The Asymptotics of Eigenvalues of a Differential Operator in the Stochastic Models with ”White Noise”

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Abstract

The asymptotics of eigenvalues for a discrete self-adjoint operator perturbed by some random process is obtained. The ”white noise” is defined as the Nelson – Gliklikh derivative of the Wiener process. It is considered as a random perturbation. This investigation is based on the resolvent method, suggested by V.A. Sadovnichy and V.V. Dubrovsky.

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1 Introduction

A lot of theoretical problems and their applications in statistical physics, population genetics, hydromechanics, control theory, theory of diffusion processes lead to the study of stochastic partial differential equations [8, 10]. The spectral problems, consisting of calculation of spectral characteristics of an operator under accidental perturbation or restoration of perturbation according to the available field data are very important in these investigations. These perturbations are most commonly stochastic. Recently there appeared a new approach in the study of stochastic equations. It is actively developing now. The term ”white noise” here refers to the Nelson – Gliklikh derivative of Wiener process [1, 2, 4], which is more adequate to theory of Brownian motion of Einstein – Smolukhowski rather than the traditional white noise [7]. In this work we consider the spectral problem for a self-adjoint operator with nuclear resolvent, perturbed by some random process such as ”white noise” [7]. The main idea is in distribution of the resolvent method proposed by V.A. Sadovnichy and V.V. Dubrovsyky for deterministic equations [5] to a stochastic case.

2 The Spaces of ”Noises”

Let \( \Omega \equiv (\Omega, \mathcal{A}, P) \) be a complete probability space, \( \mathbb{R} \) be a set of real numbers endowed with Borel \( \sigma \)-algebra. The measurable mapping \( \xi : \Omega \to \mathbb{R} \) is called a random variable. The set of random variables forms a Hilbert space \( L_2 \) with the scalar product \( (\xi_1, \xi_2) = E\xi_1\xi_2 \). Let \( J \subset \mathbb{R} \) be some interval. Consider two mappings: \( f : J \to L_2 \), which maps each \( t \in J \) to a random variable \( \xi \in L_2 \), and \( g : L_2 \times \Omega \to \mathbb{R} \), which maps every pair \( (\xi, \omega) \) to the point \( \xi(\omega) \in \mathbb{R} \). The mapping \( \eta : J \times \Omega \to \mathbb{R} \) of the form \( \eta = \eta(t, \omega) = g(f(t), \omega) \) is called a (one-dimensional) random process. The random process \( \eta \) is called continuous if almost surely (a.s.) all its trajectories are continuous, that is, for almost every (a.e.) \( \omega \in \Omega \) the trajectories \( \eta(\cdot, \omega) \) are continuous. The set of continuous random processes form a Banach space \( CL_2 \).

The (one-dimensional) Wiener process \( \beta = \beta(t) \), modelling Brownian motion on the line in Einstein – Smolukhovsky theory, is one of the most important examples of the continuous Gaussian random processes. It has the following properties:

(W1) a.s. \( \beta(0) = 0 \), a.s. all its trajectories \( \beta(t) \) are continuous, and for all \( t \in \mathbb{R}_+ (= \{0\} \cup \mathbb{R}_+) \) the random variable \( \beta(t) \) is Gaussian;

(W2) the mathematical expectation \( E(\beta(t)) = 0 \) and autocorrelation function \( E((\beta(t) - \beta(s))^2) = |t - s| \) for all \( s, t \in \mathbb{R}_+ \);

(W3) the trajectories \( \beta(t) \) are nondifferentiable at any point \( t \in \mathbb{R}_+ \) and have unlimited variation at an arbitrarily small interval.
The random process $\beta$, satisfying properties (W1) – (W3), will be called Brownian motion. Now fix $\eta \in \text{CL}_2$ and $t \in J(= (\varepsilon, \tau) \subset \mathbb{R})$ and by $\mathcal{N}_t^\eta$ denote the $\sigma$-algebra generated by the random variable $\eta(t)$. For the sake of brevity, we introduce the notation $E^\eta_t = E(\cdot | \mathcal{N}_t^\eta)$.

**Definition 2.1** Let $\eta \in \text{CL}_2$, the random variable

$$D\eta(t, \cdot) = \lim_{\Delta t \to 0^+} E^\eta_t \left( \frac{\eta(t + \Delta t, \cdot) - \eta(t, \cdot)}{\Delta t} \right)$$

is called a forward $D\eta(t, \cdot)$ (a backward $D_* \eta(t, \cdot)$) mean derivative of the random process $\eta$ at the point $t \in (\varepsilon, \tau)$ if the limit exists in the sense of uniform metric on $\mathbb{R}$.

If the random process $\eta \in \text{CL}_2$ is forward (backward) mean differentiable on $(\varepsilon, \tau)$, then a symmetric mean derivative $\overset{\circ}{\eta} = \frac{1}{2} (D + D_*) \eta$ can be defined. The mean derivative of the random process was introduced by E. Nelson [4]. The theory of such derivatives was developed by Yu.E. Gliklikh [1]. The symmetric mean derivative $\overset{\circ}{\eta}$ of the random process $\eta$ will henceforth be called the Nelson – Gliklikh derivative. By $\overset{\circ}{\eta}^{(l)}$, $l \in \mathbb{N}$ denote the $l$-th Nelson – Gliklikh derivative of the random process $\eta$. Note that, if the trajectories of the random process $\eta$ are a.s. continuously differentiable in a "common sense" on $(\varepsilon, \tau)$, then the Nelson – Gliklikh derivative of $\eta$ coincides with the "regular" derivative.

**Theorem 2.2** [2] $\beta^{(l)}(t) = (-1)^{l+1} \cdot \prod_{i=1}^{l-1} \frac{(2i-1)}{(2i)} \cdot \frac{\beta(t)}{(l)!}$ for all $t \in \mathbb{R}_+$ and $l \in \mathbb{N}$.

Introduce the space $\text{C}^l\text{L}_2$, $l \in \mathbb{N}$ of random processes of $\text{CL}_2$, whose trajectories are a.s. Nelson – Gliklikh differentiable on $J$ to order $l$ inclusively. If $J \subset \mathbb{R}_+$, then due to theorem 2.2 there exists the derivative $\overset{\circ}{\beta} \in \text{C}^1\text{L}_2$, which will be called (one-dimensional) "white noise". In [7] it is suggested that the spaces $\text{C}^l\text{L}_2$ be called spaces of differentiable "noises", where

$$\|\eta\|_{\text{C}^l\text{L}_2} = \max_{0 \leq k \leq l} \{ \sup_{t \in J} \| \overset{\circ}{\eta}^{(k)}(t, \omega) \|_{L_2} \}.$$
3 The Spectrum of a Stochastic Perturbated Operator

Let $D \subset \mathbb{R}^n$ be a bounded domain. Take the separable Hilbert space $(\mathcal{U}, [\cdot, \cdot]_{\mathcal{U}})$ of functions defined on $D$. Consider a linear operator $A : \mathcal{U} \to \mathcal{U}$.

**Definition 3.1** A linear operator $A : \mathcal{U} \to \mathcal{U}$ is called *discrete* if there exists $\nu_0 \in \mathbb{C}$ such that the resolvent $(A - \nu_0 I)^{-1}$ is a completely continuous operator in $\mathcal{U}$.

**Definition 3.2** [3] An operator $A \in \mathcal{L}(\mathcal{U})$ is called a *nuclear operator* if there exist consequences $\{a_j\}_{j \in \mathbb{N}} \subset \mathcal{U}$, $\{b_j\}_{j \in \mathbb{N}} \subset \mathcal{U}$ such that

$$\sum_{j=1}^{\infty} ||a_j|| ||b_j|| < \infty$$

and

$$Af = \sum_{j=1}^{\infty} [f, b_j] a_j \quad \forall f \in \mathcal{U}$$

are fulfilled.

**Remark 3.3** If $A$ is a self-adjoint completely continuous positively defined ($\langle Au, u \rangle \geq 0$) operator, then definition (3.2) is equivalent to the condition that the series of eigenvalues $\nu_k$ converges ($\sum_{k=1}^{\infty} \nu_k < \infty$). The expression $\sum_{k=1}^{\infty} \nu_k$ is called a (spectral) trace of operator $A$.

**Lemma 3.4** [3] Let $A \in \mathcal{L}(\mathcal{U})$ be a nuclear operator and the sequence $\{\varphi_k\}_{k \in \mathbb{N}}$ form an orthonormal basis in $\mathcal{U}$. Then the trace of operator $A$ exists, does not depend on the choice of basis $\{\varphi_k\}$ and can be represented by

$$\text{Tr} A = \sum_{k=1}^{\infty} [A \varphi_k, \varphi_k].$$

**Definition 3.5** Let $A : \mathcal{U} \to \mathcal{U}$ be a discrete self-adjoint positively defined operator, then the number $\|A\|_2 = \left( \sum_{j=1}^{\infty} \nu_k^2 \right)^{1/2}$ is called an *absolute norm* of operator $A$.

Let $T : \mathcal{U} \to \mathcal{U}$ be a discrete self-adjoint operator and the eigenvalues $\lambda_k$ of operator $T$ be numbered in increasing order. Let $B \in \mathcal{L}(\mathcal{U})$ (linear and continuous). By $R_0(\lambda) = (T - \lambda I)^{-1}$ denote the resolvent of operator $T$, by $R(\lambda) = (T + B - \lambda I)^{-1}$ denote the resolvent of the perturbed operator $T + B$. 
Consider contours $\gamma_{r_k} = \{ \lambda \in \mathbb{C} : |\lambda_k - \lambda| = r_k \}$; domains $G_{r_k} = \{ \lambda \in \mathbb{C} : |\lambda_k - \lambda| \geq r_0 \}$; $G = \bigcap_{k=1}^{\infty} G_{r_k}$, where $r_1 = \frac{1}{2}(\lambda_2 - \lambda_1)$; $r_k = \frac{1}{2}\min\{\lambda_{k+1} - \lambda_k; \lambda_k - \lambda_{k-1}\}$, $k \in \mathbb{N}, k > 1$; $r_0 = \inf_k r_k$.

**Lemma 3.6** [9] Let $T : \mathcal{U} \to \mathcal{U}$ be a discrete self-adjoint operator, the perturbing operator $B \in \mathcal{L}(\mathcal{U})$ and $\|B\|_{\mathcal{L}(\mathcal{U})} < r_0/2$ then $T + B$ is a discrete operator. Moreover

(i) if the resolvent $R_0(\lambda)$ of operator $T$ is a nuclear operator then the resolvent of perturbed operator $R(\lambda)$ is a nuclear operator as well;

(ii) if $\lambda_k \in \mathbb{C} \setminus G_{r_k}$ then $\mu_k^s \in \mathbb{C} \setminus G_{r_k}$. Here $\mu_k^s$, $s = \max(m_k)$ are the eigenvalues of a perturbed operator $T + B$ corresponding to $\lambda_k$ and $m_k$ is the multiplicity of the eigenvalue $\lambda_k$.

**Lemma 3.7** [9] Let $T : \mathcal{U} \to \mathcal{U}$ be a discrete self-adjoint operator and the operator $B \in \mathcal{L}(\mathcal{U})$ satisfy the conditions of lemma 3.5, then

$$\frac{1}{2\pi i} \int_{\gamma_{r_k}} \lambda \text{Tr} \left( R(\lambda)(BR_0(\lambda))^2 \right) d\lambda \leq r_0 r_k \|B\| \max_{\lambda \in \gamma_{r_k}} \|R_0(\lambda)\|_2^2, \quad k \in \mathbb{N}. $$

Take the discrete self-adjoint differential operator $T : \text{dom}T \subset \mathcal{U} \to \mathcal{U}$, with domain $\text{dom}T = \{u \in \mathcal{U} : Tu \in \mathcal{U} \}$ being dense in $\mathcal{U}$. Since the operator $T$ is self-adjoint it’s eigenvalues are real. Suppose that the eigenvalues of $T$ are numbered in the increasing order and have the asymptotics

$$\lambda_k \sim C k^{\frac{2\beta}{n}}, \quad (1)$$

where $\beta \geq 1$, $C = \text{const}$.

**Remark 3.8** Consider the Dirichlet boundary value problem for the operator $T = -\Delta$ in $L_2(\Pi)$, where $\Pi = \{x = (x_1, x_2, \ldots, x_n) : 0 < x_j < a_j, j = 1, \ldots, n\}$. Obviously, the eigenvalues of $T$ in this case satisfy condition (1) [6].

Let $\eta = \eta(t, \omega)$ be a one-dimensional random process with $\|\eta(t, \omega)\|_{C^k L_2} < \frac{r_0}{2}$. Denote by $\{\mu_k^s\}$ the eigenvalues of perturbed operator $T + I\eta(t, w)$, numbered in nondecreasing order of their real parts. By lemma 3.5 the operator $T + I\eta(t, w)$ is discrete, the number of eigenvalues of $T$ with their multiplicities and the number of eigenvalues of $T + B$ lying in circle $\gamma_{r_k}$ are equal. Fix $t$ and $\omega$.

**Theorem 3.9** If $T$ is a self-adjoint discrete differential operator with nuclear resolvent satisfying condition (1), $\eta \in C^k L_2$ is a one-dimensional random process with $\|\eta\|_{C^k L_2} < \frac{r_0}{2}$. Then for all $k \in \mathbb{N}$

$$\sum_{s=1}^{\nu_k} \mu_k^s(t, \omega) = m_k \lambda_k + m_k \eta(t, \omega) + O(k^{\frac{2\beta}{n} - 1}),$$

where $\{\lambda_k\}_{k=1}^{\infty} = \sigma(T)$, $\{\mu_k^s\}_{k=1}^{\infty} = \sigma(T + I\eta(t, w))$. 

Proof. Consider the operator identity:

$$R(\lambda) = R_0(\lambda) - R_0(\lambda)I\eta(t, w)R_0(\lambda) + R(\lambda)(I\eta(t, w)R_0(\lambda))^2, \quad \lambda \in G.$$  \hspace{1cm} (2)

Take the trace of both parts of this identity, multiply by $\frac{\lambda}{2\pi i}$ and take the integral over $\gamma_{rk} \subset \mathbb{C}$. Therefore we get

$$\frac{1}{2\pi i} \int_{\gamma_{rk}} \lambda \text{Tr} R(\lambda) d\lambda = \frac{1}{2\pi i} \int_{\gamma_{rk}} \lambda \text{Tr} R_0(\lambda) d\lambda - \frac{1}{2\pi i} \int_{\gamma_{rk}} \lambda \text{Tr} (R_0(\lambda) I\eta(t, w) R_0(\lambda)) d\lambda +$$

$$+ \frac{1}{2\pi i} \int_{\gamma_{rk}} \lambda \left( \text{Tr} (R(\lambda)(I\eta(t, w)R_0(\lambda))^2) \right) d\lambda. \hspace{1cm} (3)$$

Due to lemma 3.4 and orthonormality of the eigenfunction of $T$ and $T + I\eta(t, w)$ we obtain

$$\text{Tr} R(\lambda) = \sum_{j=1}^{\infty} [R(\lambda) \varphi_j, \varphi_j] = \sum_{j=1}^{\infty} \frac{1}{\mu_j - \lambda}; \quad \text{Tr} R_0(\lambda) = \sum_{j=1}^{\infty} \frac{1}{\lambda_j - \lambda}$$

for all $\lambda \in G$.

Calculate each of the integrals in (3):

$$\frac{1}{2\pi i} \int_{\gamma_{rk}} \lambda \text{Tr} R(\lambda) d\lambda = \frac{1}{2\pi i} \sum_{j=1}^{\infty} \int_{\gamma_{rk}} \frac{\lambda}{\mu_j - \lambda} d\lambda = \sum_{s=1}^{m_k} \mu_k^s, \hspace{1cm} (4)$$

by analogy

$$\frac{1}{2\pi i} \int_{\gamma_{rk}} \lambda \text{Tr} R_0(\lambda) d\lambda = m_k \lambda_k. \hspace{1cm} (5)$$

Using the equality

$$[R_0(\lambda) I\eta(t, w) R_0(\lambda) \varphi_j, \varphi_j] = \frac{[I\eta(t, w) \varphi_j, \varphi_j]}{(\lambda_j - \lambda)^2}$$

for all $\lambda \in G$ we have

$$\frac{1}{2\pi i} \int_{\gamma_{rk}} \lambda \text{Tr} (R_0(\lambda) I\eta(t, w) R_0(\lambda)) d\lambda = m_k \eta(t, \omega). \hspace{1cm} (6)$$

Obviously the absolute norm

$$\|R_0(\lambda)\|^2 \leq \frac{\text{const}}{r_k^2}.$$
Using this fact and lemma 3.6 we have for $k \in \mathbb{N}$

$$
\frac{1}{2\pi i} \int_{\gamma_k} \lambda \text{Tr} \left( R(\lambda) (I \eta(t, w) R_0(\lambda))^2 \right) d\lambda = O(k^{2\beta \pi^{-1}}). \quad (7)
$$

Substitute (4) – (7) into (3) and get the required:

$$
\sum_{s=1}^{\nu_k} \mu_k^s(t, \omega) = m_k \lambda_k + m_k \eta(t, \omega) + O(k^{2\beta \pi^{-1}}).
$$

**Corollary 3.10** If $T$ is a self-adjoint discrete differential operator with nuclear resolvent satisfying condition (1), $\beta$ is a one-dimensional "white noise" with $\| \beta \|_{C^L_2} < \frac{r_0}{2}$. Then for all $k \in \mathbb{N}$

$$
\sum_{s=1}^{\nu_k} \mu_k^s(t, \omega) = m_k \lambda_k + m_k \beta(t, \omega) + O(k^{2\beta \pi^{-1}}),
$$

where $\{\lambda_k\}_{k=1}^{\infty} = \sigma(T)$, $\{\mu_k^s\}_{k=1}^{\infty} = \sigma(T + I \beta)$.

**References**


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