Numerical and Mathematical Modeling of the Injection for Incompressible Fluids through Rigid Cylindrical Duct: Application of Melted Polymers

PPH

Hamid EL-Tourroug
Team of modelling in fluid mechanics and environment, LPT, URAC 13
Faculty of sciences, Mohammed V University, Rabat-Agdal
B.P. 1014, Rabat, Morocco

Kamal Gueraoui*
Team of modelling in fluid mechanics and environment, LPT, URAC 13
Faculty of sciences, Mohammed V University, Rabat-Agdal
B.P. 1014, Rabat, Morocco
Department of Mechanical Engineering
University of Ottawa, Ottawa, Canada
*Corresponding author

Najem Hassanain
Laboratory of Physics Materials
Faculty of sciences, Mohammed V University, Rabat-Agdal
B.P. 1014, Rabat, Morocco

Ihsane Modhaffar
Team of modelling in fluid mechanics and environment, LPT, URAC 13
Faculty of sciences, Mohammed V University, Rabat-Agdal
B.P. 1014, Rabat, Morocco
Abstract

Our melted polymer is modeled by incompressible non-Newtonian fluids. The mathematical formulation leads to very complex equations. These complexities can arise, on the one hand, from the law of behavior of the fluid of interest (Arrhenius model) and the nature of the flow and on the other hand the nature of the wall considered. We are interested in this study to establish a numerical code to study the flow of molten polymers in rigid cylindrical ducts. Using an iterative numerical finite volume method, we determine the axial profile of the velocity and the temperature.

Keywords: injection, melted polymer, modeling, Pseudo-plastic model, finite volume, numerical model, incompressible non-Newtonian fluids

Nomenclatures

\[
\begin{align*}
\rho & \quad \text{Density of volume [kg/m3]} \\
P  & \quad \text{pressure [Pa]} \\
T  & \quad \text{Temperature [K]} \\
K  & \quad \text{thermal conductivity [W/m.K]} \\
\eta & \quad \text{dynamic viscosity [kg/m.s]} \\
C_v & \quad \text{Specific heat with constant volume [J/kg. K]} \\
u, w & \quad \text{Dimensional velocity components [m/s]} \\
w_0 & \quad \text{Speed at the inlet side [m/s]} \\
L_0 & \quad \text{Length of the conduit a relaxing break [m]} \\
T_0 & \quad \text{Ambient temperature to the inlet side [K]} \\
R_0 & \quad \text{Radius a relaxing break [m]} \\
E & \quad \text{Activation energy [jmol}^{-1}] \\
\end{align*}
\]

1. Introduction

The study of polymer flow in the mold cavity brings together a great deal of phenomena. During the injection, the phenomenon is non-isothermal. The velocity of fluid along the walls is often considered null. It forms a solid cold sheath that causes the central part of the vein to be expandable to fill the section and continue
to run. The flow rate of molten polymer is principally between those ducts. The molten polymer is modeled as an incompressible non-Newtonian fluid. The mathematical formulation leads to very complex equations. These complexities can originate, on one hand, the law of behavior of the studied fluid (Model Pseudo-plastic) and the nature of the flow and the other hand, the nature of the wall in question. We are interested in this study to establish a numerical code allowing to study the flow of molten polymers in rigid cylindrical tube. Using an iterative numerical finite volume method; we determine the profiles of the radial and axial velocity in the conduct and temperature. The numerical code presented may be applied to other industrial applications [1], [2], [3], [4], [5].

2. Mathematical formulation

The movement of a fluid such as molten polymer considered incompressible can be described by a set of partial differential equations derived from the fundamental laws of mechanics and thermodynamics: continuity equation, quantity of movement equation and the equation of energy.

By projecting these equations according to the cylindrical coordinates system, we have:

\[
\begin{align*}
\frac{1}{r} \frac{\partial}{\partial r} [r u] + \frac{\partial}{\partial z} (w) &= 0 \\
\rho \frac{\partial u}{\partial t} + \rho \frac{\partial}{\partial r} (r u u) + \rho \frac{\partial}{\partial z} (w u) &= -\frac{\partial p}{\partial r} + \eta \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \eta \frac{\partial}{\partial z} \left( \frac{\partial u}{\partial z} \right) \\
&\quad + \left\{ \eta \frac{\partial}{\partial r} \left( \frac{\partial u}{\partial r} \right) - 2 \eta \frac{u}{r^2} + \eta \frac{\partial}{\partial r} \left( \frac{\partial w}{\partial r} \right) + \rho f_r \right\} \\
\rho \frac{\partial w}{\partial t} + \rho \frac{\partial}{\partial r} (r u w) + \rho \frac{\partial}{\partial z} (w w) &= -\frac{\partial p}{\partial z} + \eta \frac{\partial}{\partial r} \left( \frac{\partial w}{\partial r} \right) + \eta \frac{\partial}{\partial z} \left( \frac{\partial w}{\partial z} \right) \\
&\quad + \left\{ \eta \frac{\partial}{\partial r} \left( \frac{\partial u}{\partial r} \right) + \eta \frac{\partial}{\partial z} \left( \frac{\partial w}{\partial z} \right) + \rho f_z \right\} \\
\rho C_v \frac{\partial T}{\partial t} + \rho C_v u \frac{\partial T}{\partial r} + \rho C_v w \frac{\partial T}{\partial z} &= \frac{k}{r} \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) + \frac{\partial^2 T}{\partial z^2} \right] + 2 \eta \left( \frac{\partial u}{\partial r} \right) + \left( \frac{\partial w}{\partial z} \right)^2 \\
&\quad + \eta \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right)^2 + 2 \eta \frac{u^2}{r} - \frac{\eta}{r} \left( r u \frac{\partial u}{\partial r} \right) - \eta \frac{\partial}{\partial r} \left( u \frac{\partial u}{\partial z} \right) - 2 \eta \frac{\partial}{\partial z} \left( w \frac{\partial w}{\partial z} \right) - \eta \frac{\partial}{r} \left( r w \frac{\partial w}{\partial r} \right) \\
&\quad - \eta \frac{\partial}{r} \left( r w \frac{\partial u}{\partial z} \right) - \eta \frac{\partial}{\partial z} \left( u \frac{\partial w}{\partial r} \right)
\end{align*}
\]
with:

\[ \eta_a = \eta(R_{ref}) \exp \left( \frac{E}{R} \left( 1 - \frac{1}{R_{ref}} \right) \right) \quad \text{and} \quad \eta(R_{ref}) = \eta_0 \left( 1 + \left( \frac{\partial w}{\partial r} \right)^b \right)^{m-1} \]

(5)

Where: \( U \) is the radial component of the fluid velocity, \( w \) the axial component of the velocity, \( P \) the pressure, \( T \) the temperature, \( \rho \) the volume density of the fluid, \( \eta \) the apparent viscosity, \( c_p \) the specific heat, and \( k \) the thermal conductivity.

3. A dimensionlesslisation of flow and energy equations

With the aim to identify some dimensionless numbers characterizing the flow, one can introduce the following dimensionless variables:

\[ \hat{r} = \frac{r}{R_0}, \quad \hat{z} = \frac{z}{L_0}, \quad \hat{t} = \frac{t \omega_{\infty}}{2\pi}, \quad \hat{w} = \frac{w}{w_0}, \quad \hat{u} = \frac{u L_0}{w_0 R_0}, \quad \hat{P} = \frac{PR_0^2}{\eta \nu_0 L_0} \]

\[ \hat{R} = \frac{R}{R_0}, \quad \hat{\epsilon} = \frac{R_0}{L_0}, \quad \hat{T} = \frac{T}{T_0} \]

Equations (1), (2), (3), (4) and (5) are written in dimensionless form:

\[ \frac{1}{\hat{r}} \frac{\partial}{\partial \hat{r}} \left( \hat{r} \hat{u} \right) + \frac{\partial \hat{w}}{\partial \hat{z}} = 0 \]

(6)

\[ \epsilon^2 \beta^2 \frac{\partial^2 \hat{u}}{\partial \hat{t}^2} + \text{Re} \epsilon^2 \frac{1}{\hat{r}} \frac{\partial}{\partial \hat{r}} \left( \hat{r} \hat{u} \hat{\hat{u}} \right) + \text{Re} \epsilon^2 \frac{\partial}{\partial \hat{z}} \left( \hat{w} \hat{\hat{u}} \right) = -\frac{\partial \hat{P}}{\partial \hat{r}} + \epsilon^2 \frac{2}{\hat{r}} \frac{\partial}{\partial \hat{r}} \left( \hat{r} \frac{\partial \hat{\hat{u}}}{\partial \hat{r}} \right) \]

(7)

\[ -\epsilon^2 \frac{2}{\hat{r}} \frac{1}{\hat{r}} \frac{\partial}{\partial \hat{r}} \left[ \hat{r} \left( \frac{1}{\hat{r}} \frac{\partial}{\partial \hat{r}} \hat{u} \hat{\hat{u}} \right) + \frac{\partial \hat{w}}{\partial \hat{z}} \right] - 2\epsilon^2 \frac{\hat{u}}{\hat{r}^2} + \epsilon^2 \frac{1}{\hat{r}} \frac{\partial}{\partial \hat{r}} \left( \hat{r} \hat{\hat{u}} \right) + \epsilon^2 \frac{\partial \hat{\hat{u}}}{\partial \hat{z}} \]

\[ + \frac{\partial}{\partial \hat{z}} \left( \epsilon^2 \frac{\partial \hat{w}}{\partial \hat{r}} + \epsilon^4 \frac{\partial \hat{u}}{\partial \hat{z}} \right) + \epsilon^2 \text{Re} X_1 \]

\[ \beta^2 \frac{\partial \hat{\hat{w}}}{\partial \hat{t}} + \text{Re} \epsilon \frac{1}{\hat{r}} \frac{\partial}{\partial \hat{r}} \left( \hat{r} \hat{\hat{u}} \hat{\hat{w}} \right) + \text{Re} \epsilon \frac{\partial}{\partial \hat{z}} \left( \hat{w} \hat{\hat{w}} \right) = -\frac{\partial \hat{P}}{\partial \hat{z}} + \frac{1}{\hat{r}} \frac{\partial}{\partial \hat{r}} \left( \hat{r} \frac{\partial \hat{\hat{w}}}{\partial \hat{r}} \right) \]

(8)

\[ + \epsilon^2 \frac{\partial}{\partial \hat{z}} \left( \frac{\partial \hat{\hat{w}}}{\partial \hat{r}} \right) + \epsilon^2 \frac{1}{\hat{r}} \frac{\partial}{\partial \hat{r}} \left( \hat{r} \frac{\partial \hat{\hat{u}}}{\partial \hat{r}} \right) + \epsilon^2 \frac{\partial}{\partial \hat{z}} \left( \frac{\partial \hat{\hat{w}}}{\partial \hat{z}} \right) + X_2 \cdot \hat{f}_z \]
On one hand, we neglect the terms on $\varepsilon^2$. On the other hand, $f_z$ which represents the force of gravity is null in the direction $\hat{z}$. So the above equations become:

$$1 \frac{\partial (\hat{r} \hat{u})}{\partial \hat{r}} + \frac{\partial \hat{w}}{\partial \hat{z}} = 0$$

(10)

$$0 = \frac{\partial \hat{p}}{\partial \hat{r}}$$

(11)

$$\beta^2 \frac{\partial \hat{w}}{\partial \hat{t}} = -\frac{\partial \hat{p}}{\partial \hat{z}} + \frac{1}{\hat{r}} \frac{\partial}{\partial \hat{r}} \left( \hat{r} \frac{\partial \hat{w}}{\partial \hat{r}} \right)$$

(12)

$$\frac{\partial \hat{F}}{\partial \hat{t}} + \hat{u} \frac{\partial \hat{F}}{\partial \hat{r}} + \hat{w} \frac{\partial \hat{F}}{\partial \hat{z}} = \frac{1}{G_z} \left[ \frac{1}{\hat{r}} \frac{\partial}{\partial \hat{r}} \left( \hat{r} \frac{\partial \hat{F}}{\partial \hat{r}} \right) \right] - \frac{B_{Fr}}{G_z} \hat{w} \frac{\partial}{\partial \hat{z}} \left( \hat{r} \frac{\partial \hat{F}}{\partial \hat{r}} \right)$$

(13)

$$\eta(\tau_{ref}) = \eta_0 \left( 1 + \left( \frac{\frac{\partial \hat{w}}{\partial \hat{r}}}{\frac{\partial \hat{F}}{\partial \hat{r}}} \right)^k \right)^{m-1}$$

(14)

4. The boundary conditions

- Dynamic conditions:
  - Condition on axis:
    The axisymmetry of the flow, two homogeneous conditions imposed on the axis of the tube ($r = 0$):
    $$\frac{\partial \hat{w}}{\partial \hat{r}} = 0$$
    (14)
    $$U = 0$$
    (15)
  - Conditions of adhesion to the wall:
Thermal conditions:

The temperature is governed by the two following boundary conditions:

On the wall \((r = R)\) we have:

\[
T(r = R) = T_p
\]  

On the axis of the conduct \((r = 0)\), we have:

\[
\frac{dT}{dr}
_{r=0} = 0
\]

5. Method of solution

The equations obtained above do not admit analytical solutions. This is why the use of numerical solution becomes necessary. As part of this study, we opted for the method the finite volumes [2].

All equations were discredited by using a centered method in space and delayed in time. The algebraic equations obtained are solved using the method of double scanning.

The calculations will be initiated by an initial profile that whatever may be provided that meets the boundary conditions. The solution of the equations leads to determine the flow velocity component. By means of this component of the velocity and the equation of temperature, the temperature is determined. The obtained value of the temperature allows repeating and correcting values until convergence of the solution.

6. Convergence test

The test of convergence from the solution of the problem studied relates to the temperature. If \(m\) is the number of cycles of the approximation calculation and \(\varepsilon\) a fixed number in advance, it is necessary to check the temperature using:

\[
\sup \left| \frac{T_{\text{approx}} - T_m}{T_m} \right| \leq \varepsilon
\]

Data Program for PPH (polypropylene homopolymer) [6], [7]
- Newtonian Viscosity of reference, \(\eta_0\), is 43350 Pa.s
- The fluid behavior index, \(m\), is 0.38
- Thermal conductivity, $\lambda$, is $0.179 \text{ Wm}^{-1}\text{K}^{-1}$
- Heat capacity, $C_v$, is $2800 \text{ Jkg}^{-1}\text{K}^{-1}$
- The entrance pressure at the pipe is $4 \times 10^6 \text{ Pa}$
- The outlet pressure at the pipe is $2 \times 10^6 \text{ Pa}$
- The report of the activation energy by the gas constant is $9.73 \text{ K}^{-1}$

7. Results

Fig 1: The axial velocity profile in function of radius for two values of the viscosity $\eta$

Fig 2: The temperature profile as a function of radius for two values of the viscosity $\eta$

Fig 3: The axial velocity profile of two values for the number of Womersley $\beta$
Fig 4: The temperature profile for two values of the number of Womersley

Fig 5: The temperature profile for two values of the number of Greatz $G_z$

Fig 6: The temperature profile for two values of the number of Brinkman $B_r$
Fig 7: The axial velocity profile as a function of radius and for three different sections

Fig 8: The temperature profile as a function of radius for three different sections
Fig 9: The axial velocity profile as a function of radius for three different times

Fig 10: The temperature profile as a function of radius for three different times

Figures 1 and 2 show the axial velocity and temperature profiles for two different values of the viscosity $\eta$. It can be seen that a decrease in this parameter, which results in amore marked pseudo plasticity, is accompanied by a decrease in friction between the coaxial layers and consequently an increase of the amplitudes of the axial velocity and the temperature. We obtain qualitatively similar results to those obtained by other authors [3].
Figures 3 and 4 illustrate the profiles of the axial velocity and the temperature values of two Womersley number $\beta$. It can be seen that an increase of this parameter, is accompanied by a decrease in viscosity $\eta$, which leads to a reduced friction between the coaxial layers and the refrain increase of the amplitudes of the axial velocity and a decrease in the temperature amplitude. We obtain qualitatively similar results to those obtained by other authors [3].

Figures 5 and 6 show the temperature profiles for two values of the number of Greatz $G_z$ and Brinkman number $Br$. It can be seen that an increase of this parameter, is accompanied by an increase in viscosity $\eta$, which leads to increase the friction between the coaxial layers and to a decrease of the amplitudes of the axial velocity and an increase of the temperature amplitude. We obtain qualitatively similar results to those obtained by other authors [3].

Figures 7 and 8 are illustrated the profiles of the axial velocity and temperature as function of radius of the pipe or different sections ($z = L/4$, $z = L/2$, and $z = 3L/4$). We note that as the polymer lead in driving the amplitudes of the dimensionless axial velocity and dimensionless temperature rise. This can be explained by the fact that the polymer becomes molten and hence the molecules slide over each other more readily thus involving an increase in the axial velocity and temperature.

Figures 9 and 10 are illustrated the profiles of the dimensionless axial velocity and dimensionless temperature as a function of dimensionless radius of the duct for different times ($t = T/4$, $t = T/2$, and $t = 3T/4$). It is found that when the time elapses the axial velocity and temperature decrease. This result can be explained by the fact that close to the wall atoms are slightly displaced from their equilibrium position and the molecular movements are difficult which causes the decrease in the dimensionless axial velocity and dimensionless temperature. This result is qualitatively similar to those obtained by other authors working in similar conditions [8], [9], [10], [11].

8. Conclusion

The simulation of the thermo mechanical phenomena during the filling phase is possible to give the state of the material during and in the late of this phase. The program is carried out in the Fortran environment. This modeling thermo mechanical phenomenon was based on the equations of continuum mechanics. The use of thermo-dependent models pseudo plastic viscosity need for a coupling of heat exchange equations and the equations of mechanics. The finite volume method was used for the numerical solution of these equations.

References


Received: November 15, 2014; Published: December 18, 2014