A Realization Problem and a Characterization of Locating Total Dominating Sets in the Cartesian Product of Graphs

Benjamin N. Omamalin

Bohol Island State University-Balilihan Campus
College of Technology and Allied Sciences
Magsija, Balilihan, Bohol, Philippines

Sergio R. Canoy, Jr. and Helen M. Rara

Department of Mathematics and Statistics
Mindanao State University-Iligan Institute of Technology
Tibanga Highway, Iligan City, Philippines

Abstract

Let $G = (V(G), E(G))$ be a connected graph. A subset $S$ of $V(G)$ is a total dominating set of $G$ if every vertex of $G$ is adjacent to some vertex in $S$. The set $N_G(v)$ is the set of all vertices of $G$ adjacent to $v$. A subset $S$ of $V(G)$ is a locating set of $G$ if $N_G(u) \cap S \neq N_G(v) \cap S$ for every two distinct vertices $u$ and $v$ in $V(G) \setminus S$. A locating subset $S$ of $V(G)$ which is also a total dominating set is called a locating total dominating set of $G$. The minimum cardinality of a locating total dominating set of $G$ is called the locating total domination number of $G$. In this paper, we characterize the locating total dominating sets in the Cartesian product of graphs. We also determine the relationships between the locating total domination number and some other domination parameters such as...
domination number, total domination number, and locating domination number.

Mathematics Subject Classification: 05C69

Keywords: Cartesian product, locating total domination

1 Introduction

Let $G = (V(G), E(G))$ be a simple connected graph. The neighborhood of $v \in V(G)$ is the set $N_G(v) = \{x \in V(G) : xv \in E(G)\}$. A set $S \subseteq V(G)$ is a dominating set of $G$ if for every $v \in V(G) \setminus S$, there exists $u \in S$ such that $uv \in E(G)$, that is, $N_G[S] = V(G)$. The domination number of $G$, denoted by $\gamma(G)$, is the smallest cardinality of a dominating set of $G$. Any dominating set in $G$ of cardinality $\gamma(G)$ is referred to as a $\gamma$-set of $G$. A subset $S$ of $V(G)$ is a total dominating set of $G$ if for every $v \in V(G)$, there exists $x \in S$ such that $xv \in E(G)$. It is a locating set in $G$ if $N_G(u) \cap S \neq N_G(v) \cap S$ for every two distinct vertices $u$ and $v$ of $V(G) \setminus S$. A locating subset $S$ of $V(G)$ which is also a total dominating set is called a locating total dominating set (LTDS) in a connected graph $G$. The minimum cardinality of a locating set in $G$, denoted by $ln(G)$, is called the locating number of $G$. The minimum cardinality of a locating total dominating set in $G$, denoted by $\gamma_{LT}(G)$ is called the locating total domination number of $G$. Any LTDS of cardinality $\gamma_{LT}(G)$ is referred to as a $\gamma_{LT}$-set of $G$.

Total domination in graphs and other types of domination are found in the book by Haynes et.al. [4]. The concepts of locating set, locating total dominating set and the associated parameters are studied in [1], [2], [3] and [4].

2 Realization Problems

We begin this section by considering a remark.

Remark 2.1 For any connected graph $G$ of order $n$,

$$2 \leq \gamma(G) \leq \gamma_t(G) \leq \gamma_{LT}(G) \leq n.$$ 

Remark 2.2 For any connected graph $G$ of order $n$,

$$2 \leq \gamma(G) \leq \gamma_L(G) \leq \gamma_{LT}(G) \leq n.$$ 

Theorem 2.3 Let $a$, $b$, and $c$ be positive integers such that $2 \leq a \leq b \leq c$ and $2b = 3a$ if $a$ is even and $2b = 3a + 1$ if $a$ is odd for $a < b$. Then, there exists a connected graph $G$ such that $\gamma(G) = a$, $\gamma_t(G) = b$ and $\gamma_{LT}(G) = c$. 

Proof: Consider the following cases:

Case 1. $a = b = c \geq 2$

Let the path $P_a = \{x_1, x_2, ..., x_a\}$ and let $G_1$ be the graph obtained from $P_a$ by adding the vertices $z_i$ and edges $x_i \sim z_i$ for $i = 1, 2, ..., a$ (see Figure 1). Then, the set $\{x_i : i = 1, 2, ..., a\}$ is a $\gamma$-set, $\gamma_t$-set, and $\gamma_{LT}$-set of $G_1$. Thus, $\gamma(G_1) = a$, $\gamma_t(G_1) = b$, and $\gamma_{LT}(G_1) = c$.

**Figure 1: Graph $G_1$**

Case 2. $a = b < c$

Let $G_2$ be the graph obtained from $G_1$ by adding the vertices $y_j$ and edges $x_i \sim y_j$ for $j = 1, 2, ..., c - a$ (see Figure 2). Then, the set $A = \{x_i : i = 1, 2, ..., a\}$ is a $\gamma$-set and $\gamma_t$-set of $G_2$, and $B = A \cup \{y_j : j = 1, 2, ..., c - a\}$ is a $\gamma_{LT}$-set of $G_2$. Thus, $\gamma(G_2) = |A| = a$, $\gamma_t(G_2) = |A| = b$, and $\gamma_{LT}(G_2) = |B| = a + c - a = c$.

**Figure 2: Graph $G_2$**

Case 3. $a < b = c$

Subcase 1. Suppose $a$ is even and $2b = 3a$. Let $G_3$ be the graph obtained from $G_1$ by adding the paths $[z_{2i+1}, q_{i+1}, z_{2i+2}]$ for $i = 0, 1, 2, ..., \frac{a - 2}{2}$ (see Figure 3). The set $A = \{z_i : i = 1, 2, ..., a\}$ is a $\gamma$-set of $G_3$, and the set $B = A \cup \{q_r : r = 1, 2, ..., b - a\}$ is a $\gamma_t$-set and $\gamma_{LT}$-set of $G_3$. Thus, $\gamma(G_3) = |A| = a$, and $a < \gamma_t(G_3) = |B| = a + b - a = b$, and $\gamma_{LT}(G_3) = |B| = b = c$.

**Figure 3: Graph $G_3$**

Subcase 2. Suppose $a$ is odd and $2b = 3a + 1$. Let $G_4$ be the graph obtained
from $G_1$ by adding the paths $[z_{2i+1}, q_i, z_{2i+2}]$ for $i = 0, 1, 2, ..., \frac{a-3}{2}$ (see Figure 4). The set $A = \{z_i : i = 1, 2, ..., a\}$ is a $\gamma$-set of $G_4$, and the set $B' = A \cup \{q_r : r = 1, 2, ..., b - a - 1\} \cup \{x_a\}$ is a $\gamma_t$-set and $\gamma_{LT}$-set of $G_4$. Thus, $\gamma(G_4) = |A| = a$, and $\gamma_t(G_4) = |B'| = a + b - a = b = \gamma_{LT}(G_4)$.

Case 4. $a < b < c$

Subcase 1. Suppose $a$ is even and $2b = 3a$. Let $G_5$ be the graph obtained from $G_3$ by adding the vertices $y_i$ and the edges $z_iy_i$ for $i = 1, 2, ..., c - b$ (see Figure 5). The set $A = \{z_i : i = 1, 2, ..., a\}$ is a $\gamma$-set of $G_5$, the set $B = A \cup \{q_r : r = 1, 2, ..., b - a\}$ is a $\gamma_t$-set of $G_5$, and the set $C = B \cup \{y_j : j = 1, 2, ..., c - b\}$ is a $\gamma_{LT}$-set of $G_5$. Thus, $\gamma(G_5) = |A| = a$, $\gamma_t(G_5) = |B| = a + b - a = b$, and $\gamma_{LT}(G_5) = |B| + c - b = b + c - b = c$.

Subcase 2. Suppose $a$ is odd and $2b = 3a + 1$. Let $G_6$ be the graph obtained from $G_4$ by adding the vertices $y_i$ and the edges $z_iy_i$ for $i = 1, 2, ..., c - b$ (see Figure 6). The set $A = \{z_i : i = 1, 2, ..., a\}$ is a $\gamma$-set of $G_6$, the set $B = A \cup \{q_r : r = 1, 2, ..., b - a - 1\} \cup \{x_a\}$ is a $\gamma_t$-set of $G_6$, and the set $C = B \cup \{y_j : j = 1, 2, ..., c - b\}$ is a $\gamma_{LT}$-set of $G_6$. Thus, $\gamma(G_6) = |A| = a$, $\gamma_t(G_6) = |B| = a + b - a = b$, and $\gamma_{LT}(G_6) = |B| + c - b = b + c - b = c$.
Corollary 2.4 Given a positive integer $n$, there exists a connected graph $G$ such that $\gamma_{LT}(G) - \gamma_t(G) = n$, that is, the difference $\gamma_{LT} - \gamma_t$ can be made arbitrarily large.

**Proof:** Let $n$ be a positive integer. Set $b = 2$ and $c = n + 2$. Then, by Theorem 2.3, there exists a connected graph $G$ such that $\gamma_{LT}(G) = c$ and $\gamma_t(G) = b$. Thus, $\gamma_{LT}(G) - \gamma_t(G) = c - b = n + 2 - 2 = n$. \hfill $\Box$

Theorem 2.5 Let $a$, $b$, and $c$ be positive integers such that $2 \leq a \leq b \leq c$ and $2c = a + 2b$ if $a \leq b < c$. Then, there exists a connected graph $H$ such that $\gamma(H) = a$, $\gamma_L(H) = b$ and $\gamma_{LT}(H) = c$.

**Proof:** Consider the following cases:

**Case 1.** $a = b = c \geq 2$

Let $H_1$ be the graph obtained from $G_1$ in Figure 1 by adding the edges $z_iz_{i+1}$ for $i = 1, 2, ..., a - 1$ (see Figure 7). Then, the set $\{x_i : i = 1, 2, ..., a\}$ is a $\gamma$-set, $\gamma_L$-set, and $\gamma_{LT}$-set of $H_1$. Thus, $\gamma(H_1) = \gamma_L(H_1) = \gamma_{LT}(H_1) = a = b = c$.

![Figure 7: Graph $H_1$](image)

**Case 2.** $a < b = c$

Let $H_2$ be the graph obtained from $H_1$ by adding the edges $x_iy_i$ for $i = 1, 2, ..., c - a$ (see Figure 8). Then, the set $A = \{x_i : i = 1, 2, ..., a\}$ is a $\gamma$-set of $H_2$, and the set $B = A \cup \{y_j : j = 1, 2, ..., c - a\}$ is a $\gamma_L$-set and $\gamma_{LT}$-set of $H_2$. Thus, $\gamma(H_2) = |A| = a$, $\gamma_L(H_2) = |B| = a + c - a = c$, and $\gamma_{LT}(H_2) = |B| = c$.

![Figure 8: Graph $H_2$](image)

**Case 3.** $a = b < c$

Let $C_5^{(i)} = [x_1^{(i)}, x_2^{(i)}, x_3^{(i)}, x_4^{(i)}, x_5^{(i)}]$ be an $i$th cycle of order 5 and let $H_3$ be the graph obtained from $C_5^{(i)}$ by connecting vertices $x_2^{(i)}$ and $x_1^{(i+1)}$ by an edge for $i = 1, 2, ..., c - b - 1$ (see Figure 9). Let $2c = a + 2b$. 

![Figure 9: Graph $H_3$](image)
Then, \( A = \{ x_3^{(i)} | i = 1, \ldots, c - b \} \cup \{ x_5^{(i)} | i = 1, \ldots, c - b \} \) is a \( \gamma \)-set and \( \gamma_L \)-

Then, \( A = \{ x_3^{(i)} | i = 1, \ldots, c - b \} \cup \{ x_5^{(i)} | i = 1, \ldots, c - b \} \) is a \( \gamma \)-set and \( \gamma_L \)-

Then, \( A = \{ x_3^{(i)} | i = 1, \ldots, c - b \} \cup \{ x_5^{(i)} | i = 1, \ldots, c - b \} \) is a \( \gamma \)-set and \( \gamma_L \)-

Then, \( A = \{ x_3^{(i)} | i = 1, \ldots, c - b \} \cup \{ x_5^{(i)} | i = 1, \ldots, c - b \} \) is a \( \gamma \)-set and \( \gamma_L \)-

Then, \( A = \{ x_3^{(i)} | i = 1, \ldots, c - b \} \cup \{ x_5^{(i)} | i = 1, \ldots, c - b \} \) is a \( \gamma \)-set and \( \gamma_L \)-

Then, \( A = \{ x_3^{(i)} | i = 1, \ldots, c - b \} \cup \{ x_5^{(i)} | i = 1, \ldots, c - b \} \) is a \( \gamma \)-set and \( \gamma_L \)-

Then, \( A = \{ x_3^{(i)} | i = 1, \ldots, c - b \} \cup \{ x_5^{(i)} | i = 1, \ldots, c - b \} \) is a \( \gamma \)-set and \( \gamma_L \)-
3 Locating Total Domination in the Cartesian Product of Graphs

The Cartesian product of two graphs $G$ and $H$ is the graph $G \square H$ with vertex-set $V(G \square H) = V(G) \times V(H)$ and edge-set $E(G \square H)$ satisfying the following conditions: $(x, a)(y, b) \in E(G \square H)$ if and only if either $xy \in E(G)$ and $a = b$ or $x = y$ and $ab \in E(H)$.

**Theorem 3.1** Let $G$ and $H$ be non-trivial connected graphs. Then $C = \bigcup (\{x\} \times T_x)$ is a locating total dominating set in $G \square H$ if and only if the following properties hold:

(i) For each $x \in V(G)$, $T_x \subseteq N_H(T_x) \cup (\bigcup_{z \in N_G(x)} T_z)$

and $V(H) \setminus N_H[T_x] \subseteq \bigcup_{z \in N_G(x)} T_z$.

(ii) For each $x \in V(G)$, $T_x$ is a locating set in $H$ or there exists $u \in N_G(x)$ such that $p \in T_u$ or $q \in T_u$ whenever $p, q \in V(H) \setminus T_x$, $p \neq q$ and $N_H(p) \cap T_x = N_H(q) \cap T_x$.

(iii) For each pair of distinct vertices $x$ and $y$ of $G$,

(a) $p \in N_H(T_x) \cup (\bigcup_{z \in N_G(x) \setminus N_G(y)} T_z)$

or $p \in N_H(T_y) \cup (\bigcup_{z \in N_G(y) \setminus N_G(x)} T_z)$

for all $p \notin T_x \cup T_y$ and $N_G(y) \neq N_G(x)$.

(b) $T_x \cup T_y$ is a dominating set in $H$ whenever $N_G(x) = N_G(y)$.

(iv) For each pair of distinct adjacent vertices $x$ and $y$ of $G$ and for each pair of distinct adjacent vertices $p$ and $q$ of $H$ such that $p \notin T_x$ and $q \notin T_y$,

$p \in (\bigcup_{z \in N_G(x) \setminus \{y\}} T_z) \cup N_H(T_x \setminus \{q\})$ or

$q \in (\bigcup_{w \in N_G(y) \setminus \{x\}} T_w) \cup N_H(T_y \setminus \{p\})$.

**Proof:** Suppose $C$ is a locating total dominating set in $G \square H$. Let $x \in V(G)$. If $p \in T_x$, then $(x, p) \in C$. Since $C$ is total dominating, there exists $(z, q) \in C$ such that $(x, p)(z, q) \in E(G \square H)$. Thus, either $xz \in E(G)$ and $p = q$ or $x = z$ and $pq \in E(H)$. Hence, $z \in N_G(x)$ and $(z, p) \in C$ or $p \in N_H(T_x)$. Therefore $T_x \subseteq N_H(T_x) \cup (\bigcup_{z \in N_G(x)} T_z)$. Suppose $p \in V(H) \setminus N_H[T_x]$. Since $C$ is total dominating, there exists $(z, c) \in C$ such that $(x, p)(z, c) \in E(G \square H)$. Thus,
Therefore, $p \in E(H)$. This is a contradiction since $p \notin N_H[T_x]$. Hence, $p = c$ and $xz \in E(G)$. Thus, $p \in T_z$ for some $z \in N_G(x)$. Therefore, (i) holds.

Next, let $x \in V(G)$. If $T_x$ is locating, then we are done. Suppose that $T_x$ is not locating. Let $p, q \in V(H) \setminus T_x$ such that $p \neq q$ and $N_H(p) \cap T_x = N_H(q) \cap T_x$. Let $B = \{x\} \times (N_H(p) \setminus T_x)$. Then, $N_{G \square H}((x, p)) \cap C = B \cup \{(u, p) \in C : u \in N_G(x)\}$ and $N_{G \square H}((x, q)) \cap C = B \cup \{(u, q) \in C : u \in N_G(x)\}$. Since $C$ is locating, there exists $u \in N_G(x)$ such that $p \in T_u$ or $q \in T_u$, that is, (ii) holds.

Now, let $x, y \in V(G)$ such that $x \neq y$. Consider the following cases.

**Case 1.** Suppose $N_G(x) \neq N_G(y)$. Let $p \notin T_x \cup T_y$. Suppose that $p \notin N_H(T_y)$ and $p \notin \bigcup_{z \in N_G(y) \setminus N_G(x)} N_G(x) \cap N_G(y) \subseteq N_{G \square H}((x, p)) \cap C$. Since $C$ is locating, there exist $(v, t) \in N_{G \square H}((x, p)) \cap C \setminus N_{G \square H}((y, p)) \cap C$. This means that $(x, p)(v, t) \in E(G \square H)$, that is, $xv \in E(G)$ and $p = t$ or $x = v$ and $pt \in E(H)$. This implies that $p \in T_v$ for some $v \in N_G(x) \setminus N_G[y]$, or $p \in N_H(T_x)$.

**Case 2.** Suppose $N_G(x) = N_G(y)$. Let $p \in V(H) \setminus (T_x \cup T_y)$. Then $(x, p), (y, q) \notin C$ and $(x, p) \neq (y, q)$. Since $C$ is locating, there exists $q \in T_x \cap N_H(p)$ or $c \in T_y \cap N_H(p)$. Hence, $T_x \cup T_y$ is a dominating set of $H$. Therefore (iii) holds.

Let $x$ and $y$ be distinct adjacent vertices in $G$ and let $p$ and $q$ be distinct adjacent vertices in $H$ such that $p \notin T_x$ and $q \notin T_y$. Since $(x, p), (y, q) \notin C$ and $C$ is locating, (iv) holds.

For the converse, assume that (i), (ii), (iii) and (iv) hold. Suppose $(x, p) \in V(G \square H)$. If $p \in V(H) \setminus N_H(T_x)$, then by (i), there exists $z \in N_G(x)$ such that $p \in T_z$. Thus, $(z, p) \in N_{G \square H}((x, p)) \cap C$. Suppose $p \in N_H(T_x)$. Then $p \in T_x$ or $p \in N_H(T_x)$. If $p \in N_H(T_x)$, then there exists $q \in T_x$ such that $(x, q) \in N_{G \square H}((x, p)) \cap C$. If $p \notin N_H(T_x)$, then $p \in T_x$. By (i), there exists $z \in N_G(x)$ such that $p \in T_z$. Hence, $(z, p) \in N_{G \square H}((x, p)) \cap C$. Therefore $C$ is a total dominating set.

Now, let $(x, p), (y, q) \in V(G \square H) \setminus C$ and $(x, p) \neq (y, q)$. Consider the following cases:

**Case 1.** Suppose $x = y$. Then $p \neq q$. If $T_x$ is locating or $N_H(p) \cap T_x \neq N_H(q) \cap T_x$, then we are done. Suppose $N_H(p) \cap T_x = N_H(q) \cap T_x$. By (ii), we may assume that there exists $v \in N_G(x)$ such that $p \in T_v$. Hence, $(v, p) \in N_{G \square H}((x, p)) \cap C \setminus N_{G \square H}((x, q)) \cap C$.

**Case 2.** Suppose $x \neq y$. Consider the following subcases:

**Subcase 1.** Suppose $p = q$.

First assume that $N_G(x) \neq N_G(y)$. By (iii)(a), assume that $p \in N_H(T_x) \cup (\bigcup_{z \in N_G(x) \setminus N_G[y]} T_z)$. Then $pc \in E(H)$ for some $c \in T_z$ or $p \in T_x$ for some $z \in N_G(x) \setminus N_G[y]$. Hence, $(x, c) \in N_{G \square H}((x, p)) \cap C \setminus N_{G \square H}((y, q)) \cap C$.
or \((z, p) \in N_{G \square H}((x, p)) \cap C \setminus N_{G \square H}((y, q)) \cap C\). Similarly if \(p \in N_H(T_y) \cup ( \bigcup_{z \in N_G(y) \setminus N_G[x]} T_z)\), then \(N_{G \square H}((x, p)) \cap C \neq N_{G \square H}((y, q)) \cap C\).

Next, suppose that \(N_G(x) = N_G(y)\). By \((iii)(b)\), there exists \(c \in T_x \cup T_y\) such that \(pc \in E(H)\). Hence, \((x, c) \in N_{G \square H}((x, p)) \cap C \setminus N_{G \square H}((y, q)) \cap C\) or \((y, c) \in N_{G \square H}((y, q)) \cap C \setminus N_{G \square H}((x, p)) \cap C\).

**Subcase 2.** Suppose \(p \neq q\).

If \(p\) and \(q\) are not adjacent, then by \((i)\), there exists \(z \in N_G(x)\) such that \(p \in T_z\). Hence, \((z, p) \in N_{G \square H}((x, p)) \cap C \setminus N_{G \square H}((y, q)) \cap C\). Suppose \(p\) and \(q\) are adjacent. If \(x\) and \(y\) are not adjacent, then by \((i)\), there exists \(w \in N_G(x)\) such that \(p \in T_w\). Thus, \((w, p) \in N_{G \square H}((x, p)) \cap C \setminus N_{G \square H}((y, q)) \cap C\). Suppose \(xy \in E(G)\). Then by \((iv)\), assume first that \(p \in ( \bigcup_{z \in N_G(x) \setminus \{y\}} T_z) \cup N_H(T_x \setminus \{q\})\).

Then, \((z, p) \in N_{G \square H}((x, p)) \cap C \setminus N_{G \square H}((y, q)) \cap C\) or \((x, c) \in N_{G \square H}((x, p)) \cap C \setminus N_{G \square H}((y, q)) \cap C\) for some \(c \in (T_x \setminus \{q\}) \cap N_H(p)\).

Therefore in any case, \(N_{G \square H}((x, p)) \cap C \neq N_{G \square H}((y, q)) \cap C\).

Accordingly, \(C\) is a locating set of \(G \square H\).

Thus, \(C\) is a locating total dominating set of \(G \square H\).

**Corollary 3.2** Let \(G\) and \(H\) be non-trivial connected graphs of orders \(m\) and \(n\), respectively. Then, \(\gamma_{LT}(G \square H) \leq \min\{n \gamma_L(G), m \gamma_L(H)\}\).

**Proof:** Assume that \(m \gamma_L(H) \leq n \gamma_L(G)\). Let \(D\) be a minimum locating dominating set of \(H\). For each \(x \in V(G)\), set \(T_x = D\). By Theorem 3.1, \(C = \bigcup_{x \in V(G)} \{\{x\} \times T_x\}\) is a locating total dominating set of \(G \square H\). Hence,

\[
\gamma_{LT}(G \square H) \leq |C| = |\bigcup_{x \in V(G)} \{\{x\} \times T_x\}| = |V(G)||T_x| = m(\gamma_L(H)).
\]

Therefore, \(\gamma_{LT}(G \square H) \leq \min\{n \gamma_L(G), m \gamma_L(H)\}\).

**References**


Received: October 15, 2014; Published: December 8, 2014