of the space \( E^3_T \).

**Proof.** In the space \( E^3_T \) consider the equation

\[
z = \Phi z, \tag{35}
\]

where \( z = \{u, a\} \), the components \( P(u, p), H(u, p) \) of the operator \( \Phi(u, p) \) defined from the right sides of the equations (24), (27).

Consider the operator \( \Phi(u, p) \) in the ball \( K = K_R \) from \( E^3_T \). Analogically to (33), we get that for any \( z, z_1, z_2 \in K_R \) the following estimates are valid:

\[
\|\Phi z\|_{E^3_T} \leq A(T) + B(T) \|p(t)\|_{C[0,T]} \|u(x, t)\|_{B^3_{2,T}}, \tag{36}
\]

\[
\|\Phi z_1 - \Phi z_2\|_{E^3_T} \leq B(T) R \left( \|p_1(t) - p_2(t)\|_{C[0,T]} + \|u_1(x, t) - u_2(x, t)\|_{B^3_{2,T}} \right). \tag{37}
\]

Then for (34), from estimations (36) and (37) it follows that the operator \( \Phi \) acts in the ball \( K = K_R \) and is contractive. Therefore, in the ball \( K = K_R \) the operator \( \Phi \) has a unique fixed point \( \{u, p\} \) that is a unique solution of equation (37) in the ball \( K = K_R \), i.e. it is a unique solution of system (24), (27) in the ball \( K = K_R \).

The function \( u(x, t) \) as an element of the space \( B^3_{2,T} \) is continuous and has continuous derivatives \( u_x(x, t), u_{xx}(x, t) \) in \( D_T \).

From (20) it is easy to see that

\[
\left( \sum_{k=1}^{\infty} \left( \lambda_k^3 \|u'_k(t)\|_{C[0,T]} \right)^2 \right)^{\frac{1}{2}} \leq \frac{1}{b} \|a(t)\|_{C[0,T]} \left( \sum_{k=1}^{\infty} \left( \lambda_k^3 \|u_k(t)\|_{C[0,T]} \right)^2 \right)^{\frac{1}{2}} + \\
+ \frac{1}{b} \|f_x(x, t) + p(t)u_x(x, t)\|_{C[0,T]} \|_{L^2(0,1)}.
\]

From the last relation it is obvious that \( u_t(x, t), u_{tx}(x, t), u_{txx}(x, t) \) is continuous in \( D_T \).

It is easy to verify that the equation (1) and conditions (2), (3), (6), (7) are satisfied in the ordinary sense.

Consequently, \( \{u(x, t), p(t)\} \) is a solution of problem (1) – (3), (6), (7), and by lemma 2 it is unique in the ball \( K = K_R \). The theorem is proved.

By Lemma 1 the unique solvability of the initial problem (1)–(5) follows from the theorem.

**Theorem 2.** Let all the conditions of theorem 1 be fulfilled and

\[
\int_0^1 f(x, t)dx = 0 \ (0 \leq t \leq T), \quad \int_0^1 \varphi(x)dx = 0, \quad \varphi'(0) = \varphi'(1), \quad \varphi(x_0) = h(0)
\]

Then problem (1) – (5) has a unique classic solution in the ball \( K = K_R(\|z\|_{E^3_T}) \leq R = A(T) + 2 \) of the space \( E^3_T \).
References


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