Common Fixed Point Theorems under

Strict Contractive Conditions in

Generalized Fuzzy Metric Spaces

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Abstract

In this paper, we prove common fixed point theorems for weakly compatible mappings satisfying strict contractive condition in generalized fuzzy metric spaces by using property (E.A).

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Introduction

In 1986, Jungck [7] introduced the concept of compatible mapping and proved some common fixed point theorems of compatible mappings in metric space. In 2000, Pant et al. [10] gave two common fixed point theorems of non-compatible mappings under strict contractive conditions by using the notion of R-Weak
By using this property, some common fixed point theorems under strict contractive conditions in metric spaces have been given.

The notion of fuzzy sets was introduced by Zadeh [17]. Various concepts of fuzzy metric spaces were considered in [8, 9]. Many authors have studied fixed point theory in fuzzy metric spaces. The authors [4, 5, 13, 14] have proved fixed point theorems in fuzzy (probabilistic) metric spaces. It is well known that the probabilistic metric space is an important generalization of metric space (see [14]). Fixed point theory in probabilistic metric spaces can be considered as a part of probabilistic analysis, which is a very dynamic area of mathematical research.

**Definition 1.1:**
A binary operation \(*\) : \([0, 1] \times [0, 1] \rightarrow [0, 1] \) is a continuous t-norm if it satisfies the following conditions:
1) \(*\) is associative and commutative,
2) \(*\) is continuous,
3) \(a \ast 1 = a\) for all \(a \in [0,1]\)
4) \(a \ast b \leq c \ast d\) whenever \(a \leq c\) and \(b \leq d\), for each \(a, b, c, d \in [0,1] \).

Two typical examples of continuous t-norm are \(a \ast b = a\cdot b\) and \(a \ast b = \min \{a, b\} \).

**Definition 1.2:**
A 3-tuple \((X, \mathcal{M}, \ast)\) is called a \(\mathcal{M}\)-fuzzy metric space if \(X\) is an arbitrary (non-empty) set, \(*\) is a continuous t-norm, and \(\mathcal{M}\) is a fuzzy set on \(X^3 \times (0, \infty) \), satisfying the following conditions:
for each \(x, y, z, a \in X\) and \(t, s > 0\),
1) \(\mathcal{M}(x, y, z, t) > 0\),
2) \(\mathcal{M}(x, y, z, t) = 1\) if and only if \(x = y = z\),
3) \(\mathcal{M}(x, y, z, t) = \mathcal{M}(p\{x, y, z\}, t)\) (symmetry) where \(p\) is a permutation function,
4) \(\mathcal{M}(x, y, z, t) \ast \mathcal{M}(a, z, z, s) \leq \mathcal{M}(x, y, z, t + s)\),
5) \(\mathcal{M}(x, y, z, \cdot) : (0,\infty) \rightarrow [0,1] \) is continuous.

**Example 1.3:**
Let \(X\) be a non-empty set and \(D\) is the D-metric on \(X\). Denote \(a \ast b = a\cdot b\) for all \(a, b \in [0,1]\). For each \(t \in (0, \infty)\), define \(\mathcal{M}(x, y, z, t) = \frac{t}{t + D(x, y, z)}\) for all \(x, y, z \in X\). It is easy to see that \((X, \mathcal{M}, \ast)\) is a \(\mathcal{M}\)-fuzzy metric space.

**Definition 1.4:**
Let \((X, \mathcal{M}, \ast)\) be a \(\mathcal{M}\)-fuzzy metric space. Then \(\mathcal{M}\) is said to be continuous function on \(X^3 \times (0, \infty)\) if
\[
\lim_{n \to \infty} \mathcal{M}(x_n, y_n, z_n, t_n) = \mathcal{M}(x, y, z, t) \text{ whenever a sequence } \{x_n, y_n, z_n, t_n\} \text{ in } X^3 \times (0, \infty) \text{ converges to a point } (x, y, z, t) \in X^3 \times (0, \infty). \text{ i.e., } \lim_{n \to \infty} x_n = x, \lim_{n \to \infty} y_n = y, \lim_{n \to \infty} z_n = z \text{ and } \lim_{n \to \infty} \mathcal{M}(x, y, z, t_n) = \mathcal{M}(x, y, z, t).\]
**Lemma 1.5:**
Let $(X, \mathcal{M}, *)$ be a $\mathcal{M}$-fuzzy metric space. Then $\mathcal{M}$ is continuous function on $X^3 \times (0, \infty)$.

**Definition 1.6:**
Let $A$ and $S$ be mappings from a $\mathcal{M}$-fuzzy metric space $(X, \mathcal{M}, *)$ into itself. Then the mappings are said to be weak compatible if they commute at their coincidence point, that is $Ax = Sx$ implies that $ASx = SAx$.

**Definition 1.7:**
Let $A$ and $B$ be two self-mappings of a $\mathcal{M}$-fuzzy metric space $(X, \mathcal{M}, *)$. We say that $A$ and $B$ satisfy the property (E.A.), if there exists a sequence $\{x_n\}$ such that
$$\lim_{n \to \infty} \mathcal{M}(Ax_n, u, u, t) = \lim_{n \to \infty} \mathcal{M}(Bx_n, u, u, t) = 1$$
for some $u \in X$ and $t > 0$.

**Example 1.8:**
Let $X = \mathbb{R}$ and $\mathcal{M}(x, y, z, t) = \frac{t}{t + |x-y| + |y-z| + |x-z|}$ for every $x, y, z \in X$ and $t > 0$. Let $A$ and $B$ be defined
$$Ax = 2x + 1, \quad Bx = x + 2,$$
consider the sequence $\{x_n\}$ where $x_n = \frac{1}{n} + 1, \quad n = 1, 2, \ldots$. Thus, we have
$$\lim_{n \to \infty} \mathcal{M}(Ax_n, 3, 3, t) = \lim_{n \to \infty} \mathcal{M}(Bx_n, 3, 3, t) = 1$$
for every $t > 0$. Then $A$ and $B$ satisfying in the property (E.A.).

In the next example, we show that there are some mappings that have not property (E.A.).

**Example 1.9:**
Let $X = \mathbb{R}$ and $\mathcal{M}(x, y, z, t) = \frac{t}{t + |x-y| + |y-z| + |x-z|}$ for every $x, y, z \in X$ and $t > 0$. Let
$$Ax = x + 1 \quad \text{and} \quad Bx = x + 2,$$
if sequence $\{x_n\}$ there exist such that
$$\lim_{n \to \infty} \mathcal{M}(Ax_n, u, u, t) = \lim_{n \to \infty} \mathcal{M}(Bx_n, u, u, t) = 1$$
for some $u \in X$. We conclude that, $x_n \to u - 1$ and $x_n \to u - 2$, which is a contradiction.

Hence $A$ and $B$ do not satisfy the property (E.A.).

**Main Results**

Let $\mathcal{F}$ be the set of all fuzzy set on $X^3 \times (0, \infty)$ that is $\mathcal{F} = \{f: X^3 \times (0, \infty) \to [0,1]\}$.

**Definition 2.1:**
Let $f$ and $g \in \mathcal{F}$. The algebraic sum $f \oplus g$ of $f$ and $g$ is defined by
\( f(x, y, z, t) \oplus g(x', y', z', t) = \sup_{t_1 + t_2 = t} \min \{ f(x, y, z, t_1), g(x', y', z', t_2) \} \)

**Remarks 2.2:**
For every \( x, y, z \in X \) and every \( t > 0 \), we have

1) \( f(x, y, z, 2t) \oplus f(x, y, z, 2t) \geq \min \{ f(x, y, z, t), f(x, y, z, t) \} = f(x, y, z, t) \).

2) \( f(x, y, z, t) \oplus 1 \geq \min \{ f(x, y, z, t - \varepsilon), f(x, x, x, t - \varepsilon) \} = f(x, y, z, t - \varepsilon) \).

Letting \( \varepsilon \to 0 \), we get \( f(x, y, z, t) \oplus 1 \geq f(x, y, z, t) \).

**Throughout this section \( \Phi \) denotes a family of mappings such that for each \( \phi \in \Phi \), \( \phi: [0,1] \to [0,1] \) is continuous and increasing in each coordinate variable.** Also \( \gamma(t) = \phi(t, t, t) \geq t \) for every \( t \in [0,1] \).

**Example 2.3:**
Let \( \phi: [0,1] \to [0,1] \) be defined by \( \phi(x, y, z) = \min \{ x, y, z \} \). Clearly, \( \phi \in \Phi \).

**Proof:**
Let \( A, B, S \) and \( T \) be mappings from a \( \mathcal{M} \)-fuzzy metric space \((X, \mathcal{M}, *)\) into itself satisfying the following conditions:

1) \( A(X) \subseteq T(X), B(X) \subseteq S(X) \);

2) \( \mathcal{M}(Ax, By, Bz, t) \geq \varphi \left\{ \mathcal{M}
\left(\begin{array}{c}
Sx, Ty, Tz, \frac{2t}{k} \end{array}\right), \mathcal{M}
\left(\begin{array}{c}
Ax, Sx, Sx, \frac{2t}{k} \end{array}\right) \oplus \mathcal{M}
\left(\begin{array}{c}
By, Ty, Tz, \frac{2t}{k} \end{array}\right), \mathcal{M}
\left(\begin{array}{c}
Ax, Ty, Tz, \frac{4t}{k} \end{array}\right) \oplus \mathcal{M}
\left(\begin{array}{c}
Sx, By, Bz, \frac{4t}{k} \end{array}\right) \right\} \)

for all \( x, y, z \in X \), \( t > 0 \), \( \varphi \in \Phi \), and \( 0 \leq k < 2 \).

Suppose that one of the pairs \( (A, S) \) and \( (B, T) \) satisfies the property (E.A). \( (A, S) \) and \( (B, T) \) are weakly compatible and one of \( A(X), B(X), S(X) \) and \( T(X) \) is a complete subspace of \( X \). Then \( A, B, S \) and \( T \) have a unique common fixed point in \( X \).
\[ \mathcal{M}(A_{n}, B_{n}, B_{n}, t) \geq \begin{cases} \mathcal{M}\left(\text{Sy}_{n}, T_{n}, T_{n}, \frac{2t}{k}\right) + \mathcal{M}\left(A_{n}, S_{n}, S_{n}, \frac{2t}{k}\right) + \mathcal{M}\left(B_{n}, T_{n}, T_{n}, \frac{2t}{k}\right) \\ \mathcal{M}\left(A_{n}, T_{n}, T_{n}, \frac{4t}{k}\right) + \mathcal{M}\left(S_{n}, B_{n}, B_{n}, \frac{4t}{k}\right) \end{cases} \]

\[ = \mathcal{M}\left(B_{n}, T_{n}, T_{n}, \frac{2t}{k}\right) + \mathcal{M}\left(A_{n}, B_{n}, B_{n}, \frac{2t}{k}\right) + \mathcal{M}\left(B_{n}, T_{n}, T_{n}, \frac{2t}{k}\right) + \mathcal{M}\left(A_{n}, T_{n}, T_{n}, \frac{4t}{k}\right) + \mathcal{M}\left(B_{n}, T_{n}, T_{n}, \frac{4t}{k}\right) \]

\[ (2.3) \]

Since, \( \liminf_{n \to \infty} \mathcal{M}\left(A_{n}, B_{n}, B_{n}, \frac{2t}{k}\right) + \mathcal{M}\left(B_{n}, T_{n}, T_{n}, \frac{2t}{k}\right) \geq \liminf_{n \to \infty} \min\left\{ \mathcal{M}\left(A_{n}, B_{n}, B_{n}, \frac{2t}{k} - \epsilon\right), \mathcal{M}\left(B_{n}, T_{n}, T_{n}, \epsilon\right) \right\} \]

\[ = \liminf_{n \to \infty} \mathcal{M}\left(A_{n}, B_{n}, B_{n}, \frac{2t}{k} - \epsilon\right), \]

letting \( \epsilon \to 0 \), in the above inequality, we get

\[ \liminf_{n \to \infty} \mathcal{M}\left(A_{n}, B_{n}, B_{n}, \frac{2t}{k}\right) + \mathcal{M}\left(B_{n}, T_{n}, T_{n}, \frac{2t}{k}\right) \geq \liminf_{n \to \infty} \mathcal{M}\left(A_{n}, B_{n}, B_{n}, \frac{2t}{k}\right). \]

Also, by Remark 2.2, \( \liminf_{n \to \infty} \mathcal{M}\left(A_{n}, T_{n}, T_{n}, \frac{4t}{k}\right) + \mathcal{M}\left(S_{n}, B_{n}, B_{n}, \frac{4t}{k}\right) = \liminf_{n \to \infty} \mathcal{M}\left(A_{n}, T_{n}, T_{n}, \frac{4t}{k}\right) + \mathcal{M}\left(B_{n}, T_{n}, T_{n}, \frac{4t}{k}\right) \)

\[ \geq \liminf_{n \to \infty} \mathcal{M}\left(A_{n}, T_{n}, T_{n}, \frac{2t}{k}\right), \]

hence letting \( n \to \infty \), in inequality (2.3), we get,

\[ \liminf_{n \to \infty} \mathcal{M}\left(A_{n}, z, z, t\right) = \mathcal{M}\left(\liminf_{n \to \infty}(A_{n}, z, z, t)\right) = \liminf_{n \to \infty} \mathcal{M}(A_{n}, B_{n}, B_{n}, t) \]

\[ \geq \phi\left(1, \liminf_{n \to \infty} \mathcal{M}\left(A_{n}, B_{n}, B_{n}, \frac{2t}{k}\right), \liminf_{n \to \infty} \mathcal{M}\left(A_{n}, T_{n}, T_{n}, \frac{2t}{k}\right)\right) \]

\[ \geq \phi\left(\liminf_{n \to \infty} \mathcal{M}\left(A_{n}, B_{n}, B_{n}, \frac{2t}{k}\right), \liminf_{n \to \infty} \mathcal{M}\left(A_{n}, B_{n}, B_{n}, \frac{2t}{k}\right), \liminf_{n \to \infty} \mathcal{M}\left(A_{n}, T_{n}, T_{n}, \frac{2t}{k}\right)\right) \]

\[ = \phi\left(\liminf_{n \to \infty} \mathcal{M}(A_{n}, z, z, \frac{2t}{k}), \mathcal{M}\left(\liminf_{n \to \infty} A_{n}, z, z, \frac{2t}{k}\right), \mathcal{M}\left(\liminf_{n \to \infty} A_{n}, z, z, \frac{2t}{k}\right)\right) \]
\[ \geq \mathcal{M} \left( \lim_{n \to \infty} \inf Ay_n, z, z, \frac{2t}{k} \right) \]

\[ \vdots \]

\[ \geq \mathcal{M} \left( \lim_{n \to \infty} \inf Ay_n, z, z, \left( \frac{2}{k} \right)^n t \right) \to 1 . \]

Similarly, \( \limsup_{n \to \infty} \mathcal{M}(Ay_n, z, z, t) = \mathcal{M} \left( \limsup_{n \to \infty} Ay_n, z, z, t \right) = 1 \), hence,

\[ \lim_{n \to \infty} \mathcal{M}(Ay_n, z, z, t) = 1 . \]

Assume that \( S(X) \) is a closed subset of \( X \). Then, There exists \( u \in X \) such that \( Su = z \) using (2.2), we get

\[ \mathcal{M}(Au, Bx_n, Bx_n, t) \geq \varphi(\mathcal{M} \left( Su, Tx_n, Tx_n, \frac{2t}{k} \right)), \]

\[ \mathcal{M} \left( Au, Su, Su, \frac{2t}{k} \right) \oplus \mathcal{M} \left( Bx_n, Tx_n, Tx_n, \frac{2t}{k} \right), \]

\[ \mathcal{M} \left( Au, Tx_n, Tx_n, \frac{4t}{k} \right) \oplus \mathcal{M} \left( Su, Bx_n, Bx_n, \frac{4t}{k} \right) \]

\[ \varphi \left( \mathcal{M} \left( z, Tx_n, Tx_n, \frac{2t}{k} \right), \mathcal{M} \left( Au, z, z, \frac{2t}{k} \right) \oplus \mathcal{M} \left( Bx_n, Tx_n, Tx_n, \frac{2t}{k} \right) \right), \]...

(2.4).

In addition, it is easy to verify that

\[ \liminf_{n \to \infty} \mathcal{M} \left( Au, Su, Su, \frac{2t}{k} \right) \oplus \mathcal{M} \left( Bx_n, Tx_n, Tx_n, \frac{2t}{k} \right) \geq \mathcal{M} \left( Au, Su, Su, \frac{2t}{k} \right). \]

(2.5).

In fact, for all \( \in \) in \( \left( 0, \frac{2t}{k} \right) \), we have

\[ \mathcal{M} \left( Au, Su, Su, \frac{2t}{k} \right) \oplus \mathcal{M} \left( Bx_n, Tx_n, Tx_n, \frac{2t}{k} \right) \geq \min \left\{ \mathcal{M} \left( Au, Su, Su, \frac{2t}{k} - \in \right), \mathcal{M} \left( Bx_n, Tx_n, Tx_n, \in \right) \right\} \]

Since \( \lim_{n \to \infty} Bx_n = \lim_{n \to \infty} Tx_n = Su \), the above inequality implies that,

\[ \liminf_{n \to \infty} \left( \mathcal{M} \left( Au, Su, Su, \frac{2t}{k} \right) \oplus \mathcal{M} \left( Bx_n, Tx_n, Tx_n, \frac{2t}{k} \right) \right) \geq \mathcal{M} \left( Au, Su, Su, \frac{2t}{k} - \in \right) \).

Letting \( \in \to 0 \), in the above inequality, we get (2.5).

Also, by Remark 2.2, we get

\[ \lim inf \mathcal{M} \left( Au, Tx_n, Tx_n, \frac{4t}{k} \right) \oplus \mathcal{M} \left( Su, Bx_n, Bx_n, \frac{4t}{k} \right) = \]

\[ \lim inf \mathcal{M} \left( Au, Tx_n, Tx_n, \frac{4t}{k} \right) \oplus 1 \]

\[ \geq \]

\[ \lim inf \mathcal{M} \left( Au, Tx_n, Tx_n, \frac{2t}{k} \right) \]

\[ = \mathcal{M} \left( Au, z, z, \frac{2t}{k} \right) . \]
So, letting $n \to \infty$, in inequality (2.4), we get
\[
\mathcal{M}(Au, z, z, t) \geq \varphi \left( 1, \mathcal{M} \left( Au, z, z, \frac{2t}{k} \right), \mathcal{M} \left( Au, z, z, \frac{2t}{k} \right) \right)
\geq \varphi \left( \mathcal{M} \left( Au, z, z, \frac{2t}{k} \right), \mathcal{M} \left( Au, z, z, \frac{2t}{k} \right), \mathcal{M} \left( Au, z, z, \frac{2t}{k} \right) \right)
\geq \mathcal{M} \left( Au, z, z, \frac{2t}{k} \right)
\geq \mathcal{M} \left( Au, z, z, \left( \frac{2t}{k} \right)^n \right) \to 1.
\]

Hence, $\mathcal{M}(Au, z, z, t) = 1$.
That is, $Au = Su = z$. Since, $A(X) \subseteq T(X)$, There exists $v \in X$ such that $z = T v$.
Using (2.2), and Remark 2.2, we have

\[
\mathcal{M}(z, Bv, Bv, t) = \mathcal{M}(Au, Bv, Bv, t)
\geq \varphi \left( \mathcal{M} \left( Su, Tv, Tv, \frac{2t}{k} \right) \right),
\mathcal{M} \left( Au, Su, Su, \frac{2t}{k} \right) \oplus \mathcal{M} \left( Bv, Tv, Tv, \frac{2t}{k} \right),
\mathcal{M} \left( Au, Tv, Tv, \frac{4t}{k} \right) \oplus \mathcal{M} \left( Su, Bv, Bv, \frac{4t}{k} \right)
= \varphi \left( 1, 1 \oplus \mathcal{M} \left( Bv, z, z, \frac{2t}{k} \right), 1 \oplus \mathcal{M} \left( z, Bv, Bv, \frac{4t}{k} \right) \right)
\geq \varphi \left( \mathcal{M} \left( BV, z, z, \frac{2t}{k} \right), \mathcal{M} \left( BV, z, z, \frac{2t}{k} \right), \mathcal{M} \left( z, Bv, Bv, \frac{2t}{k} \right) \right)
\geq \mathcal{M} \left( Bv, z, z, \frac{2t}{k} \right)
\geq \mathcal{M} \left( Bv, z, z, \left( \frac{2}{k} \right)^n \right) \to 1.
\]

Hence $z = Bv = Tv$.
Since the pairs $(A, S)$ and $(B, T)$ are weakly compatible, we obtain $Az = Sz$ and $Bz = Tz$ using the inequality (2.2), we have

\[
\mathcal{M}(Az, z, z, t) = \mathcal{M}(Az, Bv, Bv, t)
\geq \varphi \left( \mathcal{M} \left( Sz, Tv, Tv, \frac{2t}{k} \right), \mathcal{M} \left( Az, Sz, Sz, \frac{2t}{k} \right) \oplus \mathcal{M} \left( Bv, Tv, Tv, \frac{2t}{k} \right), \mathcal{M} \left( Az, Tv, Tv, \frac{4t}{k} \right) \oplus \mathcal{M} \left( Sz, Bv, Bv, \frac{4t}{k} \right) \right)
\geq \varphi \left( \mathcal{M} \left( Az, z, z, \frac{2t}{k} \right), 1 \oplus \mathcal{M} \left( Az, z, z, \frac{2t}{k} \right), \mathcal{M} \left( Az, z, z, \frac{4t}{k} \right) \right)
\geq \mathcal{M} \left( Az, z, z, \frac{2t}{k} \right)
\geq \mathcal{M} \left( Az, z, z, \left( \frac{2}{k} \right)^n \right) \to 1.
\]

Then $Az = Sz = z$. 
Similarly, we can prove that \( z = Bz = Tz \). Therefore \( z \) is a common fixed point of \( A, B, S \) and \( T \).

Now, we prove the \textbf{uniqueness}, suppose if possible \( w \neq z \) be another common fixed point of \( A, B, S \) and \( T \). Then by inequality (2.2), we have

\[
\mathcal{M}(z, w, w, t) = \mathcal{M}(Az, Bw, Bw, t) \geq \varphi(\mathcal{M}(z, w, w, 2t^k), \mathcal{M}(z, w, w, 4t^k)) \geq \mathcal{M}(z, w, w, (2^k)^n) \rightarrow 1,
\]

which is a contradiction. Hence \( w = z \) is a unique common fixed point of \( A, B, S \) and \( T \).

\textbf{Remark: 2.5:}
Putting \( B = A \), and \( T = S \) in the above theorem, we get the following Corollary 2.6.

\textbf{Corollary 2.6:}
Let \( A \) and \( S \) be mappings from a \( \mathcal{M} \)-fuzzy metric space \((X, \mathcal{M}, \ast)\) into itself satisfying the following condition:

\begin{align*}
(2.6) & \quad A(X) \subseteq S(X); \\
(2.7) & \quad \mathcal{M}(Ax, Ay, Az, t) \geq \\
& \quad \varphi\left( \mathcal{M}(Sx, Sy, Sz, 2t^k), \mathcal{M}(Ax, Sx, Sx, 2t^k) \oplus \mathcal{M}(Ay, Sy, Sz, 2t^k), \mathcal{M}(Ax, Sy, Sz, 4t^k) \oplus \mathcal{M}(Sx, Ay, Az, 4t^k) \right)
\end{align*}

for all \( x, y, z \in X \), and \( t > 0 \), where \( 0 \leq k < 2 \). Suppose that the pair \((A, S)\) satisfies the property (E.A), \( (A, S) \) is weakly compatible and one of \( A(X) \) and \( S(X) \) is a complete subspace of \( X \). Then \( A \) and \( S \) have a unique common fixed point in \( X \).
References


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