Numerical Evaluation of Integrals of Analytic Functions of More than One Complex Variable

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Abstract

Seven point and eight point quadrature rules have been generated for the numerical evaluation of two dimensional integral of an analytic function of two complex variables. The truncation error associated with the rules have been analysed and it has been established that both the rules have degrees of precision five. Some examples have been considered for the purpose of numerical verification of the rules.

Keywords: Analytic function, Quadrature rules, Truncation error, Degree of precision

Mathematics Subject Classification: 65D30

1. Introduction

Multiple integrals occur quite often in mathematical physics and engineering subjects. Numerical approximation of multiple integrals is more challenging than that of one dimensional integrals. Many quadrature rules have been generated by different researchers for the numerical approximation of real multiple integrals (cf. Davis and Rabinowitz [4]). However, quite limited number of research papers are available for the numerical evaluation of complex multiple integrals of analytic
functions. Some of these papers are due to Acharya and Mohapatra [1], Acharya and Das [2], Milovanovic et al [5], Acharya et al [3], etc. The objective of the present paper is to generate some quadrature rules for the numerical evaluation of the following complex double integral of an analytic function \( f(z_1, z_2) \) over the Cartesian product \( L_1 \times L_2 \) of directed line segments \( L_j, j=1,2 \).

\[
I(f) = \int_{L_1} \int_{L_2} f(z_1, z_2) dz_1 dz_2
\]  

The function \( f(z_1, z_2) \) is assumed to be analytic in the product domain \( D_1 \times D_2 \subseteq \mathbb{C}^2 \). It is further assumed that the directed line segment \( L_j \) from \( z_{j0}-h_j \) to \( z_{j0}+h_j \), \( j=1,2 \) is contained in the domain \( D_j \). The truncation error associated with the quadrature rules would also be analysed to obtain some properties of the quadrature rules.

2. Formulation of the rules

We simultaneously derived the rules \( R_7(f) \) and \( R_8(f) \) which respectively involved the following sets of seven and eight nodes:

\[
S_1 = \{(z_{10}, z_{20}), (z_{10} \pm sh_1, z_{20} \pm th_2), (z_{10}, z_{20} \pm rh_2)\}, \quad S_2 = \{(z_{10} \pm ph_1, z_{20} \pm ph_2), (z_{10}, z_{20} \pm qh_2), (z_{10} \pm qh_1, z_{20})\}. \tag{2}
\]

Let the quadrature rules be prescribed in the following forms

\[
R_7(f) = A_1 \sum f(z_{10} \pm sh_1, z_{20} \pm th_2) + B_1 \sum f(z_{10}, z_{20} \pm rh_2) + C_1 f(z_{10}, z_{20}), \tag{3}
\]

\[
R_8(f) = A_2 \sum f(z_{10} \pm ph_1, z_{20} \pm ph_2) + B_2 \sum f(z_{10}, z_{20} + qh_2) + f(z_{10} \pm qh_1, z_{20}). \tag{4}
\]

where \( A_1, B_1, C_1, A_2, B_2, s,t,r,p,q \) are unknown quantities to be determined. Both the above rules are exact if \( f(z_1, z_2) = (z_1 - z_{10})^\mu (z_2 - z_{20})^v \) whenever at least one of \( \mu \) or \( v \) is odd. So considering \((\mu, v) = (0,0), (2,0), (0,2), (4,0), (0,4), (2,2)\) for the first rule and \((\mu, v) = (0,0), (2,0), (4,0), (2,2)\) for the second rule we have the following sets of equations. It is noteworthy that the cases \((2, 0)\) and \((0, 2)\) are indifferent and so also the cases \((4, 0)\) and \((0, 4)\) are indifferent so far as the second rule is concerned.
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\[
\begin{align*}
4A_1 + 2B_1 + C_1 &= 4h_1h_2 \\
A_1s^2 &= \frac{h_1h_2}{3} \\
2A_1t^2 + B_1r^2 &= \frac{2h_1h_2}{3} \\
A_1s^4 &= \frac{h_1h_2}{5} \\
2A_1t^4 + B_1r^4 &= \frac{2h_1h_2}{5} \\
A_1s^2t^2 &= \frac{h_1h_2}{9},
\end{align*}
\]

(5)

and

\[
\begin{align*}
A_2 + B_2 &= h_1h_2 \\
2A_2p^2 + B_2q^2 &= \frac{2h_1h_2}{3} \\
2A_2p^4 + B_2q^4 &= \frac{2h_1h_2}{5} \\
A_2p^4 &= \frac{h_1h_2}{9}
\end{align*}
\]

(6)

Solving the above system of non-linear equations simultaneously we obtain the following two quadrature rules

\[
R_7(f) = h_1h_2[8/7f(z_{10}, z_{20}) + 5/9 \sum f(z_{10} \pm h_1\sqrt{3}/\sqrt{5}, z_{20} \pm h_2\sqrt{1/3}) + 20/63 \sum f(z_{10}, z_{20} \pm h_2\sqrt{14}/\sqrt{15})]
\]

(7)

\[
R_8(f) = h_1h_2[9/49 \sum f(z_{10} \pm h_1\sqrt{7}/3, z_{20} \pm h_2\sqrt{7}/3) + 40/49 \sum \{ f(z_{10}, z_{20} \pm h_2\sqrt{7}/\sqrt{15}) + f(z_{10}, z_{20} \pm h_2\sqrt{7}/\sqrt{15}) \}]
\]

(8)

3. Analysis of truncation error

The truncation errors associated with the rules \(R_7(f)\) and \(R_8(f)\) are given by

\[
\begin{align*}
E_7(f) &= I(f) - R_7(f) \\
E_8(f) &= I(f) - R_8(f)
\end{align*}
\]

(9)
As the function \( f(z_1, z_2) \) is analytic in the product domain \( D_1 \times D_2 \) the Taylor series expansion of the function about the point \((z_{10}, z_{20})\) is given by

\[
f(z_1, z_2) = \sum_{n=0}^{\infty} \frac{1}{n!} \left( (z_1 - z_{10}) \frac{\partial}{\partial z_1} + (z_2 - z_{20}) \frac{\partial}{\partial z_2} \right)^n f(z_{10}, z_{20})
\]  

(10)

where the partial derivatives have been evaluated at \((z_{10}, z_{20})\).

From equations (1), (7) — (10) we obtain after simplification the following by setting \( h_1 = h_2 = h \):

\[
E_7(f) = \frac{16h^8}{9 \times 6!} \left\{ \frac{9}{175} f^{6+0} + f^{2+4} - \frac{32}{525} f^{0+6} \right\} + O(h^{10})
\]

(11)

and

\[
E_8(f) = \frac{16h^8}{27 \times 6!} \left\{ \frac{53}{525} (f^{6+0} + f^{0+6}) - 2(f^{2+4} + f^{4+2}) \right\} + O(h^{10})
\]

(12)

The following theorems are evident from equations (11) and (12)

**Theorem 1:**
The degree of precision of each of the rules \( R_7(f) \) and \( R_8(f) \) is five.

**Theorem 2:**
If \( |f^{\mu+v}| \leq L \) for \( \mu + v = 6 \), then \( |E_7(f)| \approx 0.0027L \) and \( |E_8(f)| \approx 0.0034L \).

It is clear from theorem 2 that the rule \( R_7(f) \) is slightly more accurate than the rule \( R_8(f) \).

4. Numerical Verification

The following integrals have been computed and the computed values have been given in Table-1.

\[
I_1(f) = \int_{-1.3+i/3}^{1.3+i/3} e^{z_1 z_2} dz_1 dz_2
\]

(13)

\[
I_2(f) = \int_{1+i/5}^{1+i/4} \sin(z_1 - z_2) dz_1 dz_2
\]

(14)

\[
I_3(f) = \int_{-0.9+i/4}^{0.9+i/2} \int_{1+i/3}^{1+i/3} \sqrt{z_1 z_2} \sec^2 z_2 dz_1 dz_2
\]

(15)
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Table 1

| Integral | Exact value | Rule | Computed value | |Error |
|----------|-------------|------|----------------|------|
| $I_1(f)$ | $-0.034690113150275 +0.071320855529817i$ | $R_7(f)$ | $-0.034690113818110 +0.071320855595101i$ | $6.71 \times 10^{-10}$ |
|          |             | $R_8(f)$ | $-0.034690112094836 +0.071320855426260i$ | $1.06 \times 10^{-9}$ |
| $I_2(f)$ | $0.017392568524770 -0.062450387956485i$ | $R_7(f)$ | $0.017392568369428 -0.062450387934521i$ | $1.56 \times 10^{-10}$ |
|          |             | $R_8(f)$ | $0.017392568417899 -0.062450387785644i$ | $2.01 \times 10^{-10}$ |
| $I_3(f)$ | $0.012671332230430 -0.125505090694779i$ | $R_7(f)$ | $0.012671229979194 -0.125505225546791i$ | $1.69 \times 10^{-7}$ |
|          |             | $R_8(f)$ | $0.012671551628294 -0.125504766593837i$ | $3.91 \times 10^{-7}$ |

The computed values in the table show that the rule $R_7(f)$ is more accurate than the eight point rule. Further if the quantity $|h|$ is not relatively small then it is necessary to apply the compound rule based on preferably the seven point rule by subdivision of the range of integration.

References


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